2 Polynomials

Lesson at a Glance

1. In the polynomial $p(x)$, the highest exponent of x is called the degree of the polynomial $p(x)$.
2. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.
3. A linear polynomial in x is of the form $ax + b$, $a \neq 0$.
4. A quadratic polynomial in x is of the form $ax^2 + bx + c$, $a \neq 0$.
5. A cubic polynomial in x is of the form $ax^3 + bx^2 + cx + d$, $a \neq 0$.
6. A constant polynomial is free from a variable. 0, 7, 8, – 4, etc., are examples of constant polynomials.
7. 0 is also called the zero polynomial.
8. The degree of a non-zero constant polynomial is 0.
9. The degree of the zero polynomial does not exist.
10. In the polynomial $3x^2 - 4x + 7$, the expressions $3x^2$, $-4x$ and 7 are called the terms of the polynomial.
11. If a polynomial consists of only one variable, then the polynomial is called polynomial in one variable.
12. Polynomials consisting of one term, two terms and three terms are called monomial, binomial and trinomial respectively.
13. A real number k is a zero of the polynomial $p(x)$, if $p(k) = 0$.
14. Every linear polynomial in one variable has a unique zero.
15. A non-zero constant polynomial has no zero.
16. Every real number is a zero of the zero polynomial.
17. A quadratic polynomial has at most two zeroes.
18. A cubic polynomial has at most three zeroes.
19. The polynomial equation of the polynomial $p(x)$ is given by $p(x) = 0$.

20. **Remainder Theorem:** If $p(x)$ is a polynomial of degree greater than or equal to 1 and $f(x)$ is divided by the linear polynomial $x - a$, then the remainder is $f(a)$.

21. Dividend = (Divisor × Quotient) + Remainder.

22. If dividend, divisor, quotient and remainder are respectively $f(x)$, $g(x)$, $q(x)$ and $r(x)$ such that degree of $f(x) \geq$ degree of $g(x)$ with $g(x) \neq 0$,

    then $$f(x) = g(x) \times q(x) + r(x)$$

    where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

23. **Factor Theorem:** If $x - a$ is a factor of $f(x)$, then $f(a) = 0$.

24. $(x + y)^2 = x^2 + 2xy + y^2$

25. $(x - y)^2 = x^2 - 2xy + y^2$

26. $x^2 - y^2 = (x + y)(x - y)$

27. $(x + a)(x + b) = x^2 + (a + b)x + ab$

28. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

29. $(x + y)^3 = x^3 + y^3 + 3xy(x + y) = x^3 + 3x^2y + 3xy^2 + y^3$

30. $(x - y)^3 = x^3 - y^3 - 3xy(x - y) = x^3 - 3x^2y + 3xy^2 - y^3$

31. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

32. $x^3 + y^3 + z^3 = 3xyz$, if $x + y + z = 0$

TEXTBOOK QUESTIONS SOLVED

Exercise 2.1 *(Page – 32)*

1. *Which of the following expressions are polynomials in one variable and which are not? State reasons for your answer.*

   (*i*) $4x^2 - 3x + 7$ (*ii*) $y^2 + \sqrt{2}$ (*iii*) $3\sqrt{t} + t\sqrt{2}$

   (*iv*) $y + \dfrac{2}{y}$ (*v*) $x^{10} + y^3 + t^{50}$.

Sol. (*i*) $4x^2 - 3x + 7$ is a polynomial in variable x, as exponents of the variable in different terms are whole numbers.

(*ii*) $y^2 + \sqrt{2}$ is a polynomial in variable y, as exponent of variable y is a whole number.

(iii) $3\sqrt{t} + t\sqrt{2} = 3t^{1/2} + \sqrt{2}\,t$ is not a polynomial, as exponent of variable t in first term, *i.e.*, $3\,t^{1/2}$ is not a whole number.

(iv) $y + \frac{2}{y} = y + 2y^{-1}$ is not a polynomial, as exponent of the variable y in the term $2y^{-1}$ is not a whole number.

(v) $x^{10} + y^3 + t^{50}$ is a polynomial in three variables.

2. *Write the coefficient of x^2 in each of the following:*

(i) $2 + x^2 + x$ (ii) $2 - x^2 + x^3$ (iii) $\frac{\pi}{2}x^2 + x$

(iv) $\sqrt{2}\,x - 1$.

Sol. (i) Coefficient of x^2 in $2 + x^2 + x$ is 1.

(ii) Coefficient of x^2 in $2 - x^2 + x^3$ is -1.

(iii) Coefficient of x^2 in $\frac{\pi}{2}x^2 + x$ is $\frac{\pi}{2}$.

(iv) Coefficient of x^2 in $\sqrt{2}\,x - 1$ is 0.

3. *Give one example each of a binomial of degree 35, and of a monomial of degree 100.*

Sol. (i) Binomial of degree 35 is $2x^{35} + x$.

(ii) Monomial of degree 100 is $-7x^{100}$.

4. *Write the degree of each of the following polynomials:*

(i) $5x^3 + 4x^2 + 7x$ (ii) $4 - y^2$

(iii) $5t - \sqrt{7}$ (iv) 3.

Sol. (i) Degree of polynomial $5x^3 + 4x^2 + 7x$ is 3.

(ii) Degree of polynomial $4 - y^2$ is 2.

(iii) Degree of polynomial $5t - \sqrt{7}$ is 1.

(iv) Degree of polynomial 3 is 0.

5. *Classify the following as linear, quadratic and cubic polynomials:*

(i) $x^2 + x$ (ii) $x - x^3$ (iii) $y + y^2 + 4$

(iv) $1 + x$ (v) $3t$ (vi) r^2

(vii) $7x^3$.

Sol. (i) Polynomial $x^2 + x$ is a quadratic polynomial.

(ii) Polynomial $x - x^3$ is a cubic polynomial.

(iii) Polynomial $y + y^2 + 4$ is a quadratic polynomial.

(iv) Polynomial $1 + x$ is a linear polynomial.

(v) Polynomial $3t$ is a linear polynomial.

(vi) Polynomial r^2 is a quadratic polynomial.

(vii) Polynomial $7x^3$ is a cubic polynomial.

Exercise 2.2 *(Pages –34-35)*

1. *Find the value of the polynomial $5x - 4x^2 + 3$ at*

(i) $x = 0$ (ii) $x = -1$ (iii) $x = 2$

Sol. Let $f(x) = 5x - 4x^2 + 3$.

(i) $f(0) = 0 - 0 + 3 = 3$ (ii) $f(-1) = -5 - 4 + 3 = -6$

(iii) $f(2) = 10 - 16 + 3 = -3$.

2. *Find p(0), p(1) and p(2) for each of the following polynomials:*

(i) $p(y) = y^2 - y + 1$ (ii) $p(t) = 2 + t + 2t^2 - t^3$

(iii) $p(x) = x^3$ (iv) $p(x) = (x - 1)(x + 1)$.

Sol. (i) $p(0) = 0 - 0 + 1 = 1$; $p(1) = 1 - 1 + 1 = 1$;

$p(2) = 4 - 2 + 1 = 3$.

(ii) $p(0) = 2 + 0 + 0 - 0 = 2$; $p(1) = 2 + 1 + 2 - 1 = 4$;

$p(2) = 2 + 2 + 8 - 8 = 4$.

(iii) $p(0) = 0$; $p(1) = 1$; $p(2) = 8$.

(iv) $p(0) = (0 - 1)(0 + 1) = -1$; $p(1) = (1 - 1)(1 + 1) = 0$;

$p(2) = (2 - 1)(2 + 1) = 3$.

3. *Verify whether the following are zeroes of the polynomial, indicated against them.*

(i) $p(x) = 3x + 1,\ x = -\frac{1}{3}$ (ii) $p(x) = 5x - \pi,\ x = \frac{4}{5}$

(iii) $p(x) = x^2 - 1,\ x = 1, -1$ (iv) $p(x) = (x + 1)(x - 2),\ x = -1, 2$

(v) $p(x) = x^2,\ x = 0$ (vi) $p(x) = lx + m,\ x = -\frac{m}{l}$

(*vii*) $p(x) = 3x^2 - 1, x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$

(*viii*) $p(x) = 2x + 1, x = \frac{1}{2}.$

Sol. (*i*) $p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right) + 1 = -1 + 1 = 0$

Hence, $x = -\frac{1}{3}$ is a zero of the polynomial $p(x)$.

(*ii*) $p\left(\frac{4}{5}\right) = 5 \times \frac{4}{5} - \pi = 4 - \pi \neq 0$

Hence, $x = \frac{4}{5}$ is not a zero of the polynomial $p(x)$.

(*iii*) $p(1) = 1 - 1 = 0$ and $p(-1) = 1 - 1 = 0$

Hence, $x = 1$ and $x = -1$ are zeroes of the polynomial $p(x)$.

(*iv*) $p(-1) = (-1 + 1)(-1 - 2) = 0$ and $p(2)$

$= (2 + 1)(2 - 2) = 0$

Hence, $x = -1$ and $x = 2$ are zeroes of the polynomial $p(x)$.

(*v*) $p(0) = 0$. Hence, $x = 0$ is a zero of the polynomial $p(x)$.

(*vi*) $p\left(-\frac{m}{l}\right) = l \cdot \left(-\frac{m}{l}\right) + m = -m + m = 0$

Hence, $x = -\frac{m}{l}$ is a zero of the polynomial $p(x)$.

(*vii*) $p\left(-\frac{1}{\sqrt{3}}\right) = 3 \times \frac{1}{3} - 1 = 1 - 1 = 0$

and $p\left(\frac{2}{\sqrt{3}}\right) = 3 \times \frac{4}{3} - 1 = 4 - 1 = 3 \neq 0$

Hence, $x = -\frac{1}{\sqrt{3}}$ is a zero and $x = \frac{2}{\sqrt{3}}$ is not a zero of the polynomial $p(x)$.

(*viii*) $p\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} + 1 = 1 + 1 = 2 \neq 0$

Hence, $x = \frac{1}{2}$ is not a zero of the polynomial $p(x)$.

4. *Find the zero of the polynomial in each of the following cases:*

(*i*) $p(x) = x + 5$ (*ii*) $p(x) = x - 5$ (*iii*) $p(x) = 2x + 5$

(*iv*) $p(x) = 3x - 2$ (*v*) $p(x) = 3x$ (*vi*) $p(x) = ax$, $a \neq 0$

(*vii*) $p(x) = cx + d$, $c \neq 0$, *c, d are real numbers.*

Sol. (*i*) For zero, $p(x) = 0 \Rightarrow x + 5 = 0$

$\Rightarrow x = -5$ is a zero of the polynomial $p(x)$.

(*ii*) For zero, $p(x) = 0 \Rightarrow x - 5 = 0$

$\Rightarrow x = 5$ is a zero of the polynomial $p(x)$.

(*iii*) For zero, $p(x) = 0 \Rightarrow 2x + 5 = 0$

$\Rightarrow x = -\frac{5}{2}$ is a zero of the polynomial $p(x)$.

(*iv*) For zero, $p(x) = 0 \Rightarrow 3x - 2 = 0$

$\Rightarrow x = \frac{2}{3}$ is a zero of the polynomial $p(x)$.

(*v*) For zero, $p(x) = 0 \Rightarrow 3x = 0$

$\Rightarrow x = 0$ is a zero of the polynomial $p(x)$.

(*vi*) For zero, $p(x) = 0 \Rightarrow ax = 0$

$\Rightarrow x = 0$, as $a \neq 0$

Therefore, $x = 0$ is a zero of the polynomial $p(x)$.

(*vii*) For zero, $p(x) = 0 \Rightarrow cx + d = 0$

$\Rightarrow x = -\frac{d}{c}$, $(c \neq 0)$

Therefore, $x = -\frac{d}{c}$ is a zero of the polynomial $p(x)$.

Exercise 2.3 *(Page – 40)*

1. *Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by*

(i) $x + 1$ (ii) $x - \frac{1}{2}$ (iii) x

(iv) $x + \pi$ (v) $5 + 2x$.

Sol. Given: Let $p(x) = x^3 + 3x^2 + 3x + 1$.

(i) When $p(x)$ is divided by $x + 1$, the remainder is $p(-1)$.

$$\therefore \quad p(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1$$
$$= -1 + 3 - 3 + 1 = 0.$$

(ii) When $p(x)$ is divided by $\left(x - \frac{1}{2}\right)$, the remainder is $p\left(\frac{1}{2}\right)$.

$$\therefore \quad p\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) + 1$$
$$= \frac{1}{8} + \frac{3}{4} + \frac{3}{2} + 1$$
$$= \frac{1 + 6 + 12 + 8}{8} = \frac{27}{8}.$$

(iii) When $p(x)$ is divided by x, the remainder is $p(0)$.

$\therefore \quad p(0) = 0 + 0 + 0 + 1 = 1.$

(iv) When $p(x)$ is divided by $x + \pi$, the remainder is $p(-\pi)$.

$$\therefore \quad p(-\pi) = (-\pi)^3 + 3(-\pi)^2 + 3(-\pi) + 1$$
$$= -\pi^3 + 3\pi^2 - 3\pi + 1.$$

(v) When $p(x)$ is divided by $5 + 2x$, the remainder is $p\left(-\frac{5}{2}\right)$

$$\therefore \quad p\left(-\frac{5}{2}\right) = \left(-\frac{5}{2}\right)^3 + 3\left(-\frac{5}{2}\right)^2 + 3\left(-\frac{5}{2}\right) + 1$$
$$= -\frac{125}{8} + \frac{75}{4} - \frac{15}{2} + 1$$
$$= \frac{-125 + 150 - 60 + 8}{8} = \frac{-27}{8}.$$

2. *Find the remainder when $x^3 - ax^2 + 6x - a$ is divided by $x - a$.*

Sol. When $p(x) = x^3 - ax^2 + 6x - a$ is divided by $(x - a)$, the remainder is $p(a)$.

$\therefore \quad p(a) = a^3 - a^3 + 6a - a = 5a.$

3. *Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.*

Sol. If $7 + 3x$ is a factor of $3x^3 + 7x$, then $p\left(-\frac{7}{3}\right) = 0$

$$\therefore \quad p\left(-\frac{7}{3}\right) = 3 \times \left(\frac{-7}{3}\right)^3 + 7 \times \frac{(-7)}{3}$$

$$= 3 \times \frac{-343}{27} - \frac{49}{3} = \frac{-343}{9} - \frac{49}{3}$$

$$= \frac{-343 - 147}{9} = \frac{-490}{9} \neq 0.$$

As remainder is not zero, hence $(7 + 3x)$ is not a factor of $3x^3 + 7x$.

Exercise 2.4 *(Pages – 43-44)*

1. *Determine which of the following polynomials has $(x + 1)$ a factor:*

(*i*) $x^3 + x^2 + x + 1$ (*ii*) $x^4 + x^3 + x^2 + x + 1$

(*iii*) $x^4 + 3x^3 + 3x^2 + x + 1$ (*iv*) $x^3 - x^2 - (2 + \sqrt{2})x + \sqrt{2}$

Sol. (*i*) If $(x + 1)$ is a factor of $p(x) = x^3 + x^2 + x + 1$, then $p(-1) = 0$

Now, $p(-1) = -1 + 1 - 1 + 1 = 0.$

Hence, $(x + 1)$ is a factor.

(*ii*) If $(x + 1)$ is a factor of $p(x) = x^4 + x^3 + x^2 + x + 1$, then $p(-1) = 0$.

Now, $p(-1) = 1 - 1 + 1 - 1 + 1 = 1 \neq 0.$

Hence, $(x + 1)$ is not a factor.

(*iii*) If $(x + 1)$ is a factor of $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$, then $p(-1) = 0$.

Now, $p(-1) = 1 - 3 + 3 - 1 + 1 = 1 \neq 0.$

Hence, $(x + 1)$ is not a factor.

(*iv*) If $(x + 1)$ is a factor of $p(x) = x^3 - x^2 - (2 + \sqrt{2})x + \sqrt{2}$, then $p(-1) = 0$.

Now, $p(-1) = (-1)^3 - (-1)^2 - (2 + \sqrt{2})(-1) + \sqrt{2}$

$= -1 - 1 + 2 + \sqrt{2} + \sqrt{2}$

$= 2\sqrt{2} \neq 0$.

Hence, $(x + 1)$ is not a factor.

2. *Use the Factor Theorem to determine whether g(x) is a factor of p(x) in each of the following cases:*

(*i*) $p(x) = 2x^3 + x^2 - 2x - 1,\ g(x) = x + 1$

(*ii*) $p(x) = x^3 + 3x^2 + 3x + 1,\ g(x) = x + 2$

(*iii*) $p(x) = x^3 - 4x^2 + x + 6,\ g(x) = x - 3.$

Sol. (*i*) If $g(x)$ is a factor of $p(x)$, then $p(-1) = 0$.

Now, $p(-1) = -2 + 1 + 2 - 1 = 0$.

Hence, $g(x)$ is a factor of $p(x)$.

(*ii*) If $g(x)$ is a factor of $p(x)$, then $p(-2) = 0$.

Now, $p(-2) = -8 + 12 - 6 + 1 = -14 + 13 = -1 \neq 0$.

Hence, $g(x)$ is not a factor of $p(x)$.

(*iii*) If $g(x)$ is a factor of $p(x)$, then $p(3) = 0$.

Now, $p(3) = 27 - 36 + 3 + 6 = 36 - 36 = 0$.

Hence, $g(x)$ is a factor of $p(x)$.

3. *Find the value of k, if x – 1 is a factor of p(x) in each of the following cases:*

(*i*) $p(x) = x^2 + x + k$ (*ii*) $p(x) = 2x^2 + kx + \sqrt{2}$

(*iii*) $p(x) = kx^2 - \sqrt{2}x + 1$ (*iv*) $p(x) = kx^2 - 3x + k.$

Sol. (*i*) If $(x - 1)$ is a factor of $p(x)$, then $p(1) = 0$.

Now, $p(1) = 0 \Rightarrow 1 + 1 + k = 0 \Rightarrow k = -2$.

(*ii*) If $(x - 1)$ is a factor of $p(x)$, then $p(1) = 0$.

Now, $p(1) = 0 \Rightarrow 2 + k + \sqrt{2} = 0 \Rightarrow k = -(2 + \sqrt{2})$.

(*iii*) If $(x - 1)$ is a factor of $p(x)$, then $p(1) = 0$.

Now, $p(1) = 0 \Rightarrow k - \sqrt{2} + 1 = 0 \Rightarrow k = \sqrt{2} - 1$.

(*iv*) If $(x - 1)$ is a factor of $p(x)$, then $p(1) = 0$.

Now, $p(1) = 0 \Rightarrow k - 3 + k = 0 \Rightarrow k = \frac{3}{2}$.

4. *Factorise*:

(*i*) $12x^2 - 7x + 1$ (*ii*) $2x^2 + 7x + 3$

(*iii*) $6x^2 + 5x - 6$ (*iv*) $3x^2 - x - 4$.

Sol. (*i*) $12x^2 - 7x + 1 = 12x^2 - 4x - 3x + 1$

$= 4x(3x - 1) - 1(3x - 1) = (4x - 1)(3x - 1)$

(*ii*) $2x^2 + 7x + 3 = 2x^2 + 6x + x + 3$

$= 2x(x + 3) + 1(x + 3) = (2x + 1)(x + 3)$.

(*iii*) $6x^2 + 5x - 6 = 6x^2 + 9x - 4x - 6$

$= 3x(2x + 3) - 2(2x + 3) = (3x - 2)(2x + 3)$.

(*iv*) $3x^2 - x - 4 = 3x^2 - 4x + 3x - 4$

$= x(3x - 4) + 1(3x - 4) = (x + 1)(3x - 4)$.

5. *Factorise*:

(*i*) $x^3 - 2x^2 - x + 2$ (*ii*) $x^3 - 3x^2 - 9x - 5$

(*iii*) $x^3 + 13x^2 + 32x + 20$ (*iv*) $2y^3 + y^2 - 2y - 1$.

Sol. (*i*) Let $p(x) = x^3 - 2x^2 - x + 2$.

Possible factors of 2 are $\pm 1, \pm 2$.

We notice $p(1) = 1 - 2 - 1 + 2 = 0$

$\Rightarrow$ $x = 1$ is a zero of polynomial $p(x)$ or $(x - 1)$ is a factor of $p(x)$.

Let us divide $p(x)$ by $(x - 1)$.

$$\begin{array}{r|l} x-1 \,\overline{)\, x^3 - 2x^2 - x + 2 \,}\; (x^2 - x - 2 \\ x^3 - x^2 \\ - \quad + \\ \hline -x^2 - x \\ -x^2 + x \\ + \quad - \\ \hline -2x + 2 \\ -2x + 2 \\ + \quad - \\ \hline 0 \end{array}$$

First term of quotient $= \frac{x^3}{x} = x^2$

Second term of quotient $= \frac{-x^2}{x} = -x$

Third term of quotient $= \frac{-2x}{x} = -2$

$\therefore \quad p(x) = x^3 - 2x^2 - x + 2$

$= (x - 1)(x^2 - x - 2) = (x - 1)(x^2 - 2x + x - 2)$

$= (x - 1)\{x(x - 2) + 1(x - 2)\}$

$= (x - 1)(x + 1)(x - 2).$

Alternative Method:

$x^3 - 2x^2 - x + 2 = x^2(x - 2) - 1(x - 2)$

$= (x - 2)(x^2 - 1)$

$= (x - 2)(x^2 - 1^2)$

$= (x - 2)(x + 1)(x - 1).$

(*ii*) Let $p(x) = x^3 - 3x^2 - 9x - 5.$

Possible factors of 5 are $\pm 1, \pm 5$

We notice $p(-1) = -1 - 3 + 9 - 5 = 0$

$\Rightarrow \quad x = -1$ is zero of polynomial $p(x)$.

$\Rightarrow \quad (x + 1)$ is a factor of $p(x)$.

Let us divide $p(x)$ by $(x + 1)$.

$$\begin{array}{r|l} x+1 \,\big)\, x^3 - 3x^2 - 9x - 5 \,\big(\, x^2 - 4x - 5 & \\ \underline{x^3 + x^2} & \\ - \quad - & \\ -4x^2 - 9x & \\ \underline{-4x^2 - 4x} & \\ + \quad + & \\ -5x - 5 & \\ \underline{-5x - 5} & \\ + \quad + & \\ 0 & \end{array}$$

First term of quotient $= \dfrac{x^3}{x} = x^2$

Second term of quotient $= \dfrac{-4x^2}{x} = -4x$

Third term of quotient $= \dfrac{-5x}{x} = -5$

$\therefore \quad p(x) = x^3 - 3x^2 - 9x - 5$

$= (x + 1)(x^2 - 4x - 5)$

$= (x + 1)(x^2 - 5x + x - 5)$

$= (x + 1)\{x(x - 5) + 1(x - 5)\}$

$= (x + 1)(x + 1)(x - 5).$

(*iii*) Let $p(x) = x^3 + 13x^2 + 32x + 20.$

Possible factors of 20 are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20.$

We notice $p(-1) = -1 + 13 - 32 + 20 = 0$

$\Rightarrow$ $x = -1$ is a zero of $p(x)$

$\Rightarrow$ $(x + 1)$ is a factor of $p(x)$.

Let us divide $p(x)$ by $(x + 1)$.

$$\begin{array}{r|l} x+1 \overline{\smash{)}\, x^3 + 13x^2 + 32x + 20} & \text{quotient } x^2 + 12x + 20 \\ x^3 + x^2 & \\ - \quad - & \\ \hline 12x^2 + 32x & \\ 12x^2 + 12x & \\ - \quad - & \\ \hline 20x + 20 & \\ 20x + 20 & \\ - \quad - & \\ \hline 0 & \end{array}$$

First term of quotient

$$= \frac{x^3}{x} = x^2$$

Second term of quotient

$$= \frac{12x^2}{x} = 12x$$

Third term of quotient

$$= \frac{20x}{x} = 20$$

$\therefore$ $p(x) = x^3 + 13x^2 + 32x + 20$

$= (x + 1)(x^2 + 12x + 20)$

$= (x + 1)(x^2 + 10x + 2x + 20)$

$= (x + 1)\{x(x + 10) + 2(x + 10)\}$

$= (x + 1)(x + 2)(x + 10)$.

(*iv*) Let $p(y) = 2y^3 + y^2 - 2y - 1$

$= y^2(2y + 1) - 1(2y + 1) = (y^2 - 1)(2y + 1)$

$= (y - 1)(y + 1)(2y + 1)$.

Exercise 2.5 *(Pages – 48-50)*

1. *Use suitable identities to find the following products:*

(*i*) $(x + 4)(x + 10)$ (*ii*) $(x + 8)(x - 10)$

(*iii*) $(3x + 4)(3x - 5)$ (*iv*) $\left(y^2 + \frac{3}{2}\right)\left(y^2 - \frac{3}{2}\right)$

(*v*) $(3 - 2x)(3 + 2x)$.

Sol. (*i*) $(x + 4)(x + 10) = x^2 + (4 + 10)x + (4 \times 10)$

[Using identity: $(x + a)(x + b) = x^2 + (a + b)x + ab$]

$= x^2 + 14x + 40$.

(*ii*) $(x + 8)(x - 10) = x^2 + (8 - 10)x + 8(-10)$

[Using identity: $(x + a)(x + b) = x^2 + (a + b)x + ab$]

$= x^2 - 2x - 80.$

(*iii*) $(3x + 4)(3x - 5) = (3x)^2 + (4 - 5)(3x) + 4 \times (-5)$

[Using identity: $(x + a)(x + b) = x^2 + (a + b)x + ab$]

$= 9x^2 - 3x - 20.$

(*iv*) $\left(y^2 + \frac{3}{2}\right)\left(y^2 - \frac{3}{2}\right) = (y^2)^2 - \left(\frac{3}{2}\right)^2$

[Using identity: $(x + a)(x - a) = x^2 - a^2$]

$= y^4 - \frac{9}{4}.$

(*v*) $(3 - 2x)(3 + 2x) = (3)^2 - (2x)^2$

[Using identity: $(x + a)(x - a) = x^2 - a^2$]

$= 9 - 4x^2.$

2. ***Evaluate the following products without multiplying directly:***

(*i*) *103 × 107* (*ii*) *95 × 96* (*iii*) *104 × 96.*

Sol. (*i*) $103 \times 107 = (100 + 3)(100 + 7)$

$= (100)^2 + (3 + 7) \times 100 + 3 \times 7$

[Using identity: $(x + a)(x + b) = x^2 + (a + b)x + ab$]

$= 10000 + 1000 + 21 = 11021.$

(*ii*) $95 \times 96 = (90 + 5)(90 + 6)$

$= (90)^2 + (5 + 6) \times 90 + 5 \times 6$

[Using identity: $(x + a)(x + b) = x^2 + (a + b)x + ab$]

$= 8100 + 990 + 30 = 9120.$

(*iii*) $104 \times 96 = (100 + 4)(100 - 4) = (100)^2 - (4)^2$

[Using identity: $(x + y)(x - y) = x^2 - y^2$]

$= 10000 - 16 = 9984.$

3. ***Factorise the following using appropriate identities:***

(*i*) $9x^2 + 6xy + y^2$ (*ii*) $4y^2 - 4y + 1$ (*iii*) $x^2 - \frac{y^2}{100}$.

Sol. (*i*) $9x^2 + 6xy + y^2 = (3x)^2 + 2 \times 3x \times y + (y)^2 = (3x + y)^2$

[Using identity: $x^2 + 2xy + y^2 = (x + y)^2$]

$= (3x + y)(3x + y)$

(ii) $4y^2 - 4y + 1 = (2y)^2 - 2 \times 2y \times 1 + (1)^2 = (2y - 1)^2$

[Using identity: $x^2 - 2xy + y^2 = (x - y)^2$]

$= (2y - 1)(2y - 1).$

(iii) $x^2 - \dfrac{y^2}{100} = (x)^2 - \left(\dfrac{y}{10}\right)^2 = \left(x - \dfrac{y}{10}\right)\left(x + \dfrac{y}{10}\right).$

[Using identity: $a^2 - b^2 = (a - b)(a + b)$]

4. *Expand each of the following, using suitable identities:*

(i) $(x + 2y + 4z)^2$ (ii) $(2x - y + z)^2$

(iii) $(-2x + 3y + 2z)^2$ (iv) $(3a - 7b - c)^2$

(v) $(-2x + 5y - 3z)^2$ (vi) $\left[\dfrac{1}{4}a - \dfrac{1}{2}b + 1\right]^2.$

Sol. (i) $(x + 2y + 4z)^2 = (x)^2 + (2y)^2 + (4z)^2 + 2(x)(2y)$
$+ 2(2y)(4z) + 2(x)(4z)$

[Using identity: $(x + y + z)^2 = x^2 + y^2$
$+ z^2 + 2xy + 2yz + 2zx$]

$= x^2 + 4y^2 + 16z^2 + 4xy + 16yz + 8xz.$

(ii) $(2x - y + z)^2 = (2x)^2 + (-y)^2 + (z)^2 + 2(2x)(-y)$
$+ 2(-y)(z) + 2(2x)(z)$

[Using identity: $(x + y + z)^2 = x^2 + y^2$
$+ z^2 + 2xy + 2yz + 2zx$]

$= 4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz.$

(iii) $(-2x + 3y + 2z)^2 = (-2x)^2 + (3y)^2 + (2z)^2$
$+ 2(-2x)(3y) + 2(3y)(2z) + 2(-2x)(2z)$

[Using identity: $(x + y + z)^2 = x^2 + y^2$
$+ z^2 + 2xy + 2yz + 2zx$]

$= 4x^2 + 9y^2 + 4z^2 - 12xy + 12yz - 8xz.$

(iv) $(3a - 7b - c)^2 = (3a)^2 + (-7b)^2 + (-c)^2 + 2(3a)(-7b)$
$+ 2(-7b)(-c) + 2(3a)(-c)$

[Using identity: $(x + y + z)^2 = x^2 + y^2$
$+ z^2 + 2xy + 2yz + 2zx$]

$= 9a^2 + 49b^2 + c^2 - 42ab + 14bc - 6ac.$

(*v*) $(-2x+5y-3z)^2 = (-2x)^2 + (5y)^2 + (-3z)^2$
$+ 2(-2x)(5y) + 2(5y)(-3z) + 2(-2x)(-3z)$
[Using identity: $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$]
$= 4x^2 + 25y^2 + 9z^2 - 20xy - 30yz + 12xz.$

(*vi*) $\left[\frac{1}{4}a - \frac{1}{2}b + 1\right]^2 = \left(\frac{1}{4}a\right)^2 + \left(-\frac{1}{2}b\right)^2 + (1)^2$
$+ 2\left(\frac{1}{4}a\right)\left(-\frac{1}{2}b\right) + 2\left(-\frac{1}{2}b\right)(1) + 2\left(\frac{1}{4}a\right)(1)$
[Using identity: $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$]
$= \frac{1}{16}a^2 + \frac{1}{4}b^2 + 1 - \frac{1}{4}ab - b + \frac{1}{2}a.$

5. ***Factorise:***

(*i*) $4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16xz$

(*ii*) $2x^2 + y^2 + 8z^2 - 2\sqrt{2}\,xy + 4\sqrt{2}\,yz - 8xz.$

Sol. (*i*) $4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16xz = (2x)^2$
$+ (3y)^2 + (-4z)^2 + 2(2x)(3y) + 2(3y)(-4z) + 2(2x)(-4z)$
[Using identity: $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x+y+z)^2$]
$= (2x + 3y - 4z)^2.$
$= (2x + 3y - 4z)(2x + 3y - 4z).$

(*ii*) $2x^2 + y^2 + 8z^2 - 2\sqrt{2}\,xy + 4\sqrt{2}\,yz - 8xz$
$= (-\sqrt{2}\,x)^2 + (y)^2 + (2\sqrt{2}\,z)^2 + 2(-\sqrt{2}\,x)(y)$
$+ 2(y)(2\sqrt{2}\,z) + 2(-\sqrt{2}\,x)(2\sqrt{2}\,z)$
[Using identity: $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x+y+z)^2$]
$= (-\sqrt{2}\,x + y + 2\sqrt{2}\,z)^2$
$= (-\sqrt{2}\,x + y + 2\sqrt{2}\,z)(-\sqrt{2}\,x + y + 2\sqrt{2}\,z).$

6. ***Write the following cubes in expanded form:***

(*i*) $(2x+1)^3$ (*ii*) $(2a-3b)^3$ (*iii*) $\left[\frac{3}{2}x+1\right]^3$ (*iv*) $\left[x - \frac{2}{3}y\right]^3$.

Sol. (*i*) $(2x + 1)^3 = (2x)^3 + (1)^3 + 3(2x)(1)(2x + 1)$

[Using identity: $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$]

$= 8x^3 + 1 + 12x^2 + 6x = 8x^3 + 12x^2 + 6x + 1.$

(*ii*) $(2a - 3b)^3 = (2a)^3 - (3b)^3 - 3(2a)(3b)(2a - 3b)$

[Using identity: $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$]

$= 8a^3 - 27b^3 - 36a^2b + 54ab^2.$

(*iii*) $\left(\frac{3}{2}x+1\right)^3 = \left(\frac{3}{2}x\right)^3 + (1)^3 + 3 \times \frac{3}{2}x \times 1\left(\frac{3}{2}x+1\right)$

[Using identity: $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$]

$= \frac{27}{8}x^3 + 1 + \frac{27}{4}x^2 + \frac{9}{2}x = \frac{27}{8}x^3 + \frac{27}{4}x^2$

$+ \frac{9}{2}x + 1.$

(*iv*) $\left(x-\frac{2}{3}y\right)^3 = (x)^3 - \left(\frac{2}{3}y\right)^3 - 3(x)\left(\frac{2}{3}y\right)\left(x-\frac{2}{3}y\right)$

[Using identity: $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$]

$= x^3 - \frac{8}{27}y^3 - 2x^2y + \frac{4}{3}xy^2.$

7. ***Evaluate the following using suitable identities:***

(*i*) *$(99)^3$* (*ii*) *$(102)^3$* (*iii*) *$(998)^3$*.

Sol. (*i*) $(99)^3 = (100 - 1)^3$

$= (100)^3 - (1)^3 - 3 \times 100 \times 1\ (100 - 1)$

[Using identity: $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$]

$= 1000000 - 1 - 30000 + 300 = 970299.$

(*ii*) $(102)^3 = (100 + 2)^3$

$= (100)^3 + (2)^3 + 3 \times 100 \times 2\ (100 + 2)$

[Using identity: $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$]

$= 1000000 + 8 + 60000 + 1200 = 1061208.$

(*iii*) $(998)^3 = (1000 - 2)^3$

$= (1000)^3 - (2)^3 - 3 \times 1000 \times 2(1000 - 2)$

[Using identity: $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$]

$= 1000000000 - 8 - 6000000 + 12000 = 994011992.$

8. *Factorise each of the following:*

(i) $8a^3 + b^3 + 12a^2b + 6ab^2$

(ii) $8a^3 - b^3 - 12a^2b + 6ab^2$

(iii) $27 - 125a^3 - 135a + 225a^2$

(iv) $64a^3 - 27b^3 - 144a^2b + 108ab^2$

(v) $27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p.$

Sol. (i) $8a^3 + b^3 + 12a^2b + 6ab^2$

$= (2a)^3 + (b)^3 + 3(2a)(b)(2a + b)$

$= (2a + b)^3$

[Using identity: $x^3 + y^3 + 3xy(x + y) = (x + y)^3$]

$= (2a + b)(2a + b)(2a + b).$

(ii) $8a^3 - b^3 - 12a^2b + 6ab^2$

$= (2a)^3 - (b)^3 - 3(2a)^2b + 3(2a)b^2$

$= (2a - b)^3$

[Using identity: $x^3 - y^3 - 3x^2y + 3xy^2 = (x - y)^3$]

$= (2a - b)(2a - b)(2a - b).$

(iii) $27 - 125a^3 - 135a + 225a^2$

$= (3)^3 - (5a)^3 - 3 \times (3)^2 \times (5a) + 3(3)(5a)^2$

$= (3 - 5a)^3$

[Using identity: $x^3 - y^3 - 3x^2y + 3xy^2 = (x - y)^3$]

$= (3 - 5a)(3 - 5a)(3 - 5a).$

(iv) $64a^3 - 27b^3 - 144a^2b + 108ab^2$

$= (4a)^3 - (3b)^3 - 3(4a)^2(3b) + 3(4a)(3b)^2$

$= (4a - 3b)^3$

[Using identity: $x^3 - y^3 - 3x^2y + 3xy^2 = (x - y)^3$]

$= (4a - 3b)(4a - 3b)(4a - 3b).$

(v) $27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p = (3p)^3 - \left(\frac{1}{6}\right)^3 - 3 \times (3p)^2$

$\times \frac{1}{6} + 3 \times (3p) \times \left(\frac{1}{6}\right)^2$

$$= \left(3p - \frac{1}{6}\right)^3$$

[Using identity: $x^3 - y^3 - 3x^2y + 3xy^2 = (x - y)^3$]

$$= \left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right).$$

9. *Verify*:

(*i*) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

(*ii*) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Sol. (*i*) Consider the identity $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

$$\Rightarrow \quad x^3 + y^3 = (x + y)^3 - 3xy(x + y)$$

$$= (x + y)\{(x + y)^2 - 3xy\}$$

$$\Rightarrow \quad x^3 + y^3 = (x + y)\ \{x^2 + y^2 + 2xy - 3xy\}$$

$$= (x + y)(x^2 - xy + y^2).$$

(*ii*) Consider the identity $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

$$\Rightarrow \quad x^3 - y^3 = (x - y)^3 + 3xy(x - y)$$

$$= (x - y)\{(x - y)^2 + 3xy\}$$

$$\Rightarrow \quad x^3 - y^3 = (x - y)\ \{x^2 + y^2 - 2xy + 3xy\}$$

$$= (x - y)(x^2 + xy + y^2).$$

10. *Factorise each of the following*:

(*i*) $27y^3 + 125z^3$ (*ii*) $64m^3 - 343n^3$.

Sol. (*i*) Consider $27y^3 + 125z^3 = (3y)^3 + (5z)^3$

$$= (3y + 5z)\ \{(3y)^2 - (3y)(5z) + (5z)^2\}$$

[Using identity: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$]

$$= (3y + 5z)(9y^2 - 15yz + 25z^2)$$

(*ii*) Consider $64m^3 - 343n^3 = (4m)^3 - (7n)^3$

$$= (4m - 7n)\ \{(4m)^2 + (4m)(7n) + (7n)^2\}$$

[Using identity: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$]

$$= (4m - 7n)(16m^2 + 28mn + 49n^2).$$

11. *Factorise*: $27x^3 + y^3 + z^3 - 9xyz$.

Sol. Consider $27x^3 + y^3 + z^3 - 9xyz$

$$= (3x)^3 + (y)^3 + (z)^3 - 3(3x)(y)(z)$$

[Using identity: $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$]

$= (3x + y + z)\{(3x)^2 + y^2 + z^2 - (3x)y - yz - (3x)z\}$
$= (3x + y + z)(9x^2 + y^2 + z^2 - 3xy - yz - 3xz).$

12. *Verify that:*

$x^3 + y^3 + z^3 - 3xyz$

$$= \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$$

Sol. Consider identity $x^3 + y^3 + z^3 - 3xyz$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= \frac{1}{2}(x + y + z)(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx)$$

$$= \frac{1}{2}(x + y + z)\{(x^2 + y^2 - 2xy) + (y^2 + z^2 - 2yz) + (z^2 + x^2 - 2xz)\}$$

$$= \frac{1}{2}(x + y + z)\{(x - y)^2 + (y - z)^2 + (z - x)^2\}.$$

13. *If $x + y + z = 0$, show that $x^3 + y^3 + z^3 = 3xyz$.*

Sol. Consider the identity $x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ ...(*i*)

If $x + y + z = 0$, then

$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0$

$\therefore$ $x^3 + y^3 + z^3 - 3xyz = 0$ [From (*i*)]

$\Rightarrow$ $x^3 + y^3 + z^3 = 3xyz.$

14. *Without actually calculating the cubes, find the value of each of the following:*

(*i*) $(-12)^3 + (7)^3 + (5)^3$ (*ii*) $(28)^3 + (-15)^3 + (-13)^3$.

Sol. (*i*) Consider $(-12)^3 + (7)^3 + (5)^3$

Let $x = -12, y = 7, z = 5$

Now, $x + y + z = -12 + 7 + 5 = 0$

We know, if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

$\therefore$ $(-12)^3 + (7)^3 + (5)^3 = 3(-12)(7)(5)$

$= -21 \times 60 = -1260.$

(*ii*) Consider $(28)^3 + (-15)^3 + (-13)^3$

Let $x = 28, y = -15, z = -13$

Now, $x + y + z = 28 - 15 - 13 = 0$

We know, if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

$\therefore \quad (28)^3 + (-15)^3 + (-13)^3$

$= 3(28)(-15)(-13) = 16380.$

15. *Give possible expressions for the length and breadth of each of the following rectangles, in which their areas are given:*

Area: $25a^2 - 35a + 12$	*Area: $35y^2 + 13y - 12$*
(*i*)	(*ii*)

Sol. (*i*) Area $= 25a^2 - 35a + 12 = 25a^2 - 20a - 15a + 12$

$= 5a(5a - 4) - 3(5a - 4) = (5a - 3)(5a - 4).$

Possible expressions for length and breadth are $(5a - 3)$ and $(5a - 4)$ respectively.

(*ii*) Area $= 35y^2 + 13y - 12 = 35y^2 + 28y - 15y - 12$

$= 7y(5y + 4) - 3(5y + 4) = (7y - 3)(5y + 4).$

Possible expressions for length and breadth are $(7y - 3)$ and $(5y + 4)$.

16. *What are the possible expressions for the dimensions of the cuboids whose volumes are given below?*

Volume: $3x^2 - 12x$	*Volume: $12ky^2 + 8ky - 20k$*
(*i*)	(*ii*)

Sol. (*i*) Volume $= 3x^2 - 12x = 3x(x - 4)$

Possible dimensions are 3, x and $(x - 4)$.

(*ii*) Volume $= 12ky^2 + 8ky - 20k = 4k(3y^2 + 2y - 5)$

$= 4k(3y^2 + 5y - 3y - 5)$

$= 4k\,\{y(3y + 5) - 1(3y + 5)\}$

$= 4k(y - 1)(3y + 5).$

Possible dimensions are $4k$, $y - 1$ and $3y + 5$.

□□