1 Real Numbers

EXERCISE 1.1

Choose the correct answer from the given four options in the following questions:

Q1. For some integer m, every even integer is of the form

(*a*) m (*b*) $m+1$ (*c*) $2m$ (*d*) $2m+1$

Sol. (*c*): Let p be any a positive integer. On dividing p by 2, we get m as quotient and r be the remainder. Then by Euclid's division algorithm, we have

$$p = 2m + r, \quad \text{where } 0 \le r < 2,$$

So, $r = 0, 1$

$$\therefore \quad p = 2m \text{ and } p = 2m + 1$$

$p = 2m$ for any integer m, then p is even.

Alternative Method: Even integers are 2, 4, 6, ...

So, these integers can be written in the form of

$$= 2 \times 1, 2 \times 2, 2 \times 3, 2 \times 4, \ldots$$
$$= 2m \quad \text{where } m = +1, +2, +3, \ldots$$

So, $2m$ becomes $\pm 2, \pm 4, \pm 6, \pm 8 \ldots$

Q2. For some integer q, every odd integer is of the form

(*a*) q (*b*) $q+1$ (*c*) $2q$ (*d*) $2q+1$

Sol. (*d*): Let p be any positive integer. On dividing p by 2, we obtain q as quotient and r is the remainder. Then by Euclid's division algorithm, we have

$$p = 2q + r \quad \text{where } 0 \le r < 2$$

So $r = 0$, and $r = 1$

$\therefore p = 2q$ and $p = 2q + 1$

Clearly, $p = 2q + 1$ is odd integer for any integer q.

Alternative Method: Odd integers are 1, 3, 5, 7... or $0 \times 1 + 1, 1 \times 2 + 1, 2 \times 3 + 1, \ldots$ or $2q + 1$

where q is any integer so odd numbers are $q = 0, \pm 1, \pm 2, \pm 3 \ldots$ $\pm 1, \pm 3, \pm 5, \pm 7 \ldots$ are all odd integers or a number of the form $2q + 1$ is odd.

Q3. $n^2 - 1$ is divisible by 8, if n is

(*a*) an integer (*b*) a natural number

(*c*) an odd integer (*d*) an even integer

Sol. (*c*): Let $p = n^2 - 1$, where n is any integer.

Case I: Let n is even, then $n = 2k$.

$$\therefore \quad p = (2k)^2 - 1$$

$$p = 4k^2 - 1$$

Let $k = 0$, then $p = 4(0)^2 - 1 = -1$, which is not divisible by 8

$k = 2$, then $p = 4(2)^2 - 1 = 15$, which is not divisible by 8

$k = 4$, then $p = 4(4)^2 = 64 - 1 = 63$, which is not divisible by 8

So, n can not be even integer.

Case II: Let n is odd then $n = 2k + 1$

$$p = (2k+1)^2 - 1$$
$$= 4k^2 + 1 + 4k - 1$$
$$p = 4k(k+1)$$

Let $k = 1$, $p = 4(1)\,[1+1] = 8$ is divisible by 8

$k = 3$ $\quad p = 4 \times 3\,(3+1) = 48 = 8 \times 6$, is divisible by 8

$k = 5$ $\quad p = 4(5)\,(5+1) = 120, = 8 \times 15$ is divisible by 8

So $n^2 - 1$ is divisible by 8 if n is odd number.

Q4. If the HCF of 65 and 117 is expressible in the form $65\,m - 117$, then the value of m is

(*a*) 4 (*b*) 2 (*c*) 1 (*d*) 3

Sol. (*b*): Find HCF of 65, 117 by any method let by factorisation

$$65 = 13 \times 5$$
$$117 = 13 \times 3 \times 3$$

So, HCF of 65 and 117 = 13

So, $\quad 65m - 117 = 13$

$\Rightarrow \quad 65m = 130$

$\Rightarrow \quad m = 2$

Q5. The largest number which divides 70 and 125, leaving remainders 5 and 8 respectively is

(*a*) 13 (*b*) 65 (*c*) 875 (*d*) 1750

Sol. (*a*): **Main concept:** Required number is largest so problem is related to HCF.

Subtract 5 and 8 from 70 and 125 respectively.

So, $70 - 5 = 65$ and $125 - 8 = 117$

HCF of 65 and 117 is 13 (by any method). So, 13 is the largest number which leaves remainder 5 and 8 after dividing 70, and 125 by 13 respectively.

Q6. If two positive integers a and b are written as $a = x^3y^2$ and $b = xy^3$; x, y are prime numbers then HCF (a, b) is

(*a*) xy (*b*) xy^2 (*c*) x^3y^3 (*d*) x^2y^2

Sol. (*b*):
$$\left.\begin{aligned} a &= x^3y^2 \\ b &= xy^3 \end{aligned}\right\}\text{prime factorisation}$$

So, HCF of a and $b = xy^2$ [common terms from a and b]

Q7. If two positive integers p and q can be expressed as $p = ab^2$ and $q = a^3b$ where a and b being prime numbers, then LCM (p, q) is

(*a*) ab (*b*) a^2b^2 (*c*) a^3b^2 (*d*) a^3b^3

Sol. (*c*): $p = ab^2$

$q = a^3b$

LCM = Product of the highest powers of each factor.

So, LCM $= a^3b^2$.

Q8. The product of a non-zero rational and an irrational number is

(*a*) always irrational (*b*) always rational

(*c*) rational or irrational (*d*) one

Sol. (*a*): Product of a rational $\frac{5}{2}$, and an irrational $\frac{\sqrt{3}}{2} = \frac{5}{2} \times \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{4}$ is also irrational.

Q9. The least number that is divisible by all the numbers from 1 to 10 (both inclusive) is

(*a*) 10 (*b*) 100 (*c*) 504 (*d*) 2520

Sol. (*d*): As we require least number so problem is based on LCM.

Prime factor from 1 – 10

$1 = 1,\quad 2 = 2,\quad 3 = 3,$

$4 = 2 \times 2,\quad 5 = 5,\quad 6 = 2 \times 3,\quad 7 = 7,$

$8 = 2 \times 2 \times 2,\quad 9 = 3 \times 3,\quad 10 = 2 \times 5$

LCM of all numbers 1 to 10 $= 1 \times 2 \times 3 \times 2 \times 5 \times 7 \times 2 \times 3$

$$\text{LCM} = 2^3 \times 3^2 \times 5^1 \times 7^1 = 72 \times 35 = 2520$$

Q10. The decimal expansion of the rational number $\frac{14587}{1250}$ will terminate after:

(*a*) one decimal place (*b*) two decimal places

(*c*) three decimal places (*d*) four decimal places

Sol. (*d*): Number is $\frac{14587}{1250} = \frac{14587}{5^4 \times 2} = \frac{14587}{5^4 \times 2^4} \times 2^3$

$$= \frac{14587}{(10)^4} \times 8 = \frac{116696}{10000} = 11.6696$$

EXERCISE 1.2

Q1. Write whether every positive integer can be of the form $(4q + 2)$, where q is an integer. Justify your answer.

Sol. 'No'. By Euclid's division lemma, we have

dividend = divisor × quotient + remainder

$$a = bq + r$$

Let $b = 4$ then

$a = 4q + r$ where q, r are positive

Since $0 \leq r < 4 \quad \therefore r = 0, 1, 2, 3$

So, a become of the form, $4q$, $4q+1$, $4q+2$ and $4q+3$
So, all integers can be represented by all $4q$, $4q+1$, $4q+2$, and $4q+3$ not only by $4q+2$.

Q2. "The product of two consecutive positive integers is divisible by 2". Is this statement true or false? Give reasons.

Sol. Yes, from any two consecutive numbers one will be even and other will be odd i.e. n, $(n+1)$. So, their product will be even which will be divisible by 2.

Hence, the product of two consecutive positive integers is divisible by 2.

Q3. "The product of three consecutive positive integers is divisible by 6". Is this statement true or false? Justify your answer.

Sol. Yes, the given statement is true.

Three consecutive positive integers are n, $(n+1)$, $(n+2)$. Out of 3 consecutive integers, one will be even and other will be divisible by 3. So, the product of all three becomes divisible by 6,

e.g., 13, 14, 15 here 14 is even, 15 is divisible by 3.

So, $13 \times 14 \times 15$ is divisible by 6.

Q4. Write whether the square of any positive integer can be of the form of $(3m+2)$, where m is a natural number. Justify your answer.

Sol. By Euclid's division lemma, $b = aq + r$

where a, b, q, r are +ve integers and here $a = 3$ then $b = 3q + r$ then $0 \le r < 3$ or $r = 0, 1, 2$, so b becomes $b = 3q, 3q+1, 3q+2$,

$$b = 3q$$

$$\Rightarrow \quad (b)^2 = (3q)^2$$

$$\Rightarrow \quad b^2 = 3 \cdot 3q^2 = 3m \quad \text{where, } 3q^2 = m$$

So, as b^2 is perfect square so $3m$ will also be perfect square.

When $r = 1$, $\quad b = 3q + 1$

$$\Rightarrow \quad (b)^2 = (3q+1)^2$$

$$\Rightarrow \quad b^2 = 9q^2 + 1 + 2 \times 3q$$

$$\Rightarrow \quad b^2 = 3[3q^2 + q] + 1$$

$$\Rightarrow \quad b^2 = 3m + 1 \text{ and } m = 3q^2 + 2q$$

So, b^2 is perfect square or a number of the form $3m+1$ is perfect square.

When $r = 2$, $\quad b = 3q + 2$

$$\Rightarrow \quad b^2 = 9q^2 + 4 + 2 \cdot 3q \cdot 2$$

$$= 9q^2 + 3 + 3 \times 4q + 1$$

$$= 3[3q^2 + 1 + 4q] + 1$$

$$\Rightarrow \quad b^2 = 3m + 1$$

Again, a number of the form $3m+1$ is perfect square.

Hence, a number of the form $(3m+2)$ can never be perfect square. But a number of the form $3m$, and $3m+1$ are perfect squares.

Q5. A positive integer is of the form $(3q + 1)$, q being a natural number. Can you write its square in any form other than $(3m + 1)$ i.e., $3m$ or $(3m + 2)$ for some integer m? Justify your answer.

Sol. By Euclid's division lemma,

$b = aq + r$ where b, q, r are natural numbers and $a = 3$

$\therefore$ $b = 3q + r$ where $0 \le r < 3$ so $r = 0, 1, 2,$

At $r = 0$, $b = 3q$

$\Rightarrow$ $b^2 = (3q)^2 = 3 \cdot 3q^2$

$\Rightarrow$ $b^2 = 3m,$ where $m = 3q^2$

So, a number of the form $3m$ is perfect square.

At $r = 1$, $b = 3q + 1$

$\Rightarrow$ $b^2 = (3q + 1)^2$

$\Rightarrow$ $b^2 = 9q^2 + 1 + 6q$

$\Rightarrow$ $b^2 = 3[3q^2 + 2q] + 1$

$\Rightarrow$ $b^2 = 3m + 1,$ where $m = 3q^2 + 2q$

So, a number of the form $(3m + 1)$ is also perfect square.

At $r = 2$, $b = 3q + 2$

$\Rightarrow$ $b^2 = (3q)^2 + (2)^2 + 2(3q)(2)$

$= 9q^2 + 4 + 3 \times 4q$

$= 9q^2 + 3 + 3 \times 4q + 1 = 3[3q^2 + 1 + 4q] + 1$

$\Rightarrow$ $b^2 = 3m + 1,$ where $m = 3q^2 + 4q + 1$

Hence, a perfect square will be of the form $3m$ and $(3m + 1)$ for m being a natural number.

Q6. The numbers 525 and 3000 are both divisible only by 3, 5, 15, 25 and 75, what is HCF of (3000, 525)? Justify your answer.

Sol. The numbers 525 and 3000 both are divisible by 3, 5, 15, 25 and 75. So, highest common factor out of 3, 5, 15, 25 and 75 is 75 or HCF of (525, 3000) is 75.

Verification: $525 = 5 \times 5 \times 3 \times 7 = 3 \times 5^2 \times 7^1$

$3000 = 2^3 \times 5^3 \times 3^1 = 2^3 \times 3^1 \times 5^3$

$\text{HCF} = 3^1 \times 5^2 = 75$

Hence, verified.

Q7. Explain why $3 \times 5 \times 7 + 7$ is a composite number.

Sol. Main Concept: A number which is not prime is composite.

$3 \times 5 \times 7 + 7 = 7[3 \times 5 + 1] = 7[15 + 1]$

$= 7 \times 16$ have prime factors $= 7 \times 2 \times 2 \times 2 \times 2$

So, number $(3 \times 5 \times 7 + 7)$ is not prime hence, it is composite.

Q8. Can two numbers have 18 as their HCF and 380 as their LCM? Give reasons.

Sol. As we know that

$$\text{HCF}(a, b) \times \text{LCM}(a, b) = (a \times b)$$

18 must be factor of 380.

So, $\frac{380}{18}$ should be a natural number.

But $\frac{380}{18}$ is not a natural number or 380 is not divisible by 18.

So, 380 and 18 are not the LCM and HCF of any two numbers.

Q9. Without actually performing the long division, find if $\frac{987}{10500}$ will have terminating or non-terminating (repeating) decimal expansion. Give reasons for your answer.

Sol. $$\frac{987}{10500} = \frac{3 \times 7 \times 47}{2^2 \times 3^1 \times 5^3 \times 7^1} = \frac{47}{2^2 \times 5^3}$$

3	987
7	329
	47

5	10500
3	2100
7	700
5	100
5	20
2	4
	2

As denominator has prime factors only in 2 and 5 so number $\frac{987}{10500}$ is terminating decimal.

$$\frac{47}{2^3 \times 5^3} \times 2 = \frac{94}{1000} = 0.094$$

Q10. A rational number in its decimal expansion is 327.7081. What can you say about the prime factors of q, when this number is expressed in the form p/q? Give reasons.

Sol. 327.7081 is terminating decimal so in the form of

$$\frac{p}{q} = \frac{3277081}{10000}$$

$$q = 2^4 \times 5^4$$

So, q has only factors of 2 and 5 so it is terminating decimal.

EXERCISE 1.3

Q1. Show that the square of any positive integer is either of the form $4q$ or $4q + 1$ for some integer q.

Sol. Main concept: $\boxed{a = 4q + r}$ $0 \le r < 4$.

By Euclid's division lemma,

$$a = 4m + r \qquad ...(i)$$

where a, m, r are integers and $0 \le r < 4$

or $\quad r = 0, 1, 2, 3$

When $r = 0$, $\quad a = 4m$ $\quad$ [From (i)]

$\Rightarrow \quad a^2 = (4m)^2$ $\quad$ [Squaring both sides]

$\Rightarrow \quad a^2 = 4.4m^2$

$\Rightarrow \quad a^2 = 4q$, where $q = 4m^2 \quad \Rightarrow \quad 4q$ is a perfect square

When $r = 1$, $\quad a = 4m + 1$ [From (*i*)]

$\Rightarrow \quad a^2 = (4m+1)^2$ [Squaring both sides]

$= (4m)^2 + (1)^2 + 2(4m)(1)$

$= 4[4m^2 + 2m] + 1$

$\Rightarrow \quad a^2 = 4q + 1, \quad$ where $q = 4m^2 + 2m$

$\because \quad a^2$ is perfect square so $4q + 1$ is also perfect square.

When $r = 2$, $\quad a = (4m + 2)$ [From (*i*)]

$\Rightarrow \quad a^2 = (4m)^2 + (2)^2 + 2(4m)(2)$ [Squaring both sides]

$\Rightarrow \quad a^2 = 4[4m^2 + 1 + 4m]$

$\Rightarrow \quad a^2 = 4q, \quad$ where $q = 4m^2 + 4m + 1$

$\because \quad a^2$ is perfect square. So, $4q$ will also be perfect square.

When $r = 3$, then $\quad a = 4m + 3$ [From (*i*)]

$\Rightarrow \quad (a)^2 = (4m+3)^2$ [Squaring both sides]

$\Rightarrow \quad a^2 = (4m)^2 + (3)^2 + 2(4m)(3)$

$\Rightarrow \quad a^2 = 16m^2 + 9 + 24m$

$= 16m^2 + 8 + 24m + 1$

$= 4[4m^2 + 2 + 6m] + 1$

$\Rightarrow \quad a^2 = 4q + 1, \quad$ where $q = 4m^2 + 6m + 2$

As a^2 is perfect square so $4q + 1$ will also be perfect square.

Hence, number of the form $4q$ and $4q + 1$ is the perfect square.

Q2. Show that the cube of any positive integer is of the form $4m$, $4m + 1$ or $4m + 3$ for some integer m.

Sol. By Euclid's division algorithm, corresponding to the positive integer a and 4

$$a = 4q + r \qquad \text{...}(i)$$

where a, q, r are non-negative integers and $0 \le r < 4$ i.e., $r = 0, 1, 2, 3$

Now, at $r = 0$, $\quad a = 4q + 0$ [From (*i*)]

$\Rightarrow \quad a^3 = (4q)^3$ [Cubing both sides]

$\Rightarrow \quad a^3 = 4 \cdot (16q^3)$

$\Rightarrow \quad a^3 = 4m, \quad$ where $m = 16q^3$

$\because a^3$ is perfect cube so $4m$ will also be perfect cube for some specified value of m.

Now, at $r = 1$, $\quad a = 4q + 1$ [From (*i*)]

$\Rightarrow \quad a^3 = (4q+1)^3$ [Cubing both sides]

$\Rightarrow \quad a^3 = (4q)^3 + (1)^3 + 3(4q)^2(1) + 3(4q)(1)^2$

$= 4 \cdot 16q^3 + 1 + 4 \cdot 12q^2 + 4 \cdot 3q$

$= 4(16q^3 + 12q^2 + 3q) + 1$

$\Rightarrow \quad a^3 = 4m + 1, \quad$ where $m = 16q^3 + 12q^2 + 3q$

$\because a^3$ is perfect cube so $4m + 1$ will also be perfect cube for some specified value of m.

At $r = 2$, $a = 4q + 2$ [From (*i*)]

$\Rightarrow$ $a^3 = (4q + 2)^3$ [Cubing both sides]

$\Rightarrow$ $a^3 = (4q)^3 + (2)^3 + 3(4q)^2 (2) + 3(4q) (2)^2$

$= 4 \cdot 16q^3 + 8 + 4 \times 24q^2 + 4 \times 12q$

$= 4[16q^3 + 2 + 24q^2 + 12q]$

$\Rightarrow$ $a^3 = 4m$, where $m = 16q^3 + 24q^2 + 12q + 2$

As a^3 is perfect cube so, $4m$ is also perfect cube for some value of positive integer m.

At $r = 3$, $a = 4q + 3$ [From (*i*)]

$\Rightarrow$ $a^3 = (4q + 3)^3$ [Cubing both sides]

$\Rightarrow$ $a^3 = (4 \cdot q)^3 + (3)^3 + 3(4q)^2 (3) + 3(4q) (3)^2$

$\Rightarrow$ $a^3 = 4 \times 16q^3 + 27 + 4 \times 36q^2 + 4q \times 27$

$\Rightarrow$ $a^3 = 4 \times 16q^3 + 24 + 3 + 4 \times 36 \cdot q^2 + 4 \times 27q$

$= 4[16q^3 + 6 + 36q^2 + 27q] + 3$

$\Rightarrow$ $a^3 = 4m + 3$, where $m = 16q^3 + 36q^2 + 27q + 6$

Hence, a number of the form $4m$, $4m + 1$ and $4m + 3$ is perfect cube for specified natural value of m.

Q3. Show that the square of any positive integer cannot be of the form $5q + 2$ or $5q + 3$ for any integer q.

Sol. By Euclid's division algorithm, consider the positive integer a and 5

$$a = 5m + r \quad \text{...}(i)$$

where, a, m, r are positive integers and $0 \le r < 5$ or $r = 0, 1, 2, 3, 4$

Squaring (*i*) both sides, we get

$a^2 = (5m)^2 + (r)^2 + 2(5m)(r) = 25m^2 + r^2 + 10mr$

$\Rightarrow$ $a^2 = 5(5m^2 + 2mr) + r^2$ ...(*ii*)

At $r = 0$, $a^2 = 5[5m^2 + 2m \cdot 0] + 0$ [From (*ii*)]

$\Rightarrow$ $a^2 = 5(5m^2)$

$\Rightarrow$ $a^2 = 5q$, where $q = 5m^2$

At $r = 1$, $a^2 = 5[5m^2 + 2m] + 1$ [From (*ii*)]

$\Rightarrow$ $a^2 = 5q + 1$, where $q = 5m^2 + 2m$

At $r = 2$, $a^2 = 5[5m^2 + 2.2m] + (2)^2$ [From (*ii*)]

$\Rightarrow$ $a^2 = 5q + 4$, where $q = 5m^2 + 4m$

At $r = 3$, $a^2 = 5[5m^2 + 2m \cdot 3] + 3^2$ [From (*ii*)]

$\Rightarrow$ $a^2 = 5[5m^2 + 6m] + 5 + 4$

$= 5[5m^2 + 6m + 1] + 4$

$\Rightarrow$ $a^2 = 5q + 4$, where $q = 5m^2 + 6m + 1$

At $r = 4$, $a^2 = 5[5m^2 + 2m \cdot 4] + 4^2$ [From (*ii*)]

$\Rightarrow$ $a^2 = 5(5m^2 + 8m) + 15 + 1$

$= 5[5m^2 + 8m + 3] + 1$

$\Rightarrow$ $a^2 = 5q + 1$, where $q = 5m^2 + 8m + 3$

Hence, the numbers of the form $5q$, $5q + 1$, $5q + 4$ are perfect squares and the numbers of the form $(5q + 2)$, $(5q + 3)$ are not perfect squares for some positive integers.

Q4. Show that the square of any positive integer cannot be of the form $(6m + 2)$, or $(6m + 5)$ for any integer m.

Sol. By Euclid's division algorithm, we have

$$a = 6q + r, \quad \text{where } 0 \le r < 6$$

or $\quad r = 0, 1, 2, 3, 4, 5$

Consider $\quad a = 6q + r$

$\Rightarrow \quad a^2 = (6q)^2 + (r)^2 + 2(6q)(r)$ [Squaring both sides]

$\Rightarrow \quad a^2 = 6[6q^2 + 2qr] + r^2$ ...(i)

At $r = 0$, $\quad a^2 = 6[6q^2 + 2q \times 0] + 0^2$ [From (i)]

$\Rightarrow \quad a^2 = 36q^2$

$\Rightarrow \quad a^2 = 6m, \quad \text{where } m = 6q^2$

At $r = 1$, $\quad a^2 = 6[6q^2 + 2q \times 1] + 1^2$ [From (i)]

$\quad = 6[6q^2 + 2q] + 1$

$\Rightarrow \quad a^2 = 6m + 1, \quad \text{where } m = 6q^2 + 2q$

At $r = 2$, $\quad a^2 = 6[6q^2 + 2q \cdot 2] + 2^2$ [From (i)]

$\quad a^2 = 6m + 4, \quad \text{where } m = (6q^2 + 4q)$

At $r = 3$, $\quad a^2 = 6[6q^2 + 2q \cdot 3] + 3^2$ [From (i)]

$\quad = 6[6q^2 + 6q] + 6 + 3$

$\quad = 6[6q^2 + 6q + 1] + 3$

$\Rightarrow \quad a^2 = 6m + 3, \quad \text{where } m = 6q^2 + 6q + 1$

At $r = 4$, $\quad a^2 = 6[6q^2 + 2q \cdot 4] + 4^2$

$\Rightarrow \quad a^2 = 6[6q^2 + 8q] + 12 + 4$

$\quad = 6[6q^2 + 8q + 2] + 4$

$\Rightarrow \quad a^2 = 6m + 4$ is perfect square, where $m = 6q^2 + 8q + 2$

At $r = 5$, $\quad a^2 = 6[6q^2 + 2q \cdot 5] + 5^2$ [From (i)]

$\Rightarrow \quad a^2 = 6[6q^2 + 10q] + 24 + 1$

$\quad = 6[6q^2 + 10q + 4] + 1$

$\Rightarrow \quad a^2 = 6m + 1$ is perfect square, where $m = 6q^2 + 10q + 4$

Hence, the numbers of the form $6m$, $(6m + 1)$, $(6m + 3)$ and $(6m + 4)$ are perfect squares and $(6m + 2)$, and $(6m + 5)$ are not perfect squares for some value of m.

Q5. Show that the square of any odd integer is of the form $(4q + 1)$ for some integer q.

Sol. By Euclid's division algorithm, $a = bq + r$ where a, b, q, r are non-negative integers and $0 \le r < 4$.

On putting $b = 4$ we get

$$\boxed{a = 4q + r} \quad \text{...(i)}$$

When $r = 0$, $a = 4q$ which is even (as it is divisible by 2)
When $r = 1$, $a = 4q + 1$ which is odd ($\because$ it is not divisible by 2)
Squaring the odd number $(4q + 1)$, we get

$$= (4q+1)^2$$
$$= (4q)^2 + (1^2) + 2(4q)$$
$$= 4[4q^2 + 2q] + 1$$
$$= 4m + 1 \text{ is perfect square for } m = 4q^2 + 2q$$

When $r = 2$, $a = 4q + 2$ [From (*i*)]
$\Rightarrow$ $a = 2(2q + 1)$ is divisible by 2 so it is even.
When $r = 3$, $a = 4q + 3 = 4q + 2 + 1$
$= 2[2q + 1] + 1$ is not divisible by 2 so it is odd.
Squaring the odd number $(4q + 3)$, we get

$$(4q+3)^2 = (4q)^2 + (3)^2 + 2(4q)(3)$$
$$= 16q^2 + 9 + 24q$$
$$= 16q^2 + 24q + 8 + 1$$
$$= 4[4q^2 + 6q + 2] + 1$$
$$= 4m + 1 \text{ is perfect square for some value of } m.$$

Q6. If n is an odd integer, then show that $n^2 - 1$ is divisible by 8.
Sol. Let $a = n^2 - 1$ ...(*i*)
Where n is odd number, i.e., $n = 1, 3, 5, 7$
When $n = 1$, $a = 1^2 - 1 = 0$, which is divisible by 8. [From eq. (*i*)]
When $n = 3$, $a = 3^2 - 1 = 9 - 1 = 8$, which is also divisible by 8. [From eq. (*i*)]
When $n = 5$,
$a = 5^2 - 1 = 25 - 1 = 24 = 8 \times 3$, which is divisible by 8. [From eq. (*i*)]
Hence, $n^2 - 1$ is divisible by 8 when n is odd.

Q7. Prove that, if x and y, both are odd positive integers, then $(x^2 + y^2)$ is even but not divisible by 4.
Sol. Let we have any two odd numbers $x = (2m + 1)$ and $y = (2m + 5)$.
Then,

$$x^2 + y^2 = (2m+1)^2 + (2m+5)^2$$
$$= 4m^2 + 1 + 4m + 4m^2 + 25 + 20m$$
$$= 8m^2 + 24m + 26$$
$$= 2[4m^2 + 12m + 13]$$

So, $x^2 + y^2$ is even but it is not divisible by 4.

Q8. Use Euclid's division algorithm to find HCF of 441, 567 and 693.
Sol. Let $a = 693$ and $b = 567$
By Euclid's division algorithm, $a = bq + r$

$$\therefore \quad 693 = 567 \times 1 + 126$$
$$567 = 126 \times 4 + 63$$
$$126 = 63 \times 2 + 0$$

Hence, HCF (693 and 567) = 63.
Now, take 441 and HCF = 63

By Euclid's division algorithm, $c = dq + r$

$$c = 441 \text{ and } d = 63$$

$$\Rightarrow \quad 441 = 63 \times 7 + 0$$

Hence, HCF (693, 567, 441) = 63.

Q9. Using Euclid's division algorithm, find the largest number that divides 1251, 9377 and 15628 leaving remainders, 1, 2, and 3 respectively.

Sol. As 1, 2, and 3 are the remainders when required largest number (HCF) divides 1251, 9377 and 15628 respectively.

We have the numbers for HCF (1251 – 1), (9377 – 2) and (15628 – 3) i.e., 1250, 9375, 15625

For HCF of 1250, 9375, 15625 let $a = 15625$, $b = 9375$

By Euclid's division algorithm, $a = bq + r$

$$\therefore \quad 15625 = 9375 \times 1 + 6250$$
$$9375 = 6250 \times 1 + 3125$$
$$6250 = 3125 \times 2 + 0$$

$\therefore$ HCF (15625, 9375) = 3125

Now, let $d = 1250$ and $c = 3125$

By Euclid's division algorithm, $c = dq + r$

$$\therefore \quad 3125 = 1250 \times 2 + 625$$
$$1250 = 625 \times 2 + 0$$

Hence, required HCF (15625, 1250 and 9375) is 625.

Q10. Prove that $\sqrt{3} + \sqrt{5}$ is irrational.

Sol. Let us consider $\sqrt{3} + \sqrt{5}$ is a rational number that can be written as

$$\sqrt{3} + \sqrt{5} = a$$

$$\Rightarrow \quad \sqrt{5} = a - \sqrt{3}$$

Squaring both sides, we get

$$(\sqrt{5})^2 = (a - \sqrt{3})^2$$

$$\Rightarrow \quad 5 = (a)^2 + (\sqrt{3})^2 - 2(a)(\sqrt{3})$$

$$\Rightarrow \quad 2a\sqrt{3} = a^2 + 3 - 5$$

$$\Rightarrow \quad \sqrt{3} = \frac{a^2 - 2}{2a}$$

As $a^2 - 2$, $2a$ are rational so $\frac{a^2 - 2}{2a}$ is also rational but $\sqrt{3}$ is not rational which contradicts our consideration. So, $\sqrt{3} + \sqrt{5}$ is irrational.

Q11. Show that 12^n cannot end with the digit 0 or 5 for any natural number n.

Sol. Number ending at 0 or 5 is divisible by 5.

Now, $\quad (12)^n = (2 \times 2 \times 3)^n = 2^{2n} \times 3^n$

It has no any 5 in its prime factorisation. So, 12^n can never end with 5 and zero.

Q12. On a morning walk, three persons, step off together and their steps measure 40 cm, 42 cm and 45 cm respectively. What is the minimum distance each should walk, so that each can cover the same distance in complete steps?

Sol. We have to find minimum distance (i.e., LCM) covered by steps.

$$40 = 2^3 \times 5$$
$$42 = 2 \times 3 \times 7$$
$$45 = 3^2 \times 5$$

LCM (40, 42, 45) $= 2^3 \times 3^2 \times 5 \times 7 = 2520$ cm

So, the minimum distance that each should walk is 2520 cm.

Q13. Write the denominator of rational number $\frac{257}{5000}$ in the form of $2^m \times 5^n$, where m, n are non-negative integers. Hence, write its decimal expansion, without actual division.

Sol. Denominator of the rational number $\frac{257}{5000}$ is 5000.

$$5000 = 2^3 \times 5^4 \quad \text{which is of the form } 2^m \times 5^n$$

where $m = 3$ and $n = 4$

$$\therefore \quad \frac{257}{5000} = \frac{257}{2^3 \times 5^4} \times \frac{2}{2} = \frac{257 \times 2}{(2 \times 5)^4} = \frac{514}{10000} = 0.0514$$

Q14. Prove that $\sqrt{p} + \sqrt{q}$ is irrational, where p and q are primes.

Sol. Consider $\sqrt{p} + \sqrt{q}$ is rational and can be represented as $\sqrt{p} + \sqrt{q} = a$

$$\Rightarrow \quad (\sqrt{p}) = a - \sqrt{q}$$
$$\Rightarrow \quad (\sqrt{p})^2 = (a - \sqrt{q})^2 \qquad \text{(squaring both sides)}$$
$$\Rightarrow \quad p = a^2 + q - 2a\sqrt{q}$$
$$\Rightarrow \quad 2a\sqrt{q} = a^2 + q - p$$
$$\Rightarrow \quad \sqrt{q} = \frac{a^2 + q - p}{2a}$$

As q is prime so $\sqrt{q}$ is not rational but $\frac{a^2 + q - p}{2a}$ is rational because a, p, q are non-zero integers which contradicts our consideration.

Hence, $\sqrt{p} + \sqrt{q}$ is irrational.

EXERCISE 1.4

Q1. Show that the cube of a positive integer of the form $(6q + r)$, where q is an integer and $r = 0, 1, 2, 3, 4$ and 5, which is also of the form $(6m + r)$.

Sol. By Euclid's division algorithm,

$$a = 6q + r \qquad \ldots(i)$$

where a, q and r are non-negative integers $0 \le r < 6$ i.e., $r = 0, 1, 2, 3, 4, 5$.

Cubing (*i*) both sides, we get

$$(a)^3 = (6q + r)^3$$

$\Rightarrow \quad a^3 = (6q)^3 + (r)^3 + 3(6q)^2(r) + 3(6q)(r)^2$

$= 6^3q^3 + r^3 + 3 \times 6^2q^2r + 6 \times 3qr^2$

$\Rightarrow \quad a^3 = 6[36q^3 + 18q^2r + 3qr^2] + r^3$ ...(*ii*)

When $r = 0$, then $a^3 = 6[36q^3 + 18q^2 \times 0 + 3q0^2] + 0^3$ [From (*ii*)]

$\Rightarrow \quad a^3 = 6[36q^3]$

$\Rightarrow \quad a^3 = 6m$ is perfect cube for some value of m such that $m = 36q^2$

When $r = 1$, $a^3 = 6[36q^3 + 18q^2 \times 1 + 3q1^2] + 1^3$ [From (*ii*)]

$= 6[36q^3 + 18q^2 + 3q] + 1$

$\Rightarrow \quad a^3 = 6m + 1$ is perfect cube for some value of m such that $m = (36q^3 + 18q^2 + 3q)$

When $r = 2$, $a^3 = 6[36q^3 + 18q^2 \times 2 + 3q \times 2^2] + 2^3$ [From (*ii*)]

$= 6[36q^3 + 36q^2 + 12q] + 6 + 2$

$= 6[36q^3 + 36q^2 + 12q + 1] + 2$

$\Rightarrow \quad a^3 = 6m + 2$ is perfect cube for some values of m such that $m = 36q^3 + 36q^2 + 12q + 1$

When $r = 3$, $a^3 = 6[36q^3 + 18q^2 \times 3 + 3q \times 3^2] + 3^3$ [From (*ii*)]

$\Rightarrow \quad a^3 = 6[36q^3 + 54q^2 + 27q] + 24 + 3$

$\Rightarrow \quad a^3 = 6[36q^3 + 54q^2 + 27q + 4] + 3$

$\Rightarrow \quad a^3 = 6m + 3$

So, $(6m + 3)$ is perfect cube for specified value of m such that

$$m = 36q^3 + 54q^2 + 27q + 4$$

When $r = 4$, then eq. (*ii*) becomes

$a^3 = 6[36q^3 + 18q^2(4) + 3q\,4^2] + 4^3$

$= 6[36q^3 + 72q^2 + 48\,q] + 60 + 4$

$= 6[36q^3 + 72q^2 + 48\,q + 10] + 4$

$\Rightarrow \quad a^3 = 6m + 4$

So, $(6m + 4)$ is perfect cube for specified value of m such that

$$m = 36q^3 + 72q^2 + 48q + 10$$

When $r = 5$, eq. (*ii*) becomes as

$a^3 = 6[36q^3 + 18q^2(5) + 3q(5)^2] + (5)^3$

$= 6[36q^3 + 90q^2 + 75q] + 120 + 5$

$= 6[36q^3 + 90q^2 + 75q + 20] + 5$

$\Rightarrow \quad a^3 = 6m + 5$

$\Rightarrow \quad (6m + 5)$ is perfect cube for specified value of

$$m = 36q^3 + 90q^2 + 75q + 20$$

Hence, cubes of positive integers is of the form $(6m + r)$, where m is a specified integer and $r = 0, 1, 2, 3, 4, 5$.

Q2. Prove that one and only one out of n, $(n + 2)$ and $(n + 4)$ is divisible by 3, where n is any positive integer.

Sol. Consider the given numbers n, $n + 2$ and $n + 4$.

When $n = 1$, numbers become $1, 1 + 2, 1 + 4 = (1, 3 \text{ and } 5)$

When $n = 2$, numbers become $2, 2 + 2, 2 + 4 = (2, 4, 6)$

When $n = 3$, numbers become $= (3, 5, 7)$

When $n = 4$, numbers become $= (4, 6, 8)$

When $n = 5$, numbers become $= (5, 7, 9)$

When $n = 6$, numbers become $= (6, 8, 10)$

When $n = 7$, numbers become $= (7, 9, 11)$

From above, we observe that out of 3 numbers one is divisible by 3.

Alternate Method: Consider that if a number n is divided by 3, then we get a quotient q and remainder r then by Euclid's division algorithm,

$$n = 3q + r \quad \text{where, } 0 \le r < 3$$

At	n_1	Divisible by 3	$n_2 = n_1 + 2$	Divisible by 3	$n_3 = n_1 + 4$	Divisible by 3
$r = 0$	$3q + 0 = 3q$	Yes	$3q + 2$	No	$3q + 4$ $= 3q + 3 + 1$ $= 3(q + 1) + 1$ $= 3m + 1$	No
$r = 1$	$3q + 1$	No	$3q + 1 + 2$ $= 3q + 3$ $= 3(q + 1)$	Yes	$3q + 1 + 4$ $= 3q + 3 + 2$ $= 3(q + 1) + 2$ $= 3m + 2$	No
$r = 2$	$3q + 2$	No	$3q + 2 + 2$ $= 3q + 3 + 1$ $= 3(q + 1) + 1$ $= 3m + 1$	No	$3q + 2 + 4$ $= 3q + 6$ $= 3(q + 2)$ $= 3m$	Yes

From table, out of n_1, n_2 or n_3 one number is divisible by 3 when $r = 0, 1, 2$, are taken.

Q3. Prove that one of any three consecutive positive integers must be divisible by 3.

Sol. Consider a number n. q and r are positive integers. When n is divided by 3 the quotient is q and remainder r. So, by Euclid's division algorithm,

$$n = 3q + r \quad (0 \le r < 3) \quad \text{or } r = 0, 1, 2, 3$$

At	n_1	Divisible by 3	$n_2 = n_1 + 1$	Divisible by 3	$n_3 = n_1 + 2$	Divisible by 3
$r = 0$	$3q + 0$ $= 3q$	Yes	$3q + 1$	No	$3q + 2$	No
$r = 1$	$3q + 1$	No	$3q + 2$	No	$3q + 3$ $= 3(q + 1)$ $= 3m$	Yes
$r = 2$	$3q + 2$	No	$3q + 3 = 3(q + 1)$ $= 3m$	Yes	$3q + 4$ $= 3q + 3 + 1$ $= 3(q + 1) + 1$ $= 3m + 1$	No

So, one of any three consecutive positive integers is divisible by 3.

Q4. For any positive integer n, prove that $n^3 - n$ is divisible by 6.

Sol. Let $a = n^3 - n$

$\Rightarrow \quad a = n(n^2 - 1)$

$= n(n-1)(n+1)$

$(n - 1)$, n, $(n + 1)$ are consecutive integers so out of three consecutive numbers at least one will be even. So, a is divisible by 2.

Sum of numbers $= (n-1) + n + (n+1)$

$= n - 1 + n + n + 1$

$= 3n$

Clearly, the sum of three consecutive numbers is divisible by 3, so any one of them must be divisible by 3.

So, out of n, $(n - 1)$, $(n + 1)$, one is divisible by 2 and one is divisible by 3 and

$$a = (n-1) \times n \times (n+1)$$

Hence, out of three factors of a, one is divisible by 2 and one is divisible by 3. So, a is divisible by 6 or $n^3 - n$ is divisible by 6.

Q5. Show that one and only one out of n, $(n + 4)$, $(n + 8)$, $(n + 12)$, $(n + 16)$ is divisible by 5, where n is any positive integer.

[**Hint:** Any positive integer can be written in the form $5q$, $(5q + 1)$, $(5q + 2)$, $(5q + 3)$, $(5q + 4)$]

Sol. Let a number n is divided by 5 then quotient is q and remainder is r. Then by Euclid's division algorithm,

$n = 5q + r$, where n, q, r are non-negative integers and $0 \leq r < 5$

When $r = 0$, $\quad n = 5q + 0 = 5q$

So, n is divisible by 5.

When $r = 1$, $\quad n = 5q + 1$

$n + 2 = 5q + 1 + 2 = 5q + 3$ is not divisible by 5.

$n + 4 = (5q + 1) + 4 = 5q + 5 = 5(q + 1)$ divisible by 5.

So, $(n + 4)$ is divisible by 5.

When $r = 2$, $\quad n = 5q + 2$

$(n + 8) = (5q + 2) + 8 = 5q + 10 = 5(q + 2) = 5m$ is divisible by 5.

So, $(n + 8)$ is divisible by 5.

When $r = 3$, $\quad n = 5q + 3$

$n + 12 = (5q + 3) + 12 = 5q + 15 = 5(q + 3) = 5m$ is divisible by 5.

So, $(n + 12)$ is divisible by 5.

When $r = 4$, $\quad n = 5q + 4$

$n + 16 = (5q + 4) + 16 = 5q + 20 = 5(q + 4)$

$(n + 16) = 5m$ is divisible by 5.

Hence, n, $(n + 4)$, $(n + 8)$, $(n + 12)$ and $(n + 16)$ are divisible by 5.

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