

2



Relations and Functions

Lesson at a Glance

1. A pair of elements grouped together in a particular order is called an **ordered pair**.
2. $(a, b) = (x, y)$ iff $a = x$ and $b = y$.
3. **Cartesian product** $A \times B$ of two sets A and B is defined as
$$A \times B = \{(a, b) : a \in A, b \in B\}.$$
and $n(A \times B) = n(A) \cdot n(B)$
i.e. if $n(A) = p$ and $n(B) = q$,
then $n(A \times B) = pq$
4. $R \times R = \{(x, y) : x, y \in R\}$; $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$.
5. $A \times \phi = \phi$, where ϕ is the empty set.
6. In general, $A \times B \neq B \times A$ but $n(A \times B) = n(B \times A)$ always, each $= n(A) \cdot n(B)$.
7. A **relation** R from a set A to a set B is a subset of $A \times B$.
In fact, every subset of $A \times B$ is a relation from set A to set B .
Therefore, number of relations from set A to set B
 $=$ number of subsets of $A \times B = 2^{n(A \times B)} = 2^{n(A)n(B)}$.
Every subset of $A \times A$ is called a relation from $A \rightarrow A$ or simply relation on the set A .
8. If $(x, y) \in R$, then y is called **image** of x under R and **Domain** of relation R is the set of all first elements of the ordered pairs in R .
9. **Range** of relation R is the set of all second elements of the ordered pairs in R .
10. **Function:** A relation R from $A \rightarrow B$ is called a function (i.e. $R \subset A \times B$) if **no two points of R have the same first entry** i.e. every element $x \in A$ has one and only one **image** $y \in B$.
11. If $f : A \rightarrow B$ is a function, then

Domain of $f = A$, co-domain of $f = B$.

Range of $f = \text{set } f(A) = \{f(x) : x \in A\}$. Range \subset Co-domain.

12. If $f : A \rightarrow B$ is a function such that
- range of $f = f(A) \subset \mathbb{R}$, then f is called a **real valued function**.
 - if $A \subset \mathbb{R}$ and $f(A) \subset \mathbb{R}$, then f is called a **real function**.
13. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x$ for all $x \in \mathbb{R}$ is called the **identity function**.
14. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = c$, $x \in \mathbb{R}$ is called a **constant function**. Its range = $\{c\}$, a singleton set.
15. **Polynomial function.** A function of the form $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ where n is a positive integer is called a polynomial function.
16. A **rational function** is of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain where $g(x) \neq 0$.
17. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = |x|$ for each $x \in \mathbb{R}$ is called **modulus function**.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$|x| \geq 0$ for all $x \in \mathbb{R}$. $|x| = 0$ iff $x = 0$

If $x \neq 0$, then $|x| > 0$.

18. $|x|^2 = |x^2| = x^2$
19. $|-x| = |x|$ and $||x|| = |x|$
20. $|x| = \sqrt{x^2}$

21. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \text{i.e., } f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is called the **signum function**.

22. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = [x]$ is called the **greatest integer function**.

$[x]$ = the greatest integer $\leq x$

If $x \in \mathbb{Z}$, then $[x] = x$. If $x \notin \mathbb{Z}$, then $[x] =$ the greatest integer $< x$.

23. If $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$, then

$$(i) \quad (cf)(x) = c f(x), x \in X.$$

$$(ii) \quad (f + g)(x) = f(x) + g(x), x \in X.$$

$$(iii) \quad (f - g)(x) = f(x) - g(x), x \in X.$$

$$(iv) \quad (fg)(x) = f(x)g(x), x \in X.$$

$$(v) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, x \in X, g(x) \neq 0.$$

TEXTBOOK QUESTIONS SOLVED

EXERCISE 2.1 (Page No.: 33-34)

1. If $\left(\frac{x}{3} + 1, y - \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of x and y .

Sol. Given: $\left(\frac{x}{3} + 1, y - \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$

Since the ordered pairs are equal, the corresponding elements are equal

$$\therefore \quad \frac{x}{3} + 1 = \frac{5}{3} \qquad \text{and} \qquad y - \frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{5}{3} - 1 \qquad \text{and} \qquad y = \frac{1}{3} + \frac{2}{3}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{2}{3} \qquad \text{and} \qquad y = \frac{3}{3}$$

$$\Rightarrow \quad x = 2 \qquad \text{and} \qquad y = 1.$$

2. If the set A has 3 elements and the set $B = \{3, 4, 5\}$, then find the number of elements in $(A \times B)$.

Sol. Given: $n(A) = 3$ and $B = \{3, 4, 5\}$ so that $n(B) = 3$

$$\therefore \quad n(A \times B) = n(A) \times n(B) = 3 \times 3 = 9.$$

3. If $G = \{7, 8\}$ and $H = \{5, 4, 2\}$, find $G \times H$ and $H \times G$.

Sol. Given $G = \{7, 8\}$ and $H = \{5, 4, 2\}$

$$\therefore \quad G \times H = \{(x, y) : x \in G \text{ and } y \in H\}$$

$$= \{(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)\}$$

and $H \times G = \{(x, y) : x \in H \text{ and } y \in G\}$
 $= \{(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)\}$.

4. State whether each of the following statements are true or false. If the statement is false, rewrite the given statement correctly.

- (i) If $P = \{m, n\}$ and $Q = \{n, m\}$, then $P \times Q = \{(m, n), (n, m)\}$.
- (ii) If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in A$ and $y \in B$.
- (iii) If $A = \{1, 2\}$, $B = \{3, 4\}$, then $A \times (B \cap \phi) = \phi$.

Sol. (i) False

Since $n(P) = 2$ and $n(Q) = 2$, therefore,
 $n(P \times Q) = n(P) \cdot n(Q) = 2 \cdot 2 = 4$.

Correct statement is

$P \times Q = \{(m, n), (m, m), (n, n), (n, m)\}$.

(ii) True.

(iii) $A \times (B \cap \phi) = A \times \phi = A \times \{\} = \phi$

\therefore The given statement is true.

5. If $A = \{-1, 1\}$, find $A \times A \times A$.

Sol. Given: $A = \{-1, 1\}$

$\Rightarrow A \times A = \{-1, 1\} \times \{-1, 1\}$
 $= \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$

$\therefore A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1),$
 $(-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$.

6. If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, find A and B .

Sol. $A =$ set of first elements

$= \{a, b\}$

[Dropping repetitions]

$B =$ set of second elements

$= \{x, y\}$.

[Dropping repetitions]

7. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(ii) $A \times C$ is a subset of $B \times D$.

Sol. (i) $B \cap C = \{1, 2, 3, 4\} \cap \{5, 6\} = \phi$

$\therefore A \times (B \cap C) = A \times \phi = \phi$

...(i)

Also $A \times B = \{1, 2\} \times \{1, 2, 3, 4\}$

$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1),$

$(2, 2), (2, 3), (2, 4)\}$

$$\begin{aligned}
 \text{and } A \times C &= \{1, 2\} \times \{5, 6\} \\
 &= \{(1, 5), (1, 6), (2, 5), (2, 6)\} \\
 \therefore (A \times B) \cap (A \times C) &= \{(1, 1), (1, 2), (1, 3), (1, 4), \\
 &\quad (2, 1), (2, 2), \\
 &\quad (2, 3), (2, 4)\} \cap \{(1, 5), (1, 6), (2, 5), (2, 6)\} \\
 &= \phi \qquad \dots(ii)
 \end{aligned}$$

From (i) and (ii),

$$A \times (B \cap C) = (A \times B) \cap (A \times C) \text{ is verified.}$$

$$\begin{aligned}
 (ii) \quad A \times C &= \{1, 2\} \times \{5, 6\} \\
 &= \{(1, 5), (1, 6), (2, 5), (2, 6)\} \\
 B \times D &= \{1, 2, 3, 4\} \times \{5, 6, 7, 8\} \\
 &= \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), \\
 &\quad (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), \\
 &\quad (4, 5), (4, 6), (4, 7), (4, 8)\}
 \end{aligned}$$

Every element of $A \times C$ is also an element of $B \times D$.

$\therefore A \times C \subset B \times D$ is verified.

8. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Write $A \times B$. How many subsets will $A \times B$ have? List them.

Sol. Here, $A = \{1, 2\}$ and $B = \{3, 4\}$

$$\begin{aligned}
 \therefore A \times B &= \{(a, b) : a \in A, b \in B\} \\
 &= \{(1, 3), (1, 4), (2, 3), (2, 4)\} = \{a, b, c, d\} \text{ (say)}
 \end{aligned}$$

Since $n(A \times B) = 4$, therefore, $A \times B$ will have $2^4 = 16$ subsets.

The subsets of $A \times B$ are

$$\begin{aligned}
 \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \\
 \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}
 \end{aligned}$$

Putting values of $a = (1, 3)$, $b = (1, 4)$, $c = (2, 3)$, $d = (2, 4)$; these subsets are

$$\begin{aligned}
 \phi, \{(1, 3)\}, \{(1, 4)\}, \{(2, 3)\}, \{(2, 4)\}, \{(1, 3), (1, 4)\}, \{(1, 3), (2, 3)\}, \\
 \{(1, 3), (2, 4)\}, \{(1, 4), (2, 3)\}, \{(1, 4), (2, 4)\}, \{(2, 3), (2, 4)\}, \\
 \{(1, 3), (1, 4), (2, 3)\}, \{(1, 3), (1, 4), (2, 4)\}, \{(1, 3), (2, 3), (2, 4)\}, \\
 \{(1, 4), (2, 3), (2, 4)\}, \{(1, 3), (1, 4), (2, 3), (2, 4)\}.
 \end{aligned}$$

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1)$, $(y, 2)$, $(z, 1)$ are in $A \times B$, find A and B , where x , y and z are distinct elements.

Sol. Since $n(A) = 3$ and $n(B) = 2$;
therefore,

$$n(A \times B) = n(A) \cdot n(B) = 3 \cdot 2 = 6$$

We know that elements of set A are the first elements and elements of set B are the second elements in the ordered pairs of $A \times B$. Also $(x, 1)$, $(y, 2)$, $(z, 1)$ are in $A \times B$.

\therefore Set $A = \{x, y, z\}$ and set $B = \{1, 2\}$.

- 10. The Cartesian product $A \times A$ has 9 elements among which are found $(-1, 0)$ and $(0, 1)$. Find the set A and the remaining elements of $A \times A$.**

Sol. Since $A \times A$ has nine elements, i.e., $n(A \times A) = 9 = 3 \times 3$; therefore, $n(A)$ must be 3.

Now two elements of $A \times A$ are $(-1, 0)$ and $(0, 1)$ (given)

Therefore, set $A = \{-1, 0, 1\}$ because in $A \times A$, elements at both places of ordered pairs are from A.

$$\begin{aligned} \therefore A \times A &= \{-1, 0, 1\} \times \{-1, 0, 1\} \\ &= \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), \\ &\quad (1, -1), (1, 0), (1, 1)\} \end{aligned}$$

\therefore The remaining elements of $A \times A$ other than the given two elements $(-1, 0)$ and $(0, 1)$ of $A \times A$ are $(-1, -1)$, $(-1, 1)$, $(0, -1)$, $(0, 0)$, $(1, -1)$, $(1, 0)$, $(1, 1)$.

EXERCISE 2.2 (Page No.: 35-36)

- 1. Let $A = \{1, 2, 3, \dots, 14\}$. Define a relation R from A to A by $R = \{(x, y) : 3x - y = 0, \text{ where } x, y \in A\}$. Write down its domain, codomain and range.**

Sol. Here, $A = \{1, 2, 3, \dots, 14\}$ and

$$\begin{aligned} R &= \{(x, y) : 3x - y = 0, \text{ where } x, y \in A\} \\ &= \{(x, y) : y = 3x, \text{ where } x = 1, 2, 3, 4\} \\ &\quad (\text{when } x \geq 5, y = 3x \geq 15, \text{ so that } y \notin A) \\ &= \{(1, 3), (2, 6), (3, 9), (4, 12)\} \end{aligned}$$

Domain of R = the set of first elements of the ordered pairs in R

$$= \{1, 2, 3, 4\}$$

Codomain of R = $A = \{1, 2, 3, \dots, 14\}$

Range of R = the set of second elements of the ordered pairs in R

$$= \{3, 6, 9, 12\}.$$

2. Define a relation R on the set N of natural numbers by $R = \{(x, y) : y = x + 5, x \text{ is a natural number less than } 4; x, y \in N\}$. Depict this relationship using roster form. Write down the domain and the range.

Sol. Here, $R = \{(x, y) : y = x + 5, x \in N \text{ and } x < 4\}$
 $= \{(x, y) : y = x + 5 \text{ and } x = 1, 2, 3\}$
 $= \{(1, 1 + 5), (2, 2 + 5), (3, 3 + 5)\}$
 $= \{(1, 6), (2, 7), (3, 8)\}$

Domain of $R = \{1, 2, 3\}$

Range of $R = \{6, 7, 8\}$.

3. $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$. Define a relation R from A to B by $R = \{(x, y) : \text{the difference between } x \text{ and } y \text{ is odd}; x \in A, y \in B\}$. Write R in roster form.

Sol. Here, $R = \{(x, y) : \text{the difference between } x \text{ and } y \text{ is odd}; x \in A, y \in B\}$

$$= \{(x, y) : |x - y| \text{ is odd}; x \in A, y \in B\}$$

$$= \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}.$$

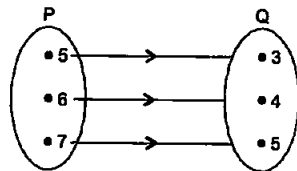
4. The Fig. shows a relationship between the sets P and Q . Write this relation

(i) in set-builder form

(ii) in roster form.

What is its domain and range?

Sol. Let R be the relation from P to Q .



Clearly from the fig., the ordered pairs $(5, 3)$, $(6, 4)$ and $(7, 5)$ belong to R . The second element of each ordered pair is two less than the corresponding first element.

Therefore,

(i) In set-builder form

$$R = \{(x, y) : y = x - 2, \text{ where } x = 5, 6, 7\}$$

(ii) In roster form

$$R = \{(5, 3), (6, 4), (7, 5)\}$$

$$\text{Domain of } R = \{5, 6, 7\}$$

$$\text{Range of } R = \{3, 4, 5\}.$$

5. Let $A = \{1, 2, 3, 4, 6\}$. Let R be the relation on A defined by $\{(a, b) : a, b \in A, b \text{ is exactly divisible by } a\}$.

- (i) Write R in roster form (ii) Find the domain of R
 (iii) Find the range of R.

Sol. Here, $A = \{1, 2, 3, 4, 6\}$ and

$$(i) \quad R = \{(a, b) : a, b \in A, b \text{ is exactly divisible by } a\}$$

$$= \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4),$$

$$(2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$$

$$(ii) \text{ Domain of } R = \{1, 2, 3, 4, 6\} = A$$

$$(iii) \text{ Range of } R = \{1, 2, 3, 4, 6\} = A.$$

- 6. Determine the domain and range of the relation R defined by $R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$.**

Sol. $R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$
 $= \{(0, 0 + 5), (1, 1 + 5), (2, 2 + 5), (3, 3 + 5),$
 $(4, 4 + 5), (5, 5 + 5)\}$
 $= \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$

$$\text{Domain of } R = \{0, 1, 2, 3, 4, 5\}$$

$$\text{Range of } R = \{5, 6, 7, 8, 9, 10\}.$$

- 7. Write the relation $R = \{(x, x^3) : x \text{ is a prime number less than } 10\}$ in roster form.**

Sol. Prime numbers less than 10 are 2, 3, 5, 7

$$\therefore x = 2, 3, 5, 7$$

$$R = \{(x, x^3) : x = 2, 3, 5, 7\}$$

$$= \{(2, 2^3), (3, 3^3), (5, 5^3), (7, 7^3)\}$$

$$= \{(2, 8), (3, 27), (5, 125), (7, 343)\}.$$

- 8. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from A to B.**

Sol. Given: $A = \{x, y, z\}$ and $B = \{1, 2\}$

$$\Rightarrow n(A) = 3 \text{ and } n(B) = 2$$

$$\text{so that } n(A \times B) = n(A) \cdot n(B) = 3 \cdot 2 = 6$$

Number of relations from A to B

$$= \text{Number of subsets of } A \times B$$

$$= 2^6 = 64.$$

- 9. Let R be the relation on Z defined by $R = \{(a, b) : a, b \in Z, a - b \text{ is an integer}\}$. Find the domain and range of R.**

Sol. $R = \{(a, b) : a, b \in Z, a - b \text{ is an integer}\}$

Since $a - b$ is an integer for all $a, b \in Z$

$$\therefore R = \{(a, b) : a, b \in Z\}$$

$$\Rightarrow \text{Domain of } R = Z$$

$$\text{Range of } R = Z.$$

EXERCISE 2.3 (Page No.: 44)

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

(i) $\{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}$

(ii) $\{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)\}$

(iii) $\{(1, 3), (1, 5), (2, 5)\}$.

Sol. (i) Domain of the relation = $\{2, 5, 8, 11, 14, 17\}$

Since every element of the domain has a unique image, **this relation is a function.**

Domain of the function = $\{2, 5, 8, 11, 14, 17\}$

Range of the function = $\{1\}$.

(ii) Domain of the relation = $\{2, 4, 6, 8, 10, 12, 14\}$

Since every element of the domain has a unique image, **this relation is a function.**

Domain of the function = $\{2, 4, 6, 8, 10, 12, 14\}$

Range of the function = $\{1, 2, 3, 4, 5, 6, 7\}$.

(iii) Domain of the relation = $\{1, 2\}$

Since the same first element 1 corresponds to two different images 3 and 5, **this relation is not a function.**

Note: Method to find domain and range of the function when rule of the function $f(x)$ is given.

Domain of a function $f(x)$ is the set of all those **real numbers** for which $f(x)$ is real and finite.

i.e. some $x \in \mathbb{R}$ does not belong to the domain of a function $f(x)$ if $f(x) \rightarrow \infty$ for this value of x or $f(x)$ assumes a value which is square root of a negative real number.

Case I If $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are

polynomial function of x ; then the points where $h(x)=0$ do not belong to the domain of $f(x)$.

Case II If $f(x) = \sqrt{g(x)}$, then only those points where $g(x) \geq 0$ belong to the domain of f .

Case III If $f(x) = \frac{1}{\sqrt{g(x)}}$, then only those points

where $g(x) > 0$ belong to the domain of f .

Range of a function is the set of values of $y = f(x)$ for values of x in the domain.

2. Find the domain and range of the following real functions:

(i) $f(x) = -|x|$ (ii) $f(x) = \sqrt{9-x^2}$.

Sol. (i) Since $|x|$ is defined i.e. is real and finite for all real numbers, $-|x|$ is also defined for all real numbers.

\therefore Domain of $f = \mathbb{R}$

Also, for all $x \in \mathbb{R}$, $|x| \geq 0$

$$\Rightarrow -|x| \leq 0 \quad \Rightarrow f(x) \leq 0$$

\therefore Range of $f = (-\infty, 0]$.

(ii) $f(x) = \sqrt{9-x^2} \quad \Rightarrow y = \sqrt{9-x^2}$

$f(x)$ is defined i.e. real and finite if $9-x^2 \geq 0$

$$\Rightarrow 3^2-x^2 \geq 0 \Rightarrow (3-x)(3+x) \geq 0$$

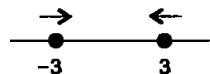
Case I $3-x \geq 0$ and $3+x \geq 0$ ($\because (+)(+) = +$)

$$\Rightarrow -x \geq -3 \text{ and } x \geq -3$$

multiplying by (-1) , (Multiplication by a negative number reverses the inequality)

$$\Rightarrow x \leq 3 \text{ and } x \geq -3$$

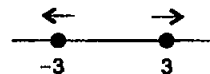
$$\therefore x \in [-3, 3] \dots (i)$$



Case II $3-x \leq 0$ and $3+x \leq 0$ ($\because (-)(-) = +$)

$$\Rightarrow -x \leq -3 \text{ and } x \leq -3$$

$$\Rightarrow x \geq 3 \text{ and } x \leq -3$$



But this is impossible

\therefore From (i), Domain of $f(x)$ is $[-3, 3]$

To find range, we should (have to) find x in terms of y .

$$y = \sqrt{9-x^2} \quad \Rightarrow y^2 = 9-x^2$$

$$\Rightarrow x^2 = 9-y^2$$

$$\Rightarrow x = \pm \sqrt{9-y^2}$$

Now x is real if $9 - y^2 \geq 0$

$$\Rightarrow (3 - y)(3 + y) \leq 0$$

Case I $3 - y \geq 0$ and $3 + y \geq 0$

$$\Rightarrow -y \geq -3 \text{ and } y \geq -3$$

$$y \leq 3 \text{ and } y \geq -3$$

$$\therefore -3 \leq y \leq 3$$

$$\therefore y \in [-3, 3] \dots(ii)$$

Case II $3 - y \leq 0$ and $3 + y \leq 0$

$$\Rightarrow -y \leq -3 \text{ and } y \leq -3$$

$$\Rightarrow y \geq 3 \text{ and } y \leq -3$$

But this is impossible.

But $y = \sqrt{9 - x^2}$ is always ≥ 0

...(iii)

From (ii) and (iii), $0 \leq y \leq 3$

\therefore Range (set) is closed interval $[0, 3]$.

3. A function f is defined by $f(x) = 2x - 5$. Write down the values of

(i) $f(0)$

(ii) $f(7)$

(iii) $f(-3)$.

Sol. Given: $f(x) = 2x - 5$

(i) Replacing x by 0, we have

$$f(0) = 2 \times 0 - 5 = -5$$

(ii) Replacing x by 7, we have

$$f(7) = 2 \times 7 - 5 = 14 - 5 = 9$$

(iii) Replacing x by -3 , we have

$$f(-3) = 2 \times (-3) - 5 = -6 - 5 = -11.$$

4. The function ' t ' which maps temperature in degree Celsius into temperature in degree Fahrenheit

is defined by $t(C) = \frac{9C}{5} + 32$. Find

(i) $t(0)$

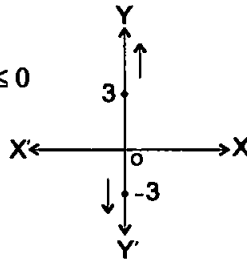
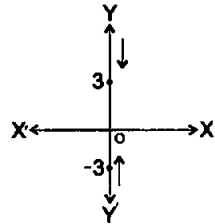
(ii) $t(28)$

(iii) $t(-10)$

(iv) The value of C , when $t(C) = 212$.

Sol. Given: $t(C) = \frac{9C}{5} + 32$

(i) Replacing C by 0, we have



$$t(0) = \frac{9 \times 0}{5} + 32 = 0 + 32 = 32$$

(ii) Replacing C by 28, we have

$$\begin{aligned} t(28) &= \frac{9 \times 28}{5} + 32 = \frac{252}{5} + 32 \\ &= 50.4 + 32 = 82.4 \end{aligned}$$

(iii) Replacing C by -10 , we have

$$t(-10) = \frac{9 \times (-10)}{5} + 32 = -18 + 32 = 14.$$

$$(iv) \quad t(C) = 212 \quad \Rightarrow \quad \frac{9C}{5} + 32 = 212$$

$$\Rightarrow \frac{9C}{5} = 212 - 32 = 180 \Rightarrow C = \frac{5}{9} \times 180 = 100.$$

5. Find the range of each of the following functions:

(i) $f(x) = 2 - 3x$, $x \in \mathbf{R}$, $x > 0$.

(ii) $f(x) = x^2 + 2$, x is a real number.

(iii) $f(x) = x$, x is a real number.

Sol. (i) $f(x) = 2 - 3x$, $x \in \mathbf{R}$, $x > 0$...(i) (given)
 Consider $x > 0$ (given)

Multiplying both sides by -3 , $-3x < 0$

Adding 2 to both sides, $2 - 3x < 2$

or (By (i)) $f(x) < 2$ for $x > 0$

$\Rightarrow f(x) \in (-\infty, 2) \Rightarrow$ Range of this $f(x)$ is $(-\infty, 2)$.

Note. Domain of this $f(x)$ is $\{x : x \in \mathbf{R}, x > 0\}$

(given) = $(0, \infty)$

(ii) $f(x) = x^2 + 2$, $x \in \mathbf{R}$...(i) (given)

$$x \in \mathbf{R} \Rightarrow x^2 \geq 0$$

Adding 2 to both sides, $x^2 + 2 \geq 2$

or (By (i)) $f(x) \geq 2 \Rightarrow f(x) \in [2, \infty)$

\therefore Range of $f(x)$ is $[2, \infty)$

(iii) $f(x) = x$; $x \in \mathbf{R}$ (given)

$\Rightarrow f(x)$ also takes all real values.

\therefore Range of $f(x)$ is \mathbf{R} .

MISCELLANEOUS EXERCISE ON CHAPTER 2

(Page No.: 46 - 47)

1. The relation f is defined by $f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 3x, & 3 \leq x \leq 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2, & 0 \leq x \leq 2 \\ 3x, & 2 \leq x \leq 10 \end{cases}$.

Show that f is a function and g is not a function.

Sol. The domain of f is $[0, 10]$.

For each $x \in [0, 3)$, $f(x) = x^2$ is uniquely defined

i.e. each $x \in [0, 3]$ has a unique image

For each $x \in (3, 10]$, $f(x) = 3x$ is uniquely defined.

At $x = 3$, $x^2 = 3^2 = 9$. Also $3x = 3 \times 3 = 9$ so that f is uniquely defined at 3.

Since every point of the domain has one and only one image under f , this relation is a function.

The domain of g is $[0, 10]$.

For each $x \in [0, 2)$, $f(x) = x^2$ is uniquely defined.

For each $x \in (2, 10]$, $f(x) = 3x$ is uniquely defined.

At $x = 2$, $g(x) = x^2 = 2^2 = 4$. Also at $x = 2$, $g(x) = 3x = 3 \times 2 = 6$.

Now, 2 is in the domain of g and the image of 2 under g is not unique since $(2, 4), (2, 6) \in g$, therefore, g is not a function.

2. If $f(x) = x^2$, find $\frac{f(1.1) - f(1)}{1.1 - 1}$.

Sol.
$$\frac{f(1.1) - f(1)}{1.1 - 1} = \frac{(1.1)^2 - (1)^2}{1.1 - 1} = \frac{(1.1 + 1)(1.1 - 1)}{1.1 - 1}$$
$$= 1.1 + 1 = 2.1.$$

3. Find the domain of the function $f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

Sol. f is a rational function.

\therefore Domain of $f = \mathbb{R} - \{x : x^2 - 8x + 12 = 0\}$

(See case I, Note after solution of Q.N.1, page 48)

$$= \mathbb{R} - \{x : (x - 2)(x - 6) = 0\}$$

$$= \mathbb{R} - \{2, 6\}.$$

4. Find the domain and the range of the real function f defined by $f(x) = \sqrt{x-1}$.

Sol. f is real only when $x - 1 \geq 0$ i.e., $x \geq 1$
(See case II, Note after solution of Q.N.1, page 48)

\therefore Domain of $f = [1, \infty)$

Let $y = f(x) = \sqrt{x-1}$

Clearly, $y \geq 0$... (i)

Also, $y^2 = x - 1$ or $x = 1 + y^2$

x is real for all $y \in \mathbb{R}$... (ii)

Range of f is the set of values common to (i) and (ii)

\therefore Range of $f = \{y : y \geq 0, y \in \mathbb{R}\}$
 $= [0, \infty).$

5. Find the domain and the range of the real function f defined by

$$f(x) = |x - 1|.$$

Sol. For all $x \in \mathbb{R}$, $f(x)$ is a unique real number.

\therefore Domain of $f = \mathbb{R}$.

Also, for all $x \in \mathbb{R}$, $|x - 1| \geq 0$

$\Rightarrow f(x) \geq 0$

\therefore Range of $f = [0, \infty).$

6. Let $f = \left\{ \left(x, \frac{x^2}{1+x^2} \right) : x \in \mathbb{R} \right\}$ be a function from \mathbb{R} into \mathbb{R} . Determine the range of f .

Sol. $y = (f(x)) = \frac{x^2}{1+x^2}$ (given) ... (i)

To find range of f

From (i), $y = \frac{x^2}{1+x^2}$ is ≥ 0 for all $x \in \mathbb{R}$ ($\because x^2 \geq 0$) ... (ii)

Again from (i), $y = \frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2}$

$$= \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2} < 1 \quad \forall x \in \mathbb{R} \quad \dots (iii)$$

From (ii) and (iii), $y \geq 0$ and $y < 1$

$$\therefore 0 \leq y = f(x) < 1$$

\therefore Range of $f(x)$ is $[0, 1)$.

7. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be defined, respectively by

$$f(x) = x + 1, \quad g(x) = 2x - 3. \text{ Find } f + g, f - g \text{ and } \frac{f}{g}.$$

Sol. $(f + g)(x) = f(x) + g(x) = (x + 1) + (2x - 3) = 3x - 2$

$$(f - g)(x) = f(x) - g(x) = (x + 1) - (2x - 3) = -x + 4$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0 = \frac{x+1}{2x-3}, \quad 2x-3 \neq 0$$

$$= \frac{x+1}{2x-3}, \quad x \neq \frac{3}{2}.$$

8. Let $f = \{ (1, 1), (2, 3), (0, -1), (-1, -3) \}$ be a function form \mathbb{Z} to \mathbb{Z} defined by $f(x) = ax + b$, for some integers a, b . Determine a, b .

Sol. Given: $f(x) = ax + b$...(i)

Now, $(1, 1) \in f \Rightarrow f(1) = 1$

Replacing x by 1 in (i), we have

$$f(1) = a \times 1 + b \quad \Rightarrow \quad 1 = a + b \quad \text{...(ii)}$$

Also $(2, 3) \in f \quad \Rightarrow \quad f(2) = 3$

Again, replacing x by 2 in (i), we have

$$f(2) = a \times 2 + b \quad \Rightarrow \quad 3 = 2a + b \quad \text{...(iii)}$$

Subtracting (ii) from (iii), we get $2 = a$

Putting this value of a in (ii), we get

$$1 = 2 + b \text{ or } b = -1$$

$$\therefore a = 2, b = -1.$$

9. Let R be a relation from \mathbb{N} to \mathbb{N} defined by

$R = \{ (a, b) : a, b \in \mathbb{N} \text{ and } a = b^2 \}$. Are the following true?

(i) $(a, a) \in R$, for all $a \in \mathbb{N}$

(ii) $(a, b) \in R$, implies $(b, a) \in R$

(iii) $(a, b) \in R, (b, c) \in R$ implies $(a, c) \in R$

Justify your answer in each case.

Sol. (i) We know that except natural number 1, none of the natural numbers is equal to its square.

$$\therefore (a, a) \notin R \text{ (It does not satisfy the condition } a = b^2)$$

$$\therefore (a, a) \in R \text{ is false.}$$

(ii) If $a = b^2$, it does not imply that $b = a^2$

i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ is not true.

For example, $(4, 2) \in R$

($\because a = 4, b = 2$ and $a = b^2$) but $(2, 4) \notin R$ ($\because 2 \neq 4^2$)

(iii) If $(a, b) \in R \Rightarrow a = b^2$..(i)

also $(b, c) \in R \Rightarrow b = c^2$..(ii)

Putting the value of b from (ii) in (i),

$$a = c^4$$

$\therefore a \neq c^2 \Rightarrow (a, c) \notin R$

For example, $(a, b) = (16, 4) \in R$ and $(b, c) = (4, 2) \in R$

(\because For each pair $a = b^2$)

but $(a, c) = (16, 2) \notin R$ ($\because 16 \neq 2^2$)

\therefore The statement is false.

10. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$. Are the following true?

(i) f is a relation from A to B

(ii) f is a function from A to B

Justify your answer in each case.

Sol. (i) Every element of f is an element of $A \times B$

$\Rightarrow f \subset A \times B$

$\Rightarrow f$ is a relation from A to B (since every subset of $A \times B$ is a relation from A to B).

(ii) $(2, 9)$ and $(2, 11)$ both belong to f

Here both these points of f have the same first entry

$\Rightarrow f$ -image of 2 is not unique

$\Rightarrow f$ is not a function from A to B .

11. Let f be the subset of $Z \times Z$ defined by $f = \{(ab, a + b) : a, b \in Z\}$. Is f a function from Z to Z ? Justify your answer.

Sol. Given: $f = \{(ab, a + b) : a, b \in Z\}$

Taking $a = b = 1$, we have $(ab, a + b) = (1, 2) \in f$

Taking $a = b = -1$, we have $(ab, a + b) = (1, -2) \in f$

$\Rightarrow f$ -image of 1 is not unique

$\Rightarrow f$ is not a function.

12. Let $A = \{9, 10, 11, 12, 13\}$ and let $f : A \rightarrow N$ be defined by $f(n) =$ the highest prime factor of n . Find the range of f .

Sol. Here, $A = \{9, 10, 11, 12, 13\}$ and $f(n) =$ the highest prime factor of n .

The only prime factor of 9 is 3.

$$\Rightarrow f(9) = 3$$

Prime factors of 10 are 2 and 5.

$$\Rightarrow f(10) = 5$$

The only prime factor of 11 is 11

$$\Rightarrow f(11) = 11$$

Prime factors of 12 are 2 and 3

$$\Rightarrow f(12) = 3$$

The only prime factor of 13 is 13

$$\Rightarrow f(13) = 13$$

\therefore Range of $f =$ set of f -images of elements in the domain A

$$= \{f(9), f(10), f(11), f(12), f(13)\}$$

$$= \{3, 5, 11, 3, 13\}$$

$$= \{3, 5, 11, 13\}. \quad [\text{Dropping repetitions}]$$

