5 Complex Numbers and Quadratic Equations

Lesson at a Glance

1. A number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$, is called a complex number and $a + ib$ is called **standard form** of complex number.
2. If $z = a + ib$ then $Re(z) = a$ and $\text{Im}(z) = b$, **the coefficient of** $\boldsymbol{i}$
3. $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $\frac{1}{i} = -\text{i}$.
4. For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$.
5. (*i*) For any two real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only if at least one of a and b is either 0 or positive.

   (*ii*) and $\sqrt{a}\,\sqrt{b} = -\sqrt{ab}$ if both a and b are real negative numbers.
6. Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are **equal** if and only if $a = c$ and $b = d$ *i.e.*, $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = Im(z_2)$.

   We call it as "Equating real and imaginary parts" on both sides.
7. The conjugate of the complex number $z = a + ib$ is denoted by $\bar{z}$ and is given by $\bar{z} = a - ib$.
8. To express $\frac{z_1}{z_2}$, $z_2 \neq 0$, in the standard form $a + ib$, multiply and divide by $\bar{z}_2$, the conjugate of denominator.
9. The modulus of the complex number $z = a + ib$ is denoted by $|z|$ and is defined as $|z| = \sqrt{a^2 + b^2}$. and hence $|z|$ is never negative.

10. If z, z_1 and z_2 are complex numbers,

(*i*) $\overline{(\overline{z})} = z$ (*ii*) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(*iii*) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ (*iv*) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(*v*) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$

(*vi*) $|z| = 0$ *if and only if* $z = 0$

(*vii*) $|z| = |\overline{z}| = |-z|$

(*viii*) $z\overline{z} = |z|^2$

(*ix*) $|z^2| = |z|^2$

(*x*) $|z_1 + z_2| \leq |z_1| + |z_2|$. *(Triangle inequality)*

11. Multiplicative inverse (or Reciprocal) of non-zero complex number z *i.e.*,

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}. \qquad (\because z\overline{z} = |z|^2)$$

12. Every non-zero complex number z can be written as

$z = x + iy = r(\cos\theta + i\sin\theta)$

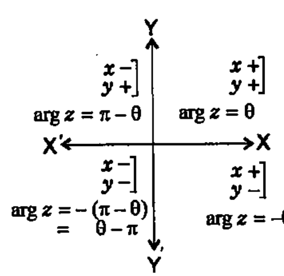
$(\because x = r\cos\theta$ and $y = r\sin\theta.)$

This is called **polar form** of z where $r = |z|$, the modulus of z and $\theta = \arg z$, where $-\pi < \theta \leq \pi$.

This angle θ is always expressed in radians.

13. If θ is the **argument** of a complex number $z = x + iy$ (as defined in 12), then θ is given by **tan θ** $= \dfrac{y}{x}$

θ mentioned in all the four cases (or as in the adjoining figure) listed below is s.t. $0 \leq \theta < \dfrac{\pi}{2}$.



(*i*) If (x, y) lies in the first quadrant, then $\arg z = \theta$

(*ii*) If (x, y) lies in the second quadrant, then $\arg z = \pi - \theta$

(*iii*) If (x, y) lies in third quadrant, then $\arg z = -(\pi - \theta) = \theta - \pi$.

(*iv*) If (x, y) lies in fourth quadrant, then $\arg z = -\theta$

14. Every real number is also a complex number. ($\because x = x + 0i$)
15. A complex number of the form $z = iy$ ($y \neq 0, y \in R$) is called a **purely imaginary number.** ($\Rightarrow$ Real part of $z = 0$)
16. Every quadratic equation has two complex roots.

TEXTBOOK QUESTIONS SOLVED

EXERCISE 5.1 *(Page No.: 103–104)*

Express each of the complex numbers given in Exercises 1 to 10 in the form *a* + *ib*:

1. $(5i)\left(-\frac{3}{5}i\right)$.

Sol. $(5i)\left(-\frac{3}{5}i\right) = (5)\left(-\frac{3}{5}\right)i^2 = -3(-1) = 3$ $[\because\ i^2 = -1]$

$= 3 + i0.$

2. $i^9 + i^{19}$.

Sol. $i^9 + i^{19} = i^8.i + i^{16}.i^3 = (i^4)^2 i + (i^4)^4 i^3 = (1)^2 i + (1)^4.(-i)$

$= i - i = 0 = 0 + 0i$

3. i^{-39}

Sol. $i^{-39} = \frac{1}{i^{39}} = \frac{1}{i^{36}.i^3} = \frac{1}{(i^4)^9.i^3}$

$= \frac{1}{(1)^9.(-i)} = \frac{1}{-i}$ $[\because\ i^4 = 1 \text{ and } i^3 = -i]$

Rationalising

$= \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{1}$ $[\because\ i^2 = -1]$

$= i = 0 + i.$

4. $3(7 + i7) + i(7 + i7)$

Sol. $3(7 + i7) + i(7 + i7) = 21 + 21i + 7i + 7i^2$

$= 21 + 28i - 7$ $[\because\ i^2 = -1]$

$= 14 + 28i.$

5. $(1 - i) - (-1 + i6)$

Sol. $(1 - i) - (-1 + i6) = 1 - i + 1 - 6i = 2 - 7i.$

6. $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$

Sol. $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)=\frac{1}{5}+\frac{2}{5}i-4-\frac{5}{2}i$

$$=\left(\frac{1}{5}-4\right)+\left(\frac{2}{5}-\frac{5}{2}\right)i$$

$$=\frac{1-20}{5}+\frac{4-25}{10}i$$

$$=-\frac{19}{5}-\frac{21}{10}i.$$

7. $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

Sol. $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

$$=\left(\frac{1}{3}+\frac{7}{3}i+4+\frac{1}{3}i\right)+\frac{4}{3}-i$$

$$=\left[\left(\frac{1}{3}+4\right)+\left(\frac{7}{3}+\frac{1}{3}\right)i\right]+\frac{4}{3}-i$$

$$=\left(\frac{13}{3}+\frac{8}{3}i\right)+\frac{4}{3}-i=\left(\frac{13}{3}+\frac{4}{3}\right)+\left(\frac{8}{3}-1\right)i$$

$$=\frac{17}{3}+\frac{5}{3}i.$$

8. $(1-i)^4$

Sol. $(1-i)^4=[(1-i)^2]^2=(1+i^2-2i)^2$

$$=(1-1-2i)^2=(-2i)^2=4i^2$$

$$=-4=-4+0i.$$

9. $\left(\frac{1}{3}+3i\right)^3$

Sol. $\left(\frac{1}{3}+3i\right)^3=\left(\frac{1}{3}\right)^3+(3i)^3+3\cdot\frac{1}{3}\cdot 3i\left(\frac{1}{3}+3i\right)$

$[\because\ (a+b)^3=a^3+b^3+3ab(a+b)]$

$$= \frac{1}{27} + 27i^3 + 3i\left(\frac{1}{3} + 3i\right)$$

$$= \frac{1}{27} - 27i + i + 9i^2 \qquad [\because\ i^3 = -i]$$

$$= \frac{1}{27} - 26i - 9 \qquad [\because\ i^2 = -1]$$

$$= \left(\frac{1}{27} - 9\right) - 26i = \frac{1-243}{27} - 26i$$

$$= -\frac{242}{27} - 26i.$$

10. $\left(-2-\frac{1}{3}i\right)^3$

Sol. $\left(-2-\frac{1}{3}i\right)^3 = \left[-\left(2+\frac{1}{3}i\right)\right]^3 = -\left(2+\frac{1}{3}i\right)^3$

$$= -\left[2^3 + \left(\frac{1}{3}i\right)^3 + 3.2.\frac{1}{3}i\left(2+\frac{1}{3}i\right)\right]$$

$$= [\because (a+b)^3 = a^3 + b^3 + 3ab\,(a+b)]$$

$$= -\left[8 + \frac{1}{27}i^3 + 2i\left(2+\frac{1}{3}i\right)\right]$$

$$= -\left[8 + \frac{1}{27}i^3 + 2i\left(2+\frac{1}{3}i\right)\right]$$

$$= -\left[8 - \frac{1}{27}i + 4i + \frac{2}{3}i^2\right] \qquad [\because\ i^3 = -i]$$

$$= -\left(8 - \frac{1}{27}i + 4i - \frac{2}{3}\right) \qquad [\because\ i^2 = -1]$$

$$= -\left[\left(8-\frac{2}{3}\right) + \left(4 - \frac{1}{27}\right)i\right]$$

$$= -\left(\frac{22}{3} + \frac{107}{27}i\right)$$

$$= -\frac{22}{3} - \frac{107}{27}\ i.$$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. 4 – 3*i*

Sol. Multiplicative inverse of $z = 4 - 3i$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{4-3i} = \frac{1}{4-3i} \times \frac{4+3i}{4+3i} = \frac{4+3i}{4^2-(3i)^2}$$

$$= \frac{4+3i}{16-9i^2} = \frac{4+3i}{16+9} = \frac{4+3i}{25}$$

$$= \frac{4}{25} + \frac{3}{25}\, i.$$

Second Solution

Multiplicative inverse of $z = 4 - 3i$ is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4+3i}{(4)^2+(-3)^2} = \frac{4+3i}{16+9} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

[∵ If $z = x + iy$, then $\bar{z} = x - iy$ and $|z| = \sqrt{x^2+y^2}$ and hence $|z|^2 = x^2 + y^2$]

12. $\sqrt{5}$ + 3*i*

Sol. Multiplicative inverse of $z = \sqrt{5} + 3i$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{\sqrt{5}+3i} = \frac{1}{\sqrt{5}+3i} \times \frac{\sqrt{5}-3i}{\sqrt{5}-3i}$$

$$= \frac{\sqrt{5}-3i}{(\sqrt{5})^2-(3i)^2} = \frac{\sqrt{5}-3i}{5-9i^2} = \frac{\sqrt{5}-3i}{5+9}$$

$$= \frac{\sqrt{5}-3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i.$$

Second Solution

Multiplicative inverse of $z = \sqrt{5} + 3i$ is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5}-3i}{(\sqrt{5})^2+(3)^2} = \frac{\sqrt{5}-3i}{5+9=14}$$

$$= \frac{\sqrt{5}}{14} - \frac{3i}{14}.$$

13. $-i$

Sol. Multiplicative inverse of $z = -i$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{-i} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = i$$

$$= 0 + i.$$

Second Solution

Multiplicative inverse of $z = -i = 0 - i$ is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{0+i}{(0)^2 + (-1)^2} = \frac{0+i}{1} = 0 + i$$

14. Express the following expression in the form $a + ib$:

$$\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}\,i)-(\sqrt{3}-i\sqrt{2})}$$

Sol. $\dfrac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}\,i)-(\sqrt{3}-i\sqrt{2})}$

Using $(a + b)(a - b) = a^2 - b^2$

$$= \frac{3^2 - (i\sqrt{5})^2}{\sqrt{3}+\sqrt{2}\,i-\sqrt{3}+\sqrt{2}\,i} = \frac{9-5i^2}{2\sqrt{2}\,i}$$

$$= \frac{9+5}{2\sqrt{2}\,i} = \frac{7}{\sqrt{2}\,i} \times \frac{-\sqrt{2}\,i}{-\sqrt{2}\,i} \quad \text{(Rationalising)}$$

$$= \frac{-7\sqrt{2}\,i}{-2i^2} = \frac{-7\sqrt{2}\,i}{2} \qquad [\because\ i^2 = -1]$$

$$= 0 - \frac{7\sqrt{2}}{2}i.$$

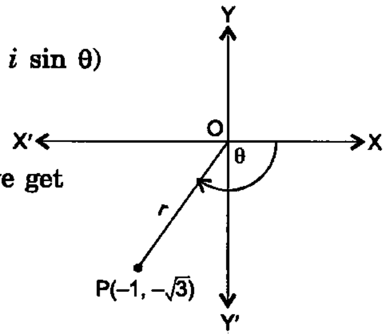
EXERCISE 5.2 *(Page No.: 108)*

Find the modulus and the arguments of each of the complex numbers in Exercises 1 to 2.

1. $z = -1 - i\sqrt{3}$

Sol. $z = -1 - i\sqrt{3} = x + iy$

is represented by the point P $(-1, -\sqrt{3})$ which lies in III quadrant.



Let $z = -1 - i\sqrt{3}$

$= r(\cos\theta + i\sin\theta)$

Then, $x = r\cos\theta = -1$

and $y = r\sin\theta = -\sqrt{3}$

Squaring and adding both, we get

$r^2(\cos^2\theta + \sin^2\theta)$

$= (-1)^2 + (-\sqrt{3})^2$

$\Rightarrow \quad r^2 = 1 + 3 = 4$

$\therefore \quad r = 2$

(or $r = |z| = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$)

Because P $(x, y) = (-1, -\sqrt{3})$ lies in third quadrant, therefore, arg z is to be of the form $\theta - \pi$

$$\Rightarrow \quad \tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3} = \tan\frac{\pi}{3}$$

$$= \tan\left(\frac{\pi}{3} - \pi\right) = \tan\left(\frac{-2\pi}{3}\right)$$

$$\Rightarrow \quad \theta = -\frac{2\pi}{3}$$

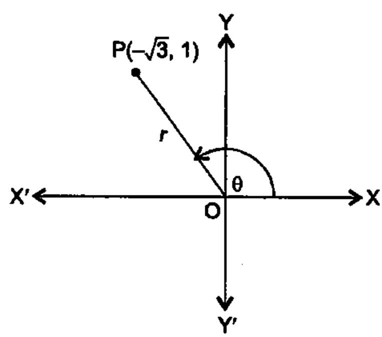
$\therefore \quad |z| = r = 2$

and $\quad \arg z = \theta = -\frac{2\pi}{3}$.

2. $z = -\sqrt{3} + i$

Sol. $z = -\sqrt{3} + i = x + iy$

is represented by the point P $(-\sqrt{3}, 1)$ which lies in second quadrant.



Let $z = -\sqrt{3} + i$

$= r(\cos\theta + i\sin\theta)$

Then, $x = r\cos\theta = -\sqrt{3}$

and $y = r\sin\theta = 1$

Squaring and adding both, we get

$r^2(\cos^2\theta + \sin^2\theta) = 3 + 1$

$\Rightarrow \quad r^2 = 4 \quad \therefore \quad r = 2$

(or $r = |z| = \sqrt{x^2+y^2} = \sqrt{(-\sqrt{3})^2+(1)^2} = \sqrt{3+1} = \sqrt{4} = 2$)

Because P $(x, y) = (-\sqrt{3}, 1)$ lies in second quadrant, therefore arg z is to be of the form $\pi - \theta$

$$\tan\theta = \frac{y}{x} = -\frac{1}{\sqrt{3}} = -\tan\frac{\pi}{6}$$

$$= \tan\left(\pi - \frac{\pi}{6}\right) = \tan\frac{5\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{5\pi}{6}$$

$\therefore \quad |z| = r = 2$ and $\arg z = \theta = \frac{5\pi}{6}$.

Convert each of the complex numbers given in Exercises 3 to 8 in the polar form:

3. $1 - i$

Sol. Let $z = 1 - i = x + iy$ *is* represented by the point P $(x, y) = (1, -1)$ which lies in fourth quadrant.

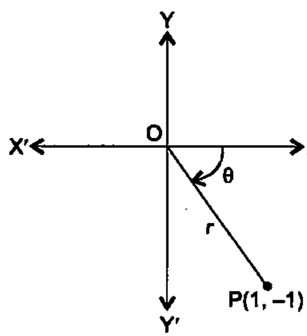
Let $z = 1 - i = r(\cos\theta + i\sin\theta)$ ... (i)

then $x = 1 = r\cos\theta$ and $y = -1 = r\sin\theta$

Squaring and adding both,

$\therefore \quad 1^2 + (-1)^2 = r^2 \Rightarrow r = \sqrt{2}$

(or $r = |z| = \sqrt{x^2+y^2} = \sqrt{(1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$)



Because P (1, –1) lies in fourth quadrant therefore, arg z is to be of the form $-\theta$.

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1 = -\tan\frac{\pi}{4} = \tan\left(-\frac{\pi}{4}\right)$$

$$\therefore \quad \theta = -\frac{\pi}{4}$$

Putting these values of r and θ in (i),

the required polar form of z is $\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$.

4. – 1 + i

Sol. $z = -1 + i = x + iy$ is represented by the point P (x, y) = P $(-1, 1)$ which lies in second quadrant.

Let $z = -1 + i = r(\cos\theta + i\sin\theta)$ ...(i)

Then, $r\cos\theta = -1 = x$

and $r\sin\theta = 1 = y$

Squaring and adding both, we get

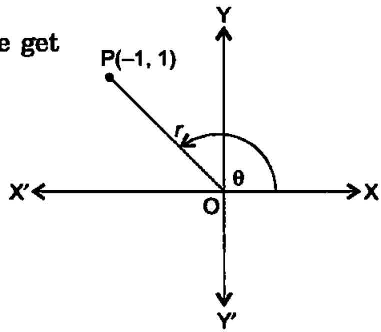
$r^2(\cos^2\theta + \sin^2\theta) = (-1)^2 + 1^2$

$\Rightarrow \quad r^2 = 1 + 1 = 2$

$\therefore \quad r = \sqrt{2}$

(or $r = |z| = \sqrt{x^2+y^2}$

$= \sqrt{(-1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$)



Because P $(-1, 1)$ lies in second quadrant, therefore arg z is to be of the form $\pi - \theta$.

$$\tan\theta = \frac{y}{x} = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi-\frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$$

$\therefore \quad \theta = \frac{3\pi}{4}$

Putting these values of r and θ in (i),

$z = \sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ is the required polar form.

5. – 1 – i

Sol. $z = -1 - i = x + iy$ is represented by the point P (x, y)= $(-1, -1)$ which lies in third quadrant.

Let $\quad z = -1 - i$

$= r(\cos\theta + i\sin\theta)$ ... (i)

Then, $r\cos\theta = -1\ (= x)$

and $r\sin\theta = -1\ (= y)$

Squaring and adding both, we get

$r^2 (\cos^2 \theta + \sin^2 \theta)$

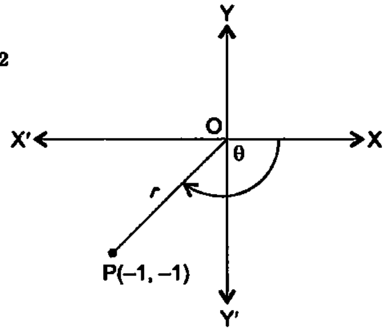
$= (-1)^2 + (-1)^2$

$\Rightarrow \quad r^2 = 1 + 1 = 2$

$\therefore \quad r = \sqrt{2}$

(or $r = |z| = \sqrt{x^2 + y^2}$

$= \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$)



Because P (–1, –1) lies in third quadrant, therefore arg z is to be of the form $-(\pi - \theta) = \theta - \pi$

$\tan \theta = \frac{y}{x} = 1 = \tan \frac{\pi}{4} = \tan (\theta - \pi)$

$= \tan \left(\frac{\pi}{4} - \pi\right)$

$= \tan \left(-\frac{3\pi}{4}\right)$

$\therefore \quad \theta = -\frac{3\pi}{4}$

Putting these values of r and θ in (*i*),

$z = \sqrt{2} \left[\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right)\right]$ is the required polar form.

6. – 3

Sol. $z = -3 = -3 + 0i = x + iy$ is represented by the point P $(x, y) = (-3, 0)$ which lies on the ray OX'.

Let $\quad z = -3 = r (\cos \theta + i \sin \theta) \quad$ ... (*i*)

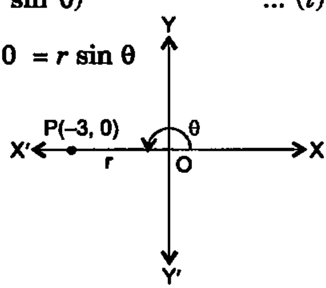
Then $x = -3 = r \cos \theta$, and $y = 0 = r \sin \theta$

Squaring and adding both,

$\therefore \; (-3)^2 + 0^2 = r^2 \Rightarrow r = 3$

(or $r = |z| = \sqrt{x^2 + y^2} =$

$\sqrt{(-3)^2 + 0^2} = \sqrt{9+0} = \sqrt{9} = 3$)



Because P (–3, 0) lies on OX' i.e. in second quadrant, therefore arg z is to be of the form $\pi - \theta$.

$\tan\theta = \dfrac{y}{x} = \dfrac{0}{-3} = 0 = \tan 0 = \tan\ (\pi - 0) = \tan\pi$

$\therefore\ \theta = \pi$

Putting these values of r and θ in (i), required polar form of

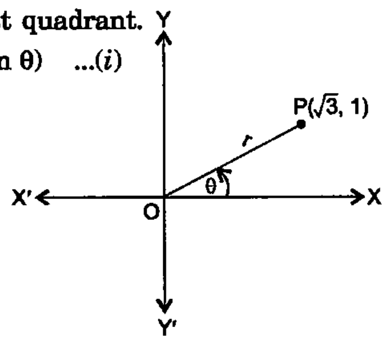
$z = 3\,(\cos\pi + i\sin\pi)$

Remark: Because P (–3, 0) lies on OX', we can also say that P lies in third quadrant and arg $z = -\pi + \theta = -\pi + 0 = -\pi$ which is rejected because by result No. 12 of" Lesson at a glance" page 118,

$-\pi < \theta \le \pi$ *i.e.* $\arg z \ne -\pi$.

7. $\sqrt{3} + i$

Sol. $z = \sqrt{3} + i = x + iy$ is represented by the point P(x, y) = P ($\sqrt{3}$,1) which lies in first quadrant.



Let $z = \sqrt{3} + i = r\,(\cos\theta + i\sin\theta)$ ...(i)

Then, $r\cos\theta = \sqrt{3} = x$

and $r\sin\theta = 1 = y$

Squaring and adding both, we get

$r^2\,(\cos^2\theta + \sin^2\theta) = 3 + 1$

$\Rightarrow\ r^2 = 4 \therefore\ r = 2$

Because the point P($\sqrt{3}$, 1) lies in first quadrant, therefore, arg z is to be of the form θ.

$\tan\theta = \dfrac{y}{x} = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$

$\therefore\ \theta = \dfrac{\pi}{6}$

Putting values of r and θ in (i),

$z = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$ is the required polar form.

8. i

Sol. $z = i = 0 + i = x + iy$ is represented by the point P (x, y) = (0, 1) which lies on OY.

Let $z = 0 + i = r(\cos\theta + i\sin\theta)$ ... (i)

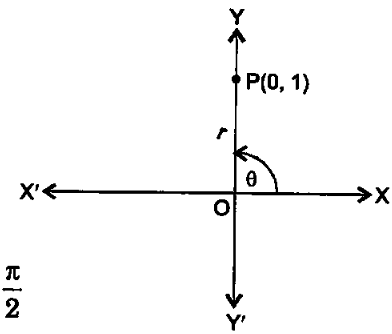
Then, $r\cos\theta = 0$ and $r\sin\theta = 1$.

Squaring and adding both, we get

$r^2(\cos^2\theta + \sin^2\theta) = 0 + 1$

$\Rightarrow \quad r^2 = 1 \therefore r = 1$

Because the point P (0, 1) lies on OY, therefore arg z is to be of the form θ.



$\tan\theta = \frac{y}{x} = \frac{1}{0} = \infty = \tan\frac{\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{2}$

Putting these values of r and θ in (i),

$z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ is the required polar form.

EXERCISE 5.3 (*Page No.: 109*)

Solve each of the following equations:

1. $x^2 + 3 = 0$

Sol. $x^2 + 3 = 0$

$\Rightarrow \quad x^2 = -3 \quad \Rightarrow \quad x = \pm\sqrt{-3} = \pm\sqrt{3(-1)}$

$\Rightarrow \quad x = \pm\sqrt{3}.\sqrt{-1}$

$\therefore \quad x = \pm\sqrt{3}\,i.$

2. $2x^2 + x + 1 = 0$

Sol. $2x^2 + x + 1 = 0$

Here, $a = 2, b = 1, c = 1$

$D = b^2 - 4ac = 1 - 4 \times 2 \times 1 = 1 - 8 = -7$

$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times 2} = \frac{-1 \pm \sqrt{7}i}{4}.$

3. $x^2 + 3x + 9 = 0$

Sol. $x^2 + 3x + 9 = 0$

Here, $a = 1, b = 3, c = 9$

$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = 9 - 36 = -27$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2 \times 1} = \frac{-3 \pm 3\sqrt{3}\,i}{2}.$$

4. $-x^2 + x - 2 = 0$

Sol. $-x^2 + x - 2 = 0$

Here, $a = -1, b = 1, c = -2$

$D = b^2 - 4ac = 1^2 - 4 \times (-1) \times (-2)$

$= 1 - 8 = -7$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times (-1)} = \frac{-1 \pm \sqrt{7}i}{-2}.$$

5. $x^2 + 3x + 5 = 0$

Sol. $x^2 + 3x + 5 = 0$

Here, $a = 1, b = 3, c = 5$

$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 5$

$= 9 - 20 = -11$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-11}}{2 \times 1} = \frac{-3 \pm \sqrt{11}\,i}{2}.$$

6. $x^2 - x + 2 = 0$

Sol. $x^2 - x + 2 = 0$

Here, $a = 1, b = -1, c = 2$

$D = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm \sqrt{-7}}{2 \times 1} = \frac{1 \pm \sqrt{7}i}{2}.$$

7. $\sqrt{2}\,x^2 + x + \sqrt{2} = 0$

Sol. $\sqrt{2}\,x^2 + x + \sqrt{2} = 0$

Here $a = \sqrt{2}, b = 1, c = \sqrt{2}$

$D = b^2 - 4ac = 1 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}} = \frac{-1 \pm \sqrt{7}\,i}{2\sqrt{2}}.$$

8. $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Sol. The given equation is $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Here $a = \sqrt{3},\quad b = -\sqrt{2},\quad c = 3\sqrt{3}$

$$D = b^2 - 4ac = (-\sqrt{2})^2 - 4(\sqrt{3})(3\sqrt{3})$$
$$= 2 - 36 = -34 < 0$$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{\sqrt{2} \pm \sqrt{-34}}{2\sqrt{3}}$$
$$= \frac{\sqrt{2} \pm \sqrt{34}\,i}{2\sqrt{3}} = \frac{\sqrt{2}\,(1 \pm \sqrt{17}i)}{2\sqrt{3}} = \frac{1 \pm \sqrt{17}\,i}{\sqrt{6}}.$$

9. $x^2 + x + \frac{1}{\sqrt{2}} = 0$

Sol. The given equation is $x^2 + x + \frac{1}{\sqrt{2}} = 0$

Here $a = 1, b = 1, c = \frac{1}{\sqrt{2}}$

$$D = b^2 - 4ac = 1^2 - 4 \times 1 \times \frac{1}{\sqrt{2}} = 1 - 2\sqrt{2}$$
$$= -(2\sqrt{2} - 1) < 0$$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-(2\sqrt{2}-1)}}{2}$$
$$= \frac{-1 \pm \sqrt{2\sqrt{2}-1}\;i}{2}.$$

10. $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

Sol. The given equation is $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

Here $a = 1,\quad b = \frac{1}{\sqrt{2}},\quad c = 1$

$$D = b^2 - 4ac = \left(\frac{1}{\sqrt{2}}\right)^2 - 4 \times 1 \times 1 = \frac{1}{2} - 4 = -\frac{7}{2} < 0$$

$$\therefore \qquad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{-\frac{7}{2}}}{2 \times 1}$$

$$= \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\frac{7}{2}}\, i}{2 \times 1} = \frac{-1 \pm \sqrt{7}\, i}{2\sqrt{2}}.$$

MISCELLANEOUS EXERCISE ON CHAPTER 5

(Page No.: 112–113)

1. Evaluate: $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$.

Sol. $i^{18} = i^{16}\, i^2 = (i^4)^4\, i^2 = (1)^4(-1) = -1$ [$\because$ $i^4 = 1$ and $i^2 = -1$]

$$\left(\frac{1}{i}\right)^{25} = \frac{1}{i^{25}} = \frac{1}{i^{24} \cdot i} = \frac{1}{(i^4)^6 \cdot i}$$

$$= \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} \qquad [\because \ (i^4)^6 = 1^6 = 1]$$

$$= \frac{i}{i^2} = \frac{i}{-1} = -i = 0 - i \qquad [\because \ i^2 = -1]$$

$$\therefore \quad \left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3 = (-1 - i)^3 = [-(1 + i)]^3$$

[$\because$ $i^{18} = i^{16}.i^2 = (i^4)^4 i^2 = (1)^4 (-1) = -1$ and $\frac{1}{i} = -i$]

$$= -(1 + i)^3$$

$$= -[1^3 + i^3 + 3 \cdot 1 \cdot i\,(1 + i)]$$

$|\because (a + b)^3 = a^3 + b^3 + 3ab\,(a + b)$

$$= -(1 - i + 3i + 3i^2)$$

$$= -(1 + 2i - 3)$$

$$= -(-2 + 2i) = 2 - 2i.$$

2. For any two complex numbers z_1 and z_2, prove that

$\text{Re}(z_1 z_2) = \text{Re}(z_1)\ \text{Re}(z_2) - \text{Im}(z_1)\ \text{Im}(z_2)$

Sol. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $\text{Re}(z_1) = x_1, \text{Im}(z_1) = y_1; \text{Re}(z_2) = x_2, \text{Im}(z_2) = y_2$

Now $z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$

$\therefore \quad \text{Re}(z_1z_2) = x_1x_2 - y_1y_2$

$= \text{Re}(z_1)\,\text{Re}(z_2) - \text{Im}(z_1)\,\text{Im}(z_2).$

3. Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

Sol. $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$

$$= \left[\frac{(1+i)-2(1-4i)}{(1-4i)(1+i)}\right]\left(\frac{3-4i}{5+i}\right)$$

$$= \left(\frac{1+i-2+8i}{1+i-4i-4i^2}\right)\left(\frac{3-4i}{5+i}\right)$$

$$= \left(\frac{-1+9i}{1-3i+4}\right)\left(\frac{3-4i}{5+i}\right)$$

$$= \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)} = \frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}$$

$$= \frac{-3+31i+36}{25-10i+3} = \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i}$$

(Rationalising)

$$= \frac{924+868i+330i+310i^2}{(28)^2-(10i)^2}$$

$$= \frac{924+1198i-310}{784-100i^2}$$

$$= \frac{614+1198i}{784+100} = \frac{2(307+599i)}{884}$$

$$= \frac{307+599i}{442} = \frac{307}{442} + \frac{599i}{442}.$$

4. **If $x - iy = \sqrt{\dfrac{a-ib}{c-id}}$, prove that $(x^2 + y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$.**

Sol. Given, $x - iy = \sqrt{\dfrac{a-ib}{c-id}}$

Squaring both sides, we get

$$(x - iy)^2 = \frac{a-ib}{c-id}$$

Taking modulus on both sides,

$$\Rightarrow \quad |(x-iy)^2| = \left|\frac{a-ib}{c-id}\right|$$

$$\Rightarrow \quad |x - iy|^2 = \frac{|a-ib|}{|c-id|} \quad \left[\because \ |z^2| = |z|^2 \text{ and } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$$\Rightarrow \quad \left(\sqrt{x^2+y^2}\right)^2 = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} \quad \left[\because |x+iy| = \sqrt{x^2+y^2}\right]$$

$$\Rightarrow \quad x^2 + y^2 = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$$

Squaring both sides, we get

$$(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}.$$

Second Solution

Given, $x - iy = \sqrt{\dfrac{a-ib}{c-id}}$

Squaring both sides, we get

$$(x - iy)^2 = \frac{a-ib}{c-id} \qquad \text{... } (i)$$

Taking conjugates on both sides of (*i*),
(*i.e.* changing i to $-i$ in (*i*))

$$(x + iy)^2 = \frac{a+ib}{c+id} \qquad \text{... } (ii)$$

Multiplying Equations (*i*), and (*ii*), we have,

$$(x - iy)^2 (x + iy)^2 = \left(\frac{a-ib}{c-id}\right)\left(\frac{a+ib}{c+id}\right)$$

or $$(x - iy)(x + iy)^2 = \frac{a^2 - i^2b^2}{c^2 - i^2d^2} \quad |\because A^2B^2 = (AB)^2$$

or $$(x^2 - i^2y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$$

or $$(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$$

5. Convert the following in the polar form:

(*i*) $\dfrac{1+7i}{(2-i)^2}$ **(*ii*)** $\dfrac{1+3i}{1-2i}$.

Sol. (*i*) $$\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4+i^2-4i} = \frac{1+7i}{3-4i} \quad [\because \; i^2 = -1]$$

$$= \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i}$$

$$= \frac{3+4i+21i+28i^2}{9-16i^2} = \frac{3+25i-28}{9+16}$$

$$= \frac{-25+25i}{25} = -1 + i$$

Now reproduce solution of Q. 4 in Exercise 5.2 page 126.

(*ii*) $$\frac{1+3i}{1-2i} = \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{1+2i+3i+6i^2}{1-4i^2} = \frac{1+5i-6}{1+4}$$

$$= \frac{-5+5i}{5} = -1 + i$$

Now again reproduce solution of Q. 4 in Exercise 5.2 page 126.

Solve each of the equations in Exercises 6 to 9.

6. $3x^2 - 4x + \frac{20}{3} = 0.$

Sol. The given equation is $3x^2 - 4x + \frac{20}{3} = 0$

Here, $a = 3, \quad b = -4, \quad c = \frac{20}{3}$

$$D = b^2 - 4ac = (-4)^2 - 4 \times 3 \times \frac{20}{3}$$
$$= 16 - 80 = -64 < 0$$

$\therefore$ $$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{4 \pm \sqrt{-64}}{6}$$
$$= \frac{4 \pm \sqrt{64}\, i}{2 \times 3} = \frac{4 \pm 8i}{6} = \frac{2}{3}(1 \pm 2i).$$

7. $x^2 - 2x + \frac{3}{2} = 0$

Sol. $x^2 - 2x + \frac{3}{2} = 0$

Here, $a = 1, b = -2, c = \frac{3}{2}$

$$D = b^2 - 4ac = (-2)^2 - 4 \times 1 \times \frac{3}{2}$$
$$= 4 - 6 = -2$$

$\therefore$ $$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm \sqrt{-2}}{2 \times 1}$$
$$= \frac{2 \pm \sqrt{2}\, i}{2} = 1 \pm \frac{\sqrt{2}}{2} i.$$

8. $27x^2 - 10x + 1 = 0$

Sol. $27x^2 - 10x + 1 = 0$

Here, $a = 27, b = -10, c = 1$

$$D = b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1$$
$$= 100 - 108 = -8$$

$\therefore$ $$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{10 \pm \sqrt{-8}}{2 \times 27}$$
$$= \frac{10 \pm 2\sqrt{2}\, i}{54} = \frac{5}{27} \pm \frac{\sqrt{2}}{27} i.$$

9. $21x^2 - 28x + 10 = 0$

Sol. $21x^2 - 28x + 10 = 0$

Here, $a = 21, b = -28, c = 10$

$$D = b^2 - 4ac = (-28)^2 - 4 \times 21 \times 10$$
$$= 784 - 840 = -56$$

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{28 \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{4(14)(-1)}}{42}$$
$$= \frac{28 \pm 2\sqrt{14}\, i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42} i$$
$$= \frac{2}{3} \pm \frac{\sqrt{14}}{21} i.$$

10. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$.

Sol. $$\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\frac{(2-i)+(1+i)+1}{(2-i)-(1+i)+1}\right|$$
$$= \left|\frac{4}{2-2i}\right| = \left|\frac{2}{1-i}\right|$$
$$= \frac{|2|}{|1-i|} = \frac{|2+i0|}{|1-i|} \qquad \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$
$$= \frac{\sqrt{4+0}}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

11. If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$.

Sol. Given, $$a + ib = \frac{(x+i)^2}{2x^2+1}$$

Taking modulus on both sides,

$$\Rightarrow \quad |a + ib| = \left|\frac{(x+i)^2}{2x^2+1}\right|$$

$$\Rightarrow \quad \sqrt{a^2+b^2} = \frac{|(x+i)^2|}{|2x^2+1|} \qquad \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$$= \frac{|x+i|^2}{2x^2+1}$$

[$\because$ $|z^2| = |z|^2$ and $2x^2 + 1$ is a positive real number]

$$= \frac{\left(\sqrt{x^2+1}\right)^2}{2x^2+1}$$

$$\Rightarrow \quad \sqrt{a^2+b^2} = \frac{x^2+1}{2x^2+1}$$

Squaring both sides, we get

$$a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}.$$

Second Solution

Given, $\quad a + ib = \dfrac{(x+i)^2}{2x^2+1}$ ... (*i*)

Taking conjugates on both sides of (*i*) (*i.e.* changing i to $-i$ in (*i*)

$$a - ib = \frac{(x-i)^2}{2x^2+1} \quad \text{... (ii)}$$

Multiplying Eqns (*i*) and (*ii*), we have

$$(a + ib)\,(a - ib) = \frac{(x+i)^2}{2x^2+1}\,\frac{(x-i)^2}{2x^2+1}$$

$$\text{or } a^2 - i^2 b^2 = \frac{\left[(x+i)(x-i)^2\right]}{\left(2x^2+1\right)^2} \quad |\because A^2B^2 = (AB)^2$$

$$\text{or } a^2 + b^2 = \frac{\left(x^2-i^2\right)^2}{\left(2x^2+1\right)^2}$$

$$\text{or } a^2 + b^2 = \frac{\left(x^2+1\right)^2}{\left(2x^2+1\right)^2}$$

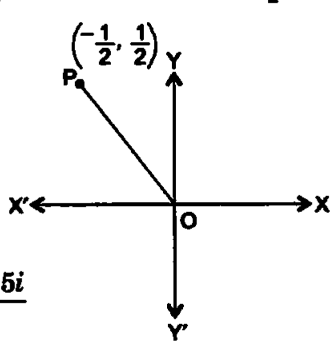
12. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find:

(*i*) $R_e\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$ (*ii*) $\text{Im}\left(\dfrac{1}{z_1\bar{z}_1}\right)$.

Sol. (*i*)
$$\frac{z_1 z_2}{\bar{z}_1} = \frac{(2-i)(-2+i)}{(2-i)}$$
$$= \frac{-4+2i+2i-i^2}{2+i} = \frac{-4+4i+1}{2+i}$$
$$= \frac{-3+4i}{2+i} \times \frac{2-i}{2-i} \qquad \text{(Rationalising)}$$
$$= \frac{-6+3i+8i-4i^2}{4-i^2}$$
$$= \frac{-6+11i+4}{4+1} = \frac{-2+11i}{5}$$
$$= -\frac{2}{5} + \frac{11}{5}i.$$
$$\therefore \quad R_e\left(\frac{z_1 z_2}{\bar{z}_1}\right) = R_e\left(-\frac{2}{5}+\frac{11}{5}i\right) = -\frac{2}{5}.$$

(*ii*)
$$\frac{1}{z_1 \bar{z}_1} = \frac{1}{(2-i)(2+i)} = \frac{1}{4-i^2}$$
$$= \frac{1}{4+1} = \frac{1}{5} = \frac{1}{5} + 0i$$
$$\therefore \quad \text{Im}\left(\frac{1}{z_1 \bar{z}_1}\right) = \text{Im}\left(\frac{1}{5}+0i\right) = 0.$$

13. Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$.



Sol.
$$\frac{1+2i}{1-3i} = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$
$$= \frac{1+3i+2i+6i^2}{1-9i^2}$$
$$= \frac{1+5i-6}{1+9} = \frac{-5+5i}{10}$$
$$= -\frac{1}{2} + \frac{1}{2}i. = x + iy$$

is represented by the point P (x, y) = P$\left(\frac{-1}{2}, \frac{1}{2}\right)$ which lies in second quadrant.

Let $z = \frac{1+2i}{1-3i} = -\frac{1}{2} + \frac{1}{2}i$

$= r(\cos\theta + i\sin\theta)$.

Then, $r\cos\theta = -\frac{1}{2} = x$ and $r\sin\theta = \frac{1}{2} = y$

Squaring and adding both, we get

$$r^2(\cos^2\theta + \sin^2\theta) = \frac{1}{4} + \frac{1}{4}$$

$$\Rightarrow \quad r^2 = \frac{1}{2} \qquad \therefore \quad r = \frac{1}{\sqrt{2}}$$

(or $r = |z| = \sqrt{x^2+y^2} = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$)

Because P $\left(\frac{-1}{2}, \frac{1}{2}\right)$ lies in second quadrant, therefore arg z is to be of the form $\pi - \theta$.

$$\tan\theta = \frac{y}{x} = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$$

$$\Rightarrow \quad \theta = \frac{3\pi}{4}$$

Hence, $|z| = r = \frac{1}{\sqrt{2}}$ and $\arg z = \theta = \frac{3\pi}{4}$.

14. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Sol. Given, $(x - iy)(3 + 5i) = \overline{-6-24i}$

$= -6 + 24i$

$$\Rightarrow \quad x - iy = \frac{-6+24i}{3+5i}$$

$$= \frac{-6+24i}{3+5i} \times \frac{3-5i}{3-5i} \quad \text{(Rationalising)}$$

$$= \frac{-18+30i+72i-120i^2}{9-25i^2}$$

$$= \frac{-18+102i+120}{9+25}$$

$$= \frac{102+102i}{34} = \frac{34(3+3i)}{34}$$

$$\Rightarrow \quad x - iy = 3 + 3i$$

Equating real and imaginary parts on both sides, we have

$$x = 3 \quad \text{and} \quad -y = 3$$

$$\Rightarrow \quad x = 3 \quad \text{and} \quad y = -3.$$

15. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

Sol. Let $z = \frac{1+i}{1-i} - \frac{1-i}{1+i}$

Taking L.C.M.,

$$= \frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)} = \frac{(1+i^2+2i)-(1+i^2-2i)}{1-i^2}$$

$$= \frac{1-1+2i-(1-1-2i)}{1+1}$$

$$= \frac{2i+2i}{2} = \frac{4i}{2} = 2i = 0 + 2i$$

$$\therefore \quad |z| = \sqrt{0+4} = 2.$$

16. If $(x + iy)^3 = u + iv$, then show that

$$\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2).$$

Sol. Given, $u + iv = (x + iy)^3$

$$= x^3 + 3x^2 \,.\, iy + 3x \,.\, (iy)^2 + (iy)^3$$

$$\left| \because \; (a+b)^3 = a^3 + b^3 + 3ab(a+b) = a^3 + 3a^2b + 3ab^2 + b^3 \right]$$

$= x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3$

$= x^3 + 3ix^2y - 3xy^2 - iy^3$

$[\because \ i^2 = -1, i^3 = -i]$

$\Rightarrow \quad u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3)$

Equating real and imaginary parts on both sides, we have

$\Rightarrow \quad u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

$\Rightarrow \quad u = x(x^2 - 3y^2)$ and $v = y(3x^2 - y^2)$

$\Rightarrow \quad \dfrac{u}{x} = x^2 - 3y^2$ and $\dfrac{v}{y} = 3x^2 - y^2$

Adding $\quad \dfrac{u}{x} + \dfrac{v}{y} = 4x^2 - 4y^2 = 4(x^2 - y^2).$

17. If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$.

Sol. $\quad |\beta| = 1 \quad$ (given) ...(*i*)

$\therefore \quad |\beta|^2 = 1$ or $\beta\bar{\beta} = 1 \quad$ ...(*ii*)

$[\because |z|^2 = z\bar{z}$ for every complex number z.]

$\therefore \quad \left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\dfrac{\beta-\alpha}{\beta\bar{\beta}-\bar{\alpha}\beta}\right| \quad$ [Putting $1 = \beta\bar{\beta}$ from (*ii*)]

$= \left|\dfrac{\beta-\alpha}{\beta(\bar{\beta}-\bar{\alpha})}\right| = \left|\dfrac{\beta-\alpha}{\beta(\overline{\beta-\alpha})}\right|$

$\left[\because \ \bar{z}_1 - \bar{z}_2 = \overline{z_1 - z_2}\right]$

$= \dfrac{|\beta-\alpha|}{|\beta|\,|(\overline{\beta-\alpha})|}$

$\left[\because \ \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \text{ and } |z_1z_2| = |z_1|\,|z_2|\right]$

$= \dfrac{|\beta-\alpha|}{|\beta||(\beta-\alpha)|} \quad [\because \ |\bar{z}| = |z|]$

$= \dfrac{1}{|\beta|} = \dfrac{1}{1} \quad$ [By (*i*)]

$= 1.$

18. Find the number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$.

Sol. Given, $|1 - i|^x = 2^x$ ...(*i*)

$$\Rightarrow \left(\sqrt{1^2 + (-1)^2}\right)^x = 2^x$$

$$\Rightarrow \quad (\sqrt{2})^x = 2 \qquad \Rightarrow \quad (2^{1/2})^x = 2^x$$

$$\Rightarrow \quad 2^{x/2} = 2^x \qquad \Rightarrow \quad \frac{x}{2} = x$$

$$\Rightarrow \quad 2x = x \qquad \Rightarrow \quad 2x - x = 0$$

$$\therefore \quad x = 0$$

$\Rightarrow$ The only solution of equation (*i*) is 0.

$\Rightarrow$ The given equation has no **non-zero** integral solution.

$\therefore$ The number of non-zero integral solutions is 0.

19. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that
$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$.

Sol. Given: $(a + ib)(c + id)(e + if)(g + ih) = A + iB$

Taking modulus on both sides,

$$\Rightarrow \quad |(a + ib)(c + id)(e + if)(g + ih)| = |A + iB|$$

$$\Rightarrow \quad |a + ib|\,|c + id|\,|e + if|\,|g + ih| = |A + iB|$$

$[\because \; |z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|]$

$$\Rightarrow \quad \sqrt{a^2 + b^2}\,.\sqrt{c^2 + d^2}\,.\sqrt{e^2 + f^2}\,.\sqrt{g^2 + h^2} = \sqrt{A^2 + B^2}$$

Squaring both sides, we get

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$.

Second Solution.

Given : $(a + ib)(c + id)(e + if)(g + ih) = A + iB$ ... (*i*)

Taking conjugates on both sides of (*i*),

$(a - ib)(c - id)(e - if)(g - ih) = A - iB$ ... (*ii*)

Multiplying Eqns (*i*) and (*ii*) we have,

$(a + ib)(a - ib)(c + id)(c - id)(e + if)(e - if)(g + ih)$
$(g - ih) = (A + iB)(A - iB)$

$\Rightarrow (a^2 - i^2 b^2)(c^2 - i^2 d^2)(e^2 - i^2 f^2)(g^2 - i^2 h^2)$
$= A^2 - i^2 B^2$

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$

20. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of *m*.

Sol. $\left(\frac{1+i}{1-i}\right)^m = 1 \Rightarrow \left(\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right)^m = 1$

$\Rightarrow \left(\frac{1+i^2+2i}{1-i^2}\right)^m = 1$

$\Rightarrow \left(\frac{1-1+2i}{1+1}\right)^m = 1$

$\Rightarrow \left(\frac{2i}{2}\right)^m = 1 \Rightarrow i^m = 1$

The least positive integral value of *m* satisfying this equation is 4 $(\because i^4 = 1)$

□□