7 Permutations and Combinations

Lesson at a Glance

1. **Fundamental principles of counting (F.P.C.)**

   (*i*) **Multiplication principle.** If an event can occur in m different ways and corresponding to each of these m ways, another event can occur in n different ways, then the total number of different ways of the occurrence of the two events (*i.e.*, 1st **and** 2nd) in succession is $m \times n$.

   (*ii*) **Addition principle.** If two events can occur independently in m and n ways respectively, then either of the two events (i.e., 1st **or** 2nd) can occur in $(m + n)$ ways.

2. $n\,! = 1 \times 2 \times 3 \times ... \times n$
   $= n(n-1)(n-2) ... 3 \times 2 \times 1.$

3. $0\,! = 1.$

4. $n\,!$ is defined only when n is whole number *i.e.*, $n = 0$ or n is a natural number.

5. $n\,! = n\,(n-1)\,! = n(n-1)\,(n-2)\,! = n(n-1)(n-2)\,(n-3)\,!$ *and so on.*

6. *If* $m, n \in$ N and $m > n$, then $m\,!$ can be expressed in terms of $n\,!$.

7. The number of **permutations (arrangements)** of n different objects taken r at a time is denoted by ${}^nP_r$ and is given by

   $${}^nP_r = \frac{n!}{(n-r)!}$$
   $$= n(n-1)(n-2) ... \text{ to } r \text{ factors where } 1 \le r \le n.$$

   ${}^nP_r$ is defined only if $n \in$ N, $r \in$ W and $r \le$ n.

8. (*a*) ${}^nP_n = n\,!$ (*b*) ${}^nP_{n-1} = n\,!$ (*c*) ${}^nP_0 = 1.$

9. The number of permutations of n objects, not all distinct, taken all at a time, when p objects are alike of one kind, q

are alike of a second kind, r are alike of a third kind and the rest, if any, are all different is $\dfrac{n!}{p!q!r!}$.

10. The number of permutations of n different objects taken r at a time, when each object may be repeated any number of times, is n^r.

11. The number of **combinations (selections)** of n different objects taken r at a time is denoted by ^nC_r and is given by

$$^nC_r = \frac{^nP_r}{r!} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{1.2.3\ldots r}.$$

^nC_r is defined only when $n \in N$, $r \in W$ and $r \leq n$.

12. $^nC_r = {}^nC_{n-r}$.

13. (*a*) $^nC_n = {}^nC_0 = 1$ (*b*) $^nC_1 = n$.

14. $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

15. If $^nC_p = {}^nC_q$ then either $p = q$ or $p + q = n$.

16. Total number of ways in which one or more things out of n different things can be selected is $2^n - 1$.

TEXTBOOK QUESTIONS SOLVED

EXERCISE 7.1 (*Page No.: 138*)

1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that

(*i*) repetition of the digits is allowed?

(*ii*) repetition of the digits is not allowed?

Sol. To form a 3-digit number, we have to arrange the given 5 digits in 3 places: the unit's place, the ten's place and the hundred's place.

(*i*) **Repetition of the digits is allowed**

Given digits are 1, 2, 3, 4 and 5 (all different and no zero).

Unit's place can be filled in 5 ways. Since repetition of digits is allowed, the ten's place can also be filled in 5 ways. For the same reason, hundred's place can be filled in 5 ways.

∴ By the multiplication principle, required number of 3-digit numbers is $5 \times 5 \times 5 = 125$.

(*ii*) **Repetition of digits is not allowed**

Unit's place can be filled in 5 different ways by anyone of the given 5 digits. The ten's place can be filled by anyone of the remaining 4 digits in 4 ways. The hundred's place can be filled by anyone of the remaining 3 digits in 3 ways. Thus, the number of ways in which the 3 places can be filled, by the multiplication principle, is $5 \times 4 \times 3 = 60$. Hence, the required number of 3-digit numbers is 60.

2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

Sol. To form a 3-digit even number, we have to arrange the given 6 digits in 3 places: the unit's place, the ten's place and the hundred's place. For even numbers, we can fill the unit's place out of the given digits with 2 or 4 or 6, *i.e.*, in 3 ways. Since the digits can be repeated, the ten's place can be filled by anyone of the 6 given digits in 6 ways. For the same reason, the hundred's place can be filled by anyone of the 6 given digits in 6 ways.

∴ By the multiplication principle, required number of 3-digit even numbers is $3 \times 6 \times 6 = 108$.

3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

Sol. There are 10 ways for choosing the first letter of the code, 9 ways for the second, 8 ways for the third and 7 ways for the fourth because no letter can be repeated (given). Using FPC (Multiplication Principle), there are $10 \times 9 \times 8 \times 7 = 5040$ required code words in all.

4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9, if each number starts with 67 and no digit appears more than once?

Sol. Out of the ten digits 0 to 9, two are fixed. Also no digit

6	7			

appears more than once. Therefore, the remaining $10 - 2 = 8$ digits can be arranged in the three places (= Total places, *i.e.*, 5 – 2 fixed places for 6 and 7) after 67 in 8, 7 and 6 ways.

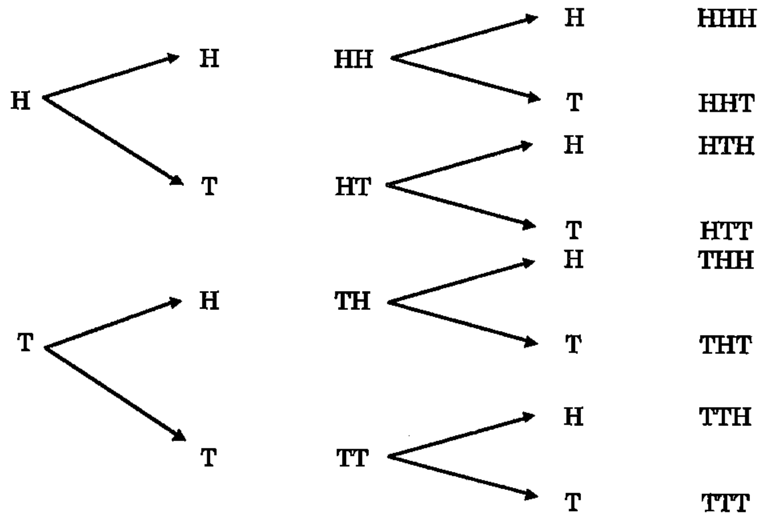
Using FPC (Multiplication Principle), total number of required 5-digit telephone numbers $= 8 \times 7 \times 6 = 336$.

5. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

Sol. On tossing a coin, it can fall either 'Head' or 'Tail' so that the number of possible outcomes on each toss is 2.

Let us denote the 'Head' by H and the 'Tail' by T. When the coin is tossed 3 times, we have the following Tree-diagram:

First Toss   Second Toss   Possible Outcomes   Third Toss   Possible Outcomes



By FPC (Multiplication Rule),

If the coin is tossed three times, then the number of possible outcomes

$$= 2 \times 2 \times 2 = 2^3 = 8.$$

Remark: The eight possible outcomes are:

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

(Here HHT is the outcome that the coin turns up 'Head' on the first toss, 'Head' on the second toss and 'Tail' on the third toss whereas HTH is the outcome that the coin turns up 'Head' on the first toss, 'Tail' on the second toss and 'Head' on the third toss. Clearly, HHT and HTH are two different outcomes.)

Note: If a coin is tossed n times, then the number of possible outcomes is z^n.

6. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

Sol. There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By multiplication rule, the number of ways is $5 \times 4 = 20$.

EXERCISE 7.2 (*Page No.: 140–141*)

1. Evaluate

(*i*) 8 ! **(*ii*) 4 ! – 3 !**

Sol. (*i*) $8\,! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320.$

(*ii*) $4\,! - 3\,! = (1 \times 2 \times 3 \times 4) - (1 \times 2 \times 3)$

$= 24 - 6 = 18.$

2. Is 3 ! + 4 ! = 7 !?

Sol. $3\,! + 4\,! = (1 \times 2 \times 3) + (1 \times 2 \times 3 \times 4) = 6 + 24 = 30$

$7\,! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

Since, $30 \neq 5040$ $\therefore$ $3\,! + 4\,! \neq 7\,!$.

3. Compute $\dfrac{8!}{6! \times 2!}$.

Sol. $$\frac{8!}{6! \times 2!} = \frac{8 \times 7 \times 6!}{6! \times 1 \times 2} = 4 \times 7 = 28.$$

4. If $\dfrac{1}{6!} + \dfrac{1}{7!} = \dfrac{x}{8!}$, find x.

Sol. Multiplying throughout by 8 !, (the largest number factorial in the denominator) we have

$$\frac{8!}{6!} + \frac{8!}{7!} = x$$

$$\Rightarrow \quad \frac{8 \times 7 \times 6!}{6!} + \frac{8 \times 7!}{7!} = x$$

$$\Rightarrow \quad 56 + 8 = x \qquad \therefore \quad x = 64.$$

5. Evaluate $\dfrac{n!}{(n-r)!}$, when

(*i*) $n = 6, r = 2$ **(*ii*) $n = 9, r = 5$.**

Sol. (*i*) When $n = 6$, $r = 2$, we have

$$\frac{n!}{(n-r)!} = \frac{6!}{(6-2)!} = \frac{6\times5\times4!}{4!} = 30$$

(*ii*) When $n = 9$, $r = 5$, we have

$$\frac{n!}{(n-r)!} = \frac{9!}{(9-5)!} = \frac{9!}{4!}$$

$$= \frac{9\times8\times7\times6\times5\times4!}{4!} = 15120.$$

EXERCISE 7.3 (*Page No.: 148*)

Note: In problems on permutations (⇒ arrangements), both number of things and their order is important.

So we must have to apply permutations formulae in the following types of problems:

(*i*) Words formed by letters (*ii*) Numbers formed by digits (iii) seating arrangements (*iv*) signals (*v*) Letters and Envelopes (*vi*) tossed (*vii*) thrown

Remark: All questions of Exercise 7.3 are questions on permutations.

1. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?

Sol. There will be as many 3-digit numbers as there are ways of filling 3 vacant places in succession by the 9 given digits.

H	T	U

This can be done in ${}^9P_3 = 9 \times 8 \times 7 = 504$ ways.

∴ The required number of 3-digit numbers = 504.

2. How many 4-digit numbers are there with no digit repeated?

Sol. We can use 10 digits 0 to 9. The number of ways of filling 4 vacant places in succession by the 10 given digits (including 0) is ${}^{10}P_4$. But these permutations will include those also where 0 is at the thousand's place. Such numbers are actually 3-digit numbers (For example 0387) and must be discarded. When 0 is fixed in the thousand's place, the remaining 9 digits can be arranged in the remaining 3 places in ${}^9P_3$ ways.

∴ The required number of 4-digit numbers

$= {}^{10}P_4 - {}^9P_3$

$= 10 \times 9 \times 8 \times 7 - 9 \times 8 \times 7$

$= 5040 - 504 = 4536.$

Th.	H	T	U

Second Solution

We know that there are 10 digits 0 to 9.

The thousand place can be filled in 9 ways (excluding 0 because otherwise number will be 3-digited only). Then hundred place can also be filled in 9 ways (including 0 and excluding the number placed in thousand place as repetition is not allowed).

Now ten's place can be filled by any one of the remaining 8 digits in 8 ways and then unit place in 7 ways.

∴ The required number of 4-digit numbers $= 9 \times 9 \times 8 \times 7$

$= 81 \times 8 \times 7 = 648 \times 7 = 4536$

(By F.P.C. (multiplication)

3. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?

Sol. For 3-digit even numbers, the unit's place can be occupied by one of the 3 digits 2, 4 or 6 (out of the given digits) The remaining 5 digits can be arranged in the remaining 2 places in ${}^5P_2$ ways.

H	T	U

∴ By the multiplication rule, the required number of 3-digit even numbers is $3 \times {}^5P_2 = 3 \times 5 \times 4 = 60$.

4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?

Sol. For 4-digit numbers, we have to arrange the given 5 digits in 4 vacant places. This can be done in ${}^5P_4 = 5 \times 4 \times 3 \times 2$ $= 120$ ways.

For 4-digit even numbers, the unit's place can be occupied by one of the 2 digits 2 or 4. The remaining 4 digits can be arranged in the remaining 3 places in ${}^4P_3$ ways.

Th.	H	T	U

∴ By the multiplication rule, the required number of 4-digit even numbers is $2 \times {}^4P_3 = 2 \times 4 \times 3 \times 2 = 48$.

5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person can not hold more than one position?

Sol. Out of 8 persons, a chairman can be chosen in 8 ways and then a vice chairman can be chosen in 7 ways.

∴ By the multiplication rule, the selection can be made in $8 \times 7 = 56$ ways. OR

we can arrange 8 persons in two vacant places in ${}^8P_2 = 8 \times 7 = 56$ ways.

Ch.	V Ch.

6. Find n if ${}^{n-1}P_3 : {}^nP_4 = 1 : 9$.

Sol. Given, $$\frac{{}^{n-1}P_3}{{}^nP_4} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-4)} = \frac{1}{9} \quad \Rightarrow \quad \frac{1}{n} = \frac{1}{9}$$

Cross-multiplying $n = 9$.

7. Find r if

(i) ${}^5P_r = 2\ {}^6P_{r-1}$ (ii) ${}^5P_r = {}^6P_{r-1}$.

Sol. (i) Given, ${}^5P_r = 2 \times {}^6P_{r-1}$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2 \times \frac{6!}{\{6-(r-1)\}!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2 \times \frac{6.5!}{(7-r)!}$$

$$\Rightarrow \quad \frac{1}{(5-r)!} = \frac{12}{(7-r)(6-r)(5-r)!} \quad (\because 7-r > 5-r)$$

$$\Rightarrow \quad 1 = \frac{12}{(7-r)(6-r)}$$

Cross-multiplying

$(7-r)(6-r) = 12 \Rightarrow 42 - 7r - 6r + r^2 = 12$

$\Rightarrow r^2 - 13r + 42 = 12$ or $r^2 - 13r + 30 = 0$

$\Rightarrow r^2 - 3r - 10r + 30 = 0$

$(r-3)(r-10) = 0$

$\therefore$ $r = 3, 10$

Now, ${}^5P_r$ and ${}^6P_{r-1}$ are meaningless when $r = 10$

($\because$ ${}^nP_r$ is defined only if $r \le n$)

$\therefore$ Rejecting $r = 10$, we have $r = 3$.

(*ii*) Given, ${}^5P_r = {}^6P_{r-1}$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{6!}{\{6-(r-1)\}!}$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{6.5!}{(7-r)!}$$

$$\Rightarrow \frac{1}{(5-r)!} = \frac{6}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow 1 = \frac{6}{(7-r)(6-r)} \Rightarrow (7-r)(6-r) = 6$$

$\Rightarrow 42 - 13r + r^2 = 6$ $\Rightarrow r^2 - 13r + 36 = 0$

$\Rightarrow (r-4)(r-9) = 0$ $\Rightarrow$ $r = 4, 9$

Now, we know that ${}^nP_r$ is meaningful only when $r \le n$.

$\therefore$ ${}^5P_r$ and ${}^6P_{r-1}$ are meaningless when $r = 9$.

$\therefore$ Rejecting $r = 9$, we have $r = 4$.

8. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

Sol. The word EQUATION has 8 distinct letters which can be arranged among themselves in ${}^8P_8$ ways.

$\therefore$ The required number of words formed

$= {}^8P_8 = 8! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$

$= 40320.$

9. How many words, with or without meaning, can be made from the letters of the word MONDAY, assuming that no letter is repeated, if

(i) 4 letters are used at a time

(ii) all letters are used at a time

(iii) all letters are used but first letter is a vowel?

Sol. The word MONDAY has 6 distinct letters.

(i) 4 letters out of 6 can be arranged in $^6P_4$ ways.

$\therefore$ The required number of words = $^6P_4$

$= 6 \times 5 \times 4 \times 3 = 360.$

(ii) 6 letters can be arranged among themselves in $^6P_6$ ways.

$\therefore$ The required number of words = $^6P_6 = 6\,!$

$= 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720.$

(iii) The first place can be filled by anyone of the two vowels O or A in 2 ways. The remaining 5 letters can be arranged in the remaining 5 places II to VI in $^5P_5$

= 5 ! ways.

I	II	III	IV	V	VI

$\therefore$ By the multiplication rule, the required number of words $= 2 \times 5\,! = 2 \times 1 \times 2 \times 3 \times 4 \times 5 = 240.$

10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?

Sol. The word MISSISSIPPI has 11 letters, not all distinct.

I occurs 4 times, S occurs 4 times, P occurs twice, M occurs once.

$\therefore$ Total number of permutations $= \dfrac{11!}{4!4!2!} \quad \Big| \quad \dfrac{n!}{p!q!r!}$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 4 \times 3 \times 2 \times 1 \times 2 \times 1}$$

$= 34650$

Now let us find the number of permutations in which the four I's come together. Treating the four I's as one letter (Units).

(I I I I) S S S S P P M

These are 1 + 4 + 2 + 1 = 8 letters (Units)

We have 8 letters (Units) which can be arranged in $\frac{8!}{4!2!}$ ways.

The four I's can be arranged among themselves in only one way $\left(\because \frac{4!}{4!}=1\right)$

∴ The number of permutations in which the four I's come together

$$= \frac{8!}{4!2!} = \frac{8\times7\times6\times5\times4!}{4!\times2\times1} = 840$$

Hence, the number of permutations in which the **four I's do not come together**

= Total number of arrangements

– Number of arrangements in which **four I's** are together.

= 34650 – 840 = 33810.

11. In how many ways can the letters of the word PERMUTATIONS be arranged if the

(i) words start with P and end with S,

(ii) vowels are all together,

(iii) there are always 4 letters between P and S?

Sol. The word PERMUTATIONS has 12 letters, not all distinct, T occurs twice.

(i) For words starting with P and ending with S, we have to arrange the remaining 10 letters with T occuring twice in the 10 vacant places between them. This can be done in $\frac{10!}{2!}$ ways.

P											S

∴ The required number of words = $\frac{10!}{2!}$

$$= \frac{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}{1\times2}$$

= 1814400.

(*ii*) The word PERMUTATIONS contains 5 distinct vowels E, U, A, I, O and 7 consonants in which T occurs twice. Since the vowels have to occur together, we assume the group of vowels (EUAIO) as a single object.

EUAIO	P	R	M	T	T	N	S

This single object together with 7 consonants become 8 objects, T occuring twice. These 8 objects can be arranged in $\frac{8!}{2!}$ ways. Corresponding to each of these arrangements, the 5 vowels, all distinct, can be arranged in $^5P_5 = 5!$ ways. Therefore, by multiplication principle, the required number of words

$$= \frac{8!}{2!} \times 5!$$

$$= \frac{(1\times2\times3\times4\times5\times6\times7\times8)(1\times2\times3\times4\times5)}{1\times2}$$

$$= 20160 \times 120 = 2419200.$$

(*iii*) Since there are always 4 letters between P and S, therefore, P and S can occupy 1st and 6th places

or 2nd and 7th places

or 3rd and 8th places

or 4th and 9th places or 5th and 10th places

or 6th and 11th places or 7th and 12th places.

Thus, P and S can occupy 2 places out of 12 in 7 different ways. Also P and S can interchange places in 2 ways. The remaining 10 places can be filled with the remaining 10 letters, T occuring twice, in $\frac{10!}{2!}$ ways.

$\therefore$ The required number of words $= 7 \times 2 \times \frac{10!}{2!}$

$= 7 \times (1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10)$

$= 7 \times 720 \times 56 \times 90 = 25401600.$

EXERCISE 7.4 *(Page No.: 153)*

Note: In problems on combinations (⇒ selections or groups) only number of things is important but not their order.

So, we must have to apply combinations formulae in the following types of problems:

(*i*) Straight lines and triangles formed by a given number of points

(*ii*) Invitations (*iii*) Selections (*iv*) Groups (*v*) Committees

(*vi*) Drawn.

1. If $^nC_8 = {}^nC_2$, find $^nC_2$.

Sol. $^nC_8 = {}^nC_2$

⇒ Either 8 = 2 which is false

$[\because \; ^nC_p = {}^nC_q \Rightarrow p = q \text{ or } p + q = n]$

or $8 + 2 = n \quad \therefore n = 10$

$\therefore \quad {}^nC_2 = {}^{10}C_2 = \dfrac{10 \times 9}{2 \times 1} = 45.$

2. Determine n if

(*i*) $^{2n}C_3 : {}^nC_3 = 12 : 1$ (*ii*) $^{2n}C_3 : {}^nC_3 = 11 : 1$.

Sol. (*i*) $^{2n}C_3 : {}^nC_3 = 12 : 1$

$$\Rightarrow \frac{2n(2n-1)(2n-2)}{3 \times 2 \times 1} : \frac{n(n-1)(n-2)}{3 \times 2 \times 1} = 12 : 1$$

$$\Rightarrow \frac{4n(2n-1)(n-1)}{n(n-1)(n-2)} = \frac{12}{1}$$

$$\Rightarrow \frac{4(2n-1)}{n-2} = 12$$

$$\Rightarrow \frac{2n-1}{n-2} = 3 \Rightarrow 2n - 1 = 3(n-2)$$

$$\Rightarrow 2n - 1 = 3n - 6$$

$$\therefore n = 5.$$

(ii) $^{2n}C_3 : {}^nC_3 = 11 : 1$

$$\Rightarrow \quad \frac{2n(2n-1)(2n-2)}{3\times2\times1} : \frac{n(n-1)(n-2)}{3\times2\times1} = 11:1$$

Taking 2 common from the factor $(2n - 2)$,

$$\Rightarrow \quad \frac{4n(2n-1)(n-1)}{n(n-1)(n-2)} = 11$$

$$\Rightarrow \quad \frac{4(2n-1)}{n-2} = 11 \quad \Rightarrow \quad 4(2n-1) = 11(n-2)$$

$$\Rightarrow \quad 8n - 4 = 11n - 22$$

$$\Rightarrow \quad -3n = -18 \quad \Rightarrow \quad 3n = 18$$

$$\therefore \quad n = 6$$

3. How many chords can be drawn through 21 points on a circle?

Sol. We know that any three points on a circle are non-collinear.

Now, a chord can be drawn by joining any two of the 21 given points.

$\therefore$ The required number of chords $= {}^{21}C_2$

[$\because$ Every two points can be joined to form a straight line]

$$= \frac{21\times20}{2\times1} = 210.$$

4. In how many ways can a team of 3 boys, and 3 girls be selected from 5 boys and 4 girls?

Sol. 3 boys can be selected out of 5 boys in $^5C_3$ ways and 3 girls can be selected out of 4 girls in $^4C_3$ ways.

$\therefore$ By multiplication principle, the required number of ways of selecting a team $= {}^5C_3 \times {}^4C_3$

$= {}^5C_2 \times {}^4C_1 \qquad [\because {}^nC_r = {}^nC_{n-r}]$

$$= \frac{5\times4}{2\times1} \times 4 = 40.$$

5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

Sol. 3 red balls out of 6 can be selected in $^6C_3$ ways.

3 white balls out of 5 can be selected in $^5C_3$ ways.

3 blue balls out of 5 can be selected in $^5C_3$ ways.

∴ By multiplication principle, 9 balls can be selected in
$^6C_3 \times {}^5C_3 \times {}^5C_3 = {}^6C_3 \times {}^5C_2 \times {}^5C_2$ $[\because \ {}^nC_r = {}^nC_{n-r}]$

$$= \frac{6\times5\times4}{3\times2\times1} \times \frac{5\times4}{2\times1} \times \frac{5\times4}{2\times1}$$

$= 20 \times 10 \times 10 = 2000$ ways.

6. Determine the number of 5-card combinations out of a deck of 52 cards if there is exactly one ace in each combination.

Sol. We know that in a deck of 52 cards, 4 are aces and 48 other cards. One ace can be selected out of 4 in $^4C_1$ ways and $5 - 1 = 4$ other cards out of 48 in $^{48}C_4$ ways.

∴ By multiplication principle, one ace and 4 other cards can be selected in $^4C_1 \times {}^{48}C_4$

$$= 4 \times \frac{48\times47\times46\times45}{4\times3\times2\times1} = 778320 \text{ ways.}$$

7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

Sol. Out of 17 players 5 are bowlers, therefore, there are $17 - 5 = 12$ other players. 4 bowlers out of 5 can be selected in $^5C_4$ ways and $11 - 4 = 7$ others players can be selected out of 12 other players in $^{12}C_7$ ways.

∴ By multiplication principle, a cricket team of 11 can be selected in $^5C_4 \times {}^{12}C_7$

$= {}^5C_1 \times {}^{12}C_5$ $[\because \ {}^nC_r = {}^nC_{n-r}]$

$$= 5 \times \frac{12\times11\times10\times9\times8}{5\times4\times3\times2\times1} = 3960.$$

8. A bag contains 5 black 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

Sol. 2 black balls out of 5 can be selected in $^5C_2$ ways and 3 red balls out of 6 can be selected in $^6C_3$ ways.

∴ By multiplication principle, the required number of selections is $^5C_2 \times {}^6C_3$

$$= \frac{5\times4}{2\times1} \times \frac{6\times5\times4}{3\times2\times1} = 200.$$

9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

Sol. Since 2 courses are compulsory, a student has to choose 3 (= 5 – 2) more courses out of the remaining 7 (= 9 – 2) courses. This he can do in $^7C_3 = \frac{7\times6\times5}{3\times2\times1}$ = 35 ways.

MISCELLANEOUS EXERCISE ON CHAPTER 7

(Page No.: 156–157)

1. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER?

Sol. In the word DAUGHTER, there are 3 vowels, namely, A, U, E and 5 consonants, namely, D, G, H, T, R.

2 vowels out of 3 can be selected in $^3C_2 = {^3C_1} = 3$ ways.

3 consonants out of 5 can be selected in $^5C_3 = {^5C_2} = \frac{5\times4}{2\times1}$ = 10 ways.

∴ The number of selections of 2 vowels and 3 consonants is 3 × 10 = 30.

Now, each of these 30 selections has 5 letters which can be arranged among themselves in $^5P_5 = 5! = 1 \times 2 \times 3 \times 4 \times 5$ = 120 ways.

∴ The required number of different words is

$$30 \times 120 = 3600.$$

Note: In the above question we first formed combinations (selections) and then permutations (arrangements) of those selections.

2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonants occur together?

Sol. The word EQUATION has 8 distinct letters in which there are 5 vowels, namely E, U, A, I, O and 3 consonants, namely Q, T and N. Since the vowels and consonants occur together, we assume the 5 vowels as one object (EUAIO) and the 3 consonants as another object (QTN). These two objects can be arranged among themselves in 2 ! = 2 ways. In each of these 2 arrangements, the 5 vowels can interchange places in 5 ! ways and the 3 consonants can interchange places in 3 ! ways.

∴ By multiplication principle, the required number of words is 2 ! × 5 ! × 3 ! = 2 × 120 × 6 = 1440.

3. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:

(*i*) exactly 3 girls? (*ii*) at least 3 girls?

(*iii*) at most 3 girls?

Sol. (*i*) 3 girls can be selected out of 4 in $^4C_3$ ways and 7 – 3 = 4 boys can be selected out of 9 in $^9C_4$ ways. Therefore, the required number of ways is

$^4C_3 \times {}^9C_4 = {}^4C_1 \times {}^9C_4$ $\quad [\because \; {}^nC_r = {}^nC_{n-r}]$

$$= 4 \times \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 504.$$

(*ii*) Since at least 3 girls (*i.e.*, 3 or more than 3) are to be there in every committee, therefore, the committee can consist of

(*a*) 3 girls and 4 boys (*b*) 4 girls and 3 boys

3 girls and 4 boys can be selected in $^4C_3 \times {}^9C_4$ ways.

4 girls and 3 boys can be selected in $^4C_4 \times {}^9C_3$ ways.

∴ The required number of ways

$$= {}^4C_3 \times {}^9C_4 + {}^4C_4 \times {}^9C_3$$

$$= {}^4C_1 \times {}^9C_4 + {}^4C_0 \times {}^9C_3$$

$$= 4 \times \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} + 1 \times \frac{9 \times 8 \times 7}{3 \times 2 \times 1}$$

$$= 504 + 84 = 588.$$

(*iii*) Since at most 3 girls (*i.e.*, 3 or less than 3) are to be there in every committee, therefore, the committee can consist of

(*a*) 3 girls and 4 boys
(*b*) 2 girls and 5 boys
(*c*) 1 girl and 6 boys
(*d*) no girl and 7 boys

3 girls and 4 boys can be selected in ${}^4C_3 \times {}^9C_4$ ways.

2 girls and 5 boys can be selected in ${}^4C_2 \times {}^9C_5$ ways.

1 girl and 6 boys can be selected in ${}^4C_1 \times {}^9C_6$ ways.

No girl and 7 boys can be selected in ${}^4C_0 \times {}^9C_7$ ways.

∴ The required number of ways

$$= {}^4C_3 \times {}^9C_4 + {}^4C_2 \times {}^9C_5 + {}^4C_1 \times {}^9C_6 + {}^4C_0 \times {}^9C_7$$

$$= {}^4C_1 \times {}^9C_4 + {}^4C_2 \times {}^9C_4 + {}^4C_1 \times {}^9C_3 + {}^4C_0 \times {}^9C_2$$

$$[\because \ {}^nC_r = {}^nC_{n-r}]$$

$$= 4 \times \frac{9\times8\times7\times6}{4\times3\times2\times1} + \frac{4\times3}{2\times1} \times \frac{9\times8\times7\times6}{4\times3\times2\times1}$$

$$+ 4 \times \frac{9\times8\times7}{3\times2\times1} + 1 \times \frac{9\times8}{2\times1}$$

$$= 504 + 756 + 336 + 36 = 1632.$$

4. If the different permutations of all the letters of the word EXAMINATION are listed as in a dictionary, how many words are there in this list before the first word starting with E?

Sol. Writing the letters of the word EXAMINATION in alphabetic order, we have AAEIIMNNOTX

Total number of letters in this word is 11.

∴ The required number of words before the first word starting with E is equal to the number of words which begin with A. Because only A appears before E in the above alphabetical order. When A is fixed in the first place, we have to arrange the remaining 10 letters in which there are two I's and two N's.

[A] × × × × × × × × × ×

∴ The required number of words

$$= \frac{10!}{2!2!}$$

$$= \frac{10\times9\times8\times7\times6\times5\times4\times3\times2!}{2\times1\times2!}$$

$$= 907200.$$

5. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9 which are divisible by 10 and no digit is repeated?

Sol. Numbers divisible by 10 must have '0' in the unit's place.

$$\times \times \times \times \times 0$$

The remaining 5 digits can be arranged in the remaining 5 vacant places in $^5P_5 = 5\,!$ ways.

$\therefore$ The required number of 6-digit numbers = 5 ! = 120.

6. The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet?

Sol. 2 vowels out of 5 can be selected in

$$^5C_2 = \frac{5\times4}{2\times1} = 10 \text{ ways.}$$

2 consonants out of 21 can be selected in

$$^{21}C_2 = \frac{21\times20}{2\times1} = 210 \text{ ways.}$$

$\therefore$ The number of selections of 2 vowels and 2 consonants is 10 × 210 = 2100.

Now, each of these 2100 selections has 4 letters which can be arranged among themselves in

$$^4P_4 = 4\,! = 1 \times 2 \times 3 \times 4 = 24 \text{ ways.}$$

Therefore, the required number of different words = 2100 × 24 = 50400.

Note: See note at the end of solution of Q.N.1 of this exercise (page 205).

7. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions?

Sol. The various possibilities of selecting 8 questions are:

	Part I	*Part II*
	5	7
(i)	3	5
(ii)	4	4
(iii)	5	3

($\because$ A student has to select at least 3 questions from each part.)

The required number of ways

$$= {}^5C_3 \times {}^7C_5 + {}^5C_4 \times {}^7C_4 + {}^5C_5 \times {}^7C_3$$
$$= {}^5C_2 \times {}^7C_2 + {}^5C_1 \times {}^7C_3 + {}^5C_0 \times {}^7C_3 \quad \left| \because \; {}^nC_r = {}^nC_{n-r} \right.$$
$$= \frac{5\times4}{2\times1}\times\frac{7\times6}{2\times1} + 5\times\frac{7\times6\times5}{3\times2\times1} + 1\times\frac{7\times6\times5}{3\times2\times1}$$
$$= 10 \times 21 + 5 \times 35 + 1 \times 35$$
$$= 210 + 175 + 35 = 420.$$

8. Determine the number of 5-card combinations out of deck of 52 cards if each selection of 5 cards has exactly one king.

Sol. In a deck of 52 cards, there are 4 kings and 48 other cards. One king can be selected out of 4 in ${}^4C_1$ ways and $5 - 1 = 4$ other cards out of 48 in ${}^{48}C_4$ ways.

$\therefore$ By multiplication principle, one king and 4 other cards can be selected in ${}^4C_1 \times {}^{48}C_4 = 4 \times 194580 = 778320$ ways.

9. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Sol. We know that in the row of 9 places, the second, fourth, sixth and the eighth places are the even places. Four women can be arranged in four even places in ${}^4P_4$ ways. Five men can be arranged in the remaining five odd places in ${}^5P_5$ ways. By the Fundamental Principle of Counting (Multiplication), the required number of seating arrangements is

$${}^4P_4 \times {}^5P_5 = 4! \times 5! = (4 \times 3 \times 2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)$$
$$= 24 \times 120 = 2880.$$

10. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will

join. In how many ways can the excursion party be chosen?

Sol. According to given, the only two possibilities are:

(*i*) the particular 3 students join

(*ii*) the particular 3 students do not join.

In the first case of inclusion, we have to choose 7 (= 10 – 3) more students out of the remaining 22 (= 25 – 3) students. This can be done in $^{22}C_7$ ways.

In the second case of exclusion, we have to choose all 10 students out of the remaining 22 (= 25 – 3) students. This can be done in $^{22}C_{10}$ ways.

∴ The required number of ways = $^{22}C_7 + {}^{22}C_{10}$ [Addition Principle]

$$= \frac{22\times21\times20\times19\times18\times17\times16}{7\times6\times5\times4\times3\times2\times1} + \frac{22\times21\times20\times19\times18\times17\times16\times15\times14\times13}{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}$$

$$= 170544 + 646646 = 817190.$$

11. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together?

Sol. The word ASSASSINATION has 13 letters, of which A appears 3 times, S appears 4 times, I appears 2 times, N appears 2 times and the rest are all different. Since all the S's are to occur together, we take them as a single object (SSSS). This single object together with 9 remaining letters become 10 objects (SSSS) AAA II NN TO which can be arranged in $\frac{10!}{3!2!2!}$ ways.

The four S's can be arranged among themselves in $\frac{4!}{4!} = 1$ way

∴ The required number of ways = $\frac{10!}{3!2!2!} \times 1$

$$= \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{(3\times2\times1)\times(2\times1)\times(2\times1)}$$

= 151200.