11 Conic Sections

Lesson at a Glance

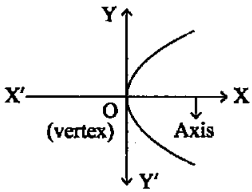
1. **Circle.** A circle is the locus (path) of a point which moves such that its distance from a given point (called **centre** of the circle) is constant (called **radius** of the circle).
2. Equation of the circle having centre (h, k) and radius r is
$$(x - h)^2 + (y - k)^2 = r^2.$$
3. Equation of the circle having centre (0, 0) and radius r is
$$x^2 + y^2 = r^2.$$
4. **Concentric circles.** Circles having the same centre are called concentric circles.
5. **Area of circle** = πr^2, where r is the radius of the circle.
6. General form of the equation of the circle is
$x^2 + y^2 + 2gx + 2fy + c = 0,$
centre of this circle is $(-g, -f)$
i.e., $\left(-\frac{1}{2}\text{ coefficient of } x, -\frac{1}{2}\text{ coefficient of } y\right)$
Radius of this circle is $\sqrt{g^2 + f^2 - c}$
i.e., $\sqrt{\text{sum of squares of co-ordinates of centre of circle} - \text{constant term}}$.
7. **Parabola.** A parabola is the locus (path) of a point which moves such that its distance from a given point (called **focus** of the parabola) is equal to its perpendicular distance from a given straight line not passing the focus (called **directrix** of the parabola).
8. **Summary of the main facts about four parabolas in standard form:**

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$

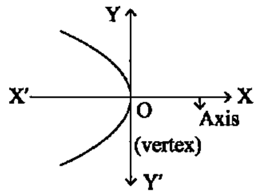
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$

Shapes(curves) of the four forms of the parabola:

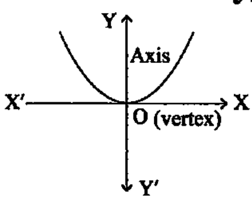
Form-I (Rightward Parabola $y^2 = 4ax$)



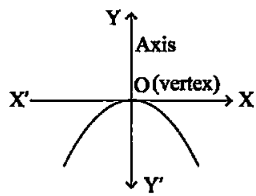
Form-II (Leftward Parabola $y^2 = -4ax$)



Form-III (Upward Parabola $x^2 = 4ay$)



Form-IV (Downward Parabola $x^2 = -4ay$)



9. Ellipse

An ellipse is the locus of a point P (say) which moves such that the sum of its distances from two given points A and B (say) (called **foci** of the ellipse) is constant say K (called length of **major axis** of the ellipse) *i.e.,* PA + PB = K.

10. Hyperbola

A hyperbola is the locus of a point P(say) which moves such that the difference of its distances from two given points A and B (say) (called **foci** of the hyperbola) is constant say K (called length of **transverse axis** of the hyperbola) *i.e.,* $|PA - PB| = K$.

11. Summary of the main facts about two ellipses in standard form:

Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ $(a > b)$
Co-ordinates of centre	(0, 0)	(0, 0)

Co-ordinates of foci	$(\pm c, 0)$	$(0, \pm c)$
Co-ordinates of vertices	$(\pm a, 0)$	$(0, \pm a)$
Length of major axis	$2a$	$2a$
Equation of major axis	$y = 0$	$x = 0$
Length of minor axis	$2b$	$2b$
Equation of minor axis	$x = 0$	$y = 0$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equations of latus rectum	$x = \pm c$	$y = \pm c$
Eccentricity	$\frac{c}{a}$	$\frac{c}{a}$

12. $c^2 = a^2 - b^2$ for each of the two forms of the ellipse.

13. Summary of the main facts about two hyperbolas in standard form:

Form	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Co-ordinates of centre	$(0, 0)$	$(0, 0)$
Co-ordinates of foci	$(\pm c, 0)$	$(0, \pm c)$
Co-ordinates of vertices	$(\pm a, 0)$	$(0, \pm a)$
Length of transverse axis	$2a$	$2a$
Equation of transverse axis	$y = 0$	$x = 0$
Length of conjugate axis	$2b$	$2b$
Equation of conjugate axis	$x = 0$	$y = 0$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equations of latus rectum	$x = \pm c$	$y = \pm c$
Eccentricity	$\frac{c}{a}$	$\frac{c}{a}$

14. $c^2 = a^2 + b^2$ for each of the two forms of the hyperbola.

EXERCISE 11.1 *(Page No.: 241)*

In each of the following Exercises 1 to 5, find the equation of the circle with

1. Centre (0, 2) and radius 2.

Sol. Here $h = 0$, $k = 2$ and $r = 2$.

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

i.e., $(x - 0)^2 + (y - 2)^2 = 2^2$

or $x^2 + y^2 - 4y + 4 = 4$

or $x^2 + y^2 - 4y = 0.$

2. Centre (– 2, 3) and radius 4.

Sol. Here $h = -2$, $k = 3$ and $r = 4$.

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

i.e., $(x + 2)^2 + (y - 3)^2 = 4^2$

or $x^2 + 4x + 4 + y^2 - 6y + 9 = 16$

or $x^2 + y^2 + 4x - 6y - 3 = 0.$

3. Centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$.

Sol. We know that the equation of the circle with centre at (h, k) and radius r is given by

$$(x - h)^2 + (y - k)^2 = r^2 \quad ...(i)$$

Here $(h, k) = \left(\frac{1}{2}, \frac{1}{4}\right)$ and $r = \frac{1}{12}$

Putting values of h, k and r in (*i*),

∴ Equation of the required circle is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{4}\right)^2 = \left(\frac{1}{12}\right)^2$$

or $x^2 + \frac{1}{4} - x + y^2 + \frac{1}{16} - \frac{1}{2}y = \frac{1}{144}$

or $x^2 + y^2 - x - \frac{1}{2}y + \frac{11}{36} = 0$

$$\left[\because \frac{1}{4} + \frac{1}{16} - \frac{1}{144} = \frac{36 + 9 - 1}{144} = \frac{11}{36}\right]$$

Multiplying by 36, we have

$36x^2 + 36y^2 - 36x - 18y + 11 = 0.$

4. Centre (1, 1) and radius $\sqrt{2}$.

Sol. Here $h = 1$, $k = 1$ and $r = \sqrt{2}$.

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

i.e., $(x - 1)^2 + (y - 1)^2 = (\sqrt{2})^2$

or $x^2 - 2x + 1 + y^2 - 2y + 1 = 2$

or $x^2 + y^2 - 2x - 2y = 0.$

5. Centre (– a, – b) and radius $\sqrt{a^2 - b^2}$.

Sol. Here $h = -a$, $k = -b$ and $r = \sqrt{a^2 - b^2}$.

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

i.e., $(x + a)^2 + (y + b)^2 = a^2 - b^2$

or $x^2 + 2ax + a^2 + y^2 + 2by + b^2 = a^2 - b^2$

or $x^2 + y^2 + 2ax + 2by + 2b^2 = 0.$

In each of the following Exercises 6 to 9, find the centre and radius of the circles.

6. $(x + 5)^2 + (y - 3)^2 = 36$.

Sol. The given equation is $[x - (-5)]^2 + (y - 3)^2 = 6^2$.

Comparing it with $(x - h)^2 + (y - k)^2 = r^2$, we have

$h = -5$, $k = 3$ and $r = 6$.

∴ The given circle has centre at $(h, k) = (-5, 3)$ and radius $r = 6$.

7. $x^2 + y^2 - 4x - 8y - 45 = 0$.

Sol. The given equation is

$(x^2 - 4x) + (y^2 - 8y) = 45.$

Completing the squares within the parenthesis, we get

[Adding $(\frac{1}{2}\text{coeff. of } x)^2$ and $(\frac{1}{2}\text{coeff. of } y)^2$ to both sides]

$(x^2 - 4x + 2^2) + (y^2 - 8y + 4^2) = 45 + 4 + 16$

$\Rightarrow$ $(x - 2)^2 + (y - 4)^2 = 65$

Comparing it with $(x - h)^2 + (y - k)^2 = r^2$, we have

$h = 2$, $k = 4$ and $r = \sqrt{65}$.

∴ The given circle has centre at $(h, k) = (2, 4)$ and radius $r = \sqrt{65}$.

8. $x^2 + y^2 - 8x + 10y - 12 = 0.$

Sol. The given equation is

$$(x^2 - 8x) + (y^2 + 10y) = 12.$$

Completing the squares within the parenthesis, we get

[Adding $(\frac{1}{2}\text{coeff. of } x)^2$ and $(\frac{1}{2}\text{coeff. of } y)^2$ to both sides]

$$(x^2 - 8x + 4^2) + (y^2 + 10y + 5^2) = 12 + 16 + 25$$

$\Rightarrow$ $$(x - 4)^2 + (y + 5)^2 = 53$$

$\Rightarrow$ $$(x - 4)^2 + [y - (-5)]^2 = (\sqrt{53})^2$$

Comparing it with $(x - h)^2 + (y - k)^2 = r^2$, we have

$$h = 4,\ k = -5 \text{ and } r = \sqrt{53}.$$

$\therefore$ The given circle has centre at $(h, k) = (4, -5)$ and radius $r = \sqrt{53}$.

9. $2x^2 + 2y^2 - x = 0.$

Sol. The given equation is

$$2x^2 + 2y^2 - x = 0$$

Dividing every term by 2, to make coefficient of x^2 and y^2 unity,

or $$x^2 + y^2 - \frac{1}{2}x = 0$$

or $$\left(x^2 - \frac{1}{2}x\right) + y^2 = 0$$

Completing the square within parenthesis,

Adding $\left(\frac{1}{2}\text{coeff. of } x\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$ to both sides,

we have $$\left(x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2\right) + y^2 = \frac{1}{16}$$

or $$\left(x - \frac{1}{4}\right)^2 + y^2 = \left(\frac{1}{4}\right)^2$$

Comparing it with $(x - h)^2 + (y - k)^2 = r^2$, we have

$$h = \frac{1}{4},\ k = 0 \text{ and } r = \frac{1}{4}$$

$\therefore$ The centre of the given circle is $(h, k) = \left(\frac{1}{4}, 0\right)$ and radius is $r = \frac{1}{4}$.

10. Find the equation of the circle passing through the points (4, 1) and (6, 5) and whose centre is on the line $4x + y = 16$.

Sol. Let the equation of the circle be

$$(x - h)^2 + (y - k)^2 = r^2 \quad ...(i)$$

Since the points (4, 1) and (6, 5) lie on it, we have

$$(4 - h)^2 + (1 - k)^2 = r^2 \quad ...(ii)$$

and $$(6 - h)^2 + (5 - k)^2 = r^2 \quad ...(iii)$$

From (*ii*) and (*iii*) equating the two values of r^2, we have

$$(4 - h)^2 + (1 - k)^2 = (6 - h)^2 + (5 - k)^2$$

or $$(16 - 8h + h^2) + (1 - 2k + k^2) = (36 - 12h + h^2) + (25 - 10k + k^2)$$

or $-8h - 2k + 17 = -12h - 10k + 61 \Rightarrow 4h + 8k = 44$

Dividing by 4, $$h + 2k = 11 \quad ...(iv)$$

Since the centre (h, k) lies on the line $4x + y = 16$, we have $$4h + k = 16 \quad ...(v)$$

Let us solve eqns. (*iv*) and (*v*) for h and k.

Eqn. (*v*) $-$ 4 $\times$ eqn. (*iv*) gives $k - 8k = 16 - 44$ or $-7k = -28$ or $k = 4$

Putting $k = 4$ in eqn. (*iv*), $h + 8 = 11$ or $h = 3$

Putting these values of h and k in (*ii*), we have

$$r^2 = (4 - 3)^2 + (1 - 4)^2 = 1 + 9 = 10$$

Putting these values of h, k and r^2 in (*i*), the equation of required circle is

$$(x - 3)^2 + (y - 4)^2 = 10$$

or $$x^2 + 9 - 6x + y^2 + 16 - 8y = 10$$

or $$x^2 + y^2 - 6x - 8y + 15 = 0.$$

11. Find the equation of the circle passing through the points (2, 3) and (– 1, 1) and whose centre is on the line $x - 3y - 11 = 0$.

Sol. Let the equation of the circle be

$$(x - h)^2 + (y - k)^2 = r^2 \quad ...(i)$$

Since the points (2, 3) and (– 1, 1) lie on it, we have

$$(2 - h)^2 + (3 - k)^2 = r^2 \quad ...(ii)$$

and $$(-1 - h)^2 + (1 - k)^2 = r^2 \quad ...(iii)$$

From (*ii*) and (*iii*), equating the two values of r^2, we have

$$(2 - h)^2 + (3 - k)^2 = (-1 - h)^2 + (1 - k)^2$$

or $$(4 - 4h + h^2) + (9 - 6k + k^2) = (1 + 2h + h^2) + (1 - 2k + k^2)$$

or $$13 - 4h - 6k = 2 + 2h - 2k$$

or $$6h + 4k = 11 \quad ...(iv)$$

Given: The centre (h, k) lies on the line $x - 3y - 11 = 0$

$$\therefore \quad h - 3k = 11 \quad ...(v)$$

Eqn. (*iv*) – 6 × eqn. (*v*) gives $4k + 18k = 11 - 66$

$$\Rightarrow \quad 22k = -55 \quad \Rightarrow k = \frac{-55}{22} = -\frac{5}{2}$$

Putting $k = -\frac{5}{2}$ in (*v*), we have

$$h - 3\left(-\frac{5}{2}\right) = 11 \quad \text{or} \quad h = 11 - \frac{15}{2} = \frac{7}{2}$$

Putting the values of h and k in (*ii*), we get

$$\left(2 - \frac{7}{2}\right)^2 + \left(3 + \frac{5}{2}\right)^2 = r^2$$

$$\Rightarrow \left(\frac{4-7}{2}\right)^2 + \left(\frac{6+5}{2}\right)^2 = r^2 \quad \Rightarrow r^2 = \frac{9}{4} + \frac{121}{4} = \frac{130}{4}$$

Putting values of h, k and r^2 in (*i*), required equation of circle is $\left(x - \frac{7}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{130}{4}$

or $$\left(x^2 - 7x + \frac{49}{4}\right) + \left(y^2 + 5y + \frac{25}{4}\right) = \frac{130}{4}$$

or $$x^2 + y^2 - 7x + 5y - 14 = 0.$$

$$\left(\because \frac{49}{4} + \frac{25}{4} - \frac{130}{4} = \frac{74 - 130}{4} = \frac{-56}{4} = -14\right)$$

12. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2, 3).

Sol. Let the centre of the circle be C $(h, 0)$

[$\because$ Every point on x-axis has its y-coordinate 0]

Radius of circle = 5

$\therefore$ Equation of circle is $(x - h)^2 + (y - 0)^2 = 5^2$

or $\quad x^2 + y^2 - 2hx + h^2 - 25 = 0 \quad$ ...(*i*)

Since it passes through the point (2, 3), we have by putting

$$x = 2, \; y = 3,$$

$$4 + 9 - 4h + h^2 - 25 = 0$$

or $\quad h^2 - 4h - 12 = 0 \quad$ or $\quad (h - 6)(h + 2) = 0$

$\therefore \quad h = 6 \quad$ or $\quad -2$

When $h = 6$, from (*i*), the equation of circle is

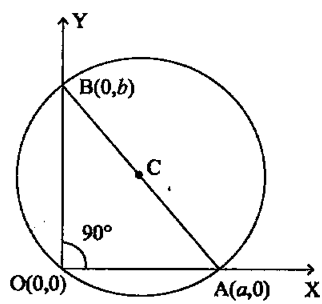
$$x^2 + y^2 - 12x + 11 = 0. \; (\because 36 - 25 = 11)$$

When $h = -2$, from (*i*) the equation of circle is

$$x^2 + y^2 + 4x - 21 = 0. \; (\because 4 - 25 = -21)$$

13. Find the equation of the circle passing through (0, 0) and making intercepts *a* and *b* on the coordinate axes.

Sol. Given: The required circle passes through the origin O(0,0) and makes intercepts a and b on the co-ordinates axes, therefore the circle also passes through the points A(a,0) and B(0,b). Join AB. Let C be the centre of required circle.



We know that angle between the co-ordinate axes *i.e.*, $\angle AOB = 90°$.

$\therefore$ AB is a diameter of the circle. [Angle in a semi-circle]

$\therefore$ Centre C of the circle is the mid-point of diameter AB.

$\therefore$ By mid-point formula, Centre C is $\left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$.

Also radius r of the circle = AC = BC = $\frac{1}{2}$ (diameter AB)

$$= \frac{1}{2}\sqrt{(a-0)^2 + (0-b)^2} = \frac{1}{2}\sqrt{(a^2 + b)^2}$$

$\therefore$ Equation of required circle is

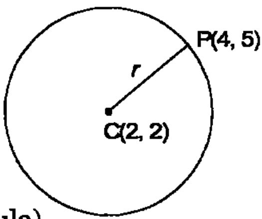
$(x-h)^2+(y-k)^2=r^2$, i.e., $\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\left(\frac{1}{2}\sqrt{a^2+b^2}\right)^2$

i.e., $\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\frac{1}{4}(a^2+b^2)$

or $x^2+\frac{a^2}{4}-ax+y^2+\frac{b^2}{4}-by=\frac{a^2}{4}+\frac{b^2}{4}$

or $x^2+y^2-ax-by=0.$

14. Find the equation of a circle with centre (2, 2) and passing through the point (4, 5).



Sol. Centre of circle is C(2, 2).

Since the circle passes through the point P(4, 5), its radius

$r = CP = \sqrt{(4-2)^2+(5-2)^2}$ (Distance Formula)

$= \sqrt{4+9} = \sqrt{13}$

∴ The equation of circle is $(x-h)^2+(y-k)^2=r^2$

$(x-2)^2+(y-2)^2=(\sqrt{13})^2$

or $(x^2-4x+4)+(y^2-4y+4)=13$

or $x^2+y^2-4x-4y=5.$

15. Does the point (– 2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

Sol. The given circle $x^2+y^2=25$

or $x^2+y^2=5^2$

has its centre at O(0, 0) and radius 5.

Distance of the given point P(– 2.5, 3.5) from centre O(0, 0) is

$OP = \sqrt{(2.5)^2+(3.5)^2} = \sqrt{6.25+12.25}$

$= \sqrt{18.50} < \sqrt{25} = 5 = \text{radius}$

Since, OP is less than the radius of the circle, the given point lies inside the circle.

EXERCISE 11.2 *(Page No.: 246–247)*

In each of the following Exercise 1 to 6, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum.

1. $y^2 = 12x$.

Sol. Comparing $y^2 = 12x$ with $y^2 = 4ax$, [Standard Form I]
we have

$$4a = 12 \quad \Rightarrow \quad a = 3$$

Focus is $(a, 0) = (3, 0)$

Axis is x-axis.

Equation of the directrix is $x = -a$ *i.e.*, $x = -3$.

Length of the latus rectum $= 4a = 4 \times 3 = 12$.

2. $x^2 = 6y$.

Sol. The given equation is $x^2 = 6y$

It is of the form $x^2 = 4ay$ [Standard Form III]

Here $4a = 6 \quad \therefore a = \frac{6}{4} = \frac{3}{2}$

The co-ordinates of the focus are $(0, a)$

$$= \left(0, \frac{3}{2}\right)$$

Equation of directrix is $y = -a$ *i.e.*, $y = -\frac{3}{2}$

Length of latus rectum $= 4a = 6$

3. $y^2 = -8x$.

Sol. Comparing $y^2 = -8x$ with $y^2 = -4ax$, [Standard Form II]
we have

$$4a = 8 \quad \Rightarrow \quad a = 2$$

Focus is $(-a, 0) = (-2, 0)$

Axis is x-axis.

Equation of directrix is $x = a$, *i.e.*, $x = 2$.

Length of latus rectum $= 4a = 4 \times 2 = 8$.

4. $x^2 = -16y$.

Sol. The given equation is $x^2 = -16y$

It is of the form $x^2 = -4ay$ (Standard Form IV)

Here $4a = 16 \quad \therefore \quad a = 4$

The co-ordinates of the focus are $(0, -a) = (0, -4)$

Equation of directrix is $y = a$ *i.e.*, $y = 4$

Length of latus rectum $= 4a = 16$.

5. $y^2 = 10x$.

Sol. Comparing $y^2 = 10x$ with $y^2 = 4ax$, [Standard Form I] we have

$$4a = 10 \quad \Rightarrow \quad a = \frac{5}{2}$$

Focus is $(a, 0) = \left(\frac{5}{2}, 0\right)$

Axis is x-axis.

Equation of directrix is $x = -a$, *i.e.*, $x = -\frac{5}{2}$

Length of latus rectum $= 4a = 4 \times \frac{5}{2} = 10$.

6. $x^2 = -9y$.

Sol. Comparing $x^2 = -9y$ with $x^2 = -4ay$, [Standard Form IV] we have

$$4a = 9 \quad \Rightarrow \quad a = \frac{9}{4}$$

Focus is $(0, -a) = \left(0, -\frac{9}{4}\right)$.

Axis is y-axis.

Equation of directrix is $y = a$, *i.e.*, $y = \frac{9}{4}$.

Length of latus rectum $= 4a = 4 \times \frac{9}{4} = 9$.

In each of the following Exercises 7 to 12, find the equation of the parabola that satisfies the given conditions:

7. Focus (6, 0); directrix $x = -6$.

Sol. Since the focus (6, 0) lies on the x-axis, ($\because y = 0$), the x-axis itself is the axis of the parabola. Therefore, the equation of the parabola is either

$y^2 = 4ax$ or $y^2 = -4ax$ $(a > 0)$

(Standard Form I or II)

Since the focus (6, 0) lies to the right of the vertex (0, 0); the parabola is right hand parabola $y^2 = 4ax$.

Since focus $(a, 0)$ is (6, 0). Comparing $a = 6$

Hence, the required equation of parabola is $y^2 = 4ax$

or $y^2 = 4(6)x$ or $y^2 = 24x$.

8. Focus (0, – 3); directrix y = 3.

Sol. Since the focus (0, – 3) lies on the y-axis, ($\because x = 0$), the y-axis itself is the axis of the parabola. Therefore, the equation of the parabola is

either $x^2 = 4ay$ or $x^2 = -4ay$ $(a > 0)$

Since the focus (0, – 3) lies below the origin and directrix is $y = 3$, the parabola is downward parabola $x^2 = -4ay$ with $a = 3$. Hence the required equation is

$$x^2 = -4(3)y \quad \text{or} \quad x^2 = -12y.$$

9. Vertex (0, 0); focus (3, 0).

Sol. Since the focus (3, 0) lies on the x-axis, the x-axis itself is the axis of the parabola. Therefore, the equation of the parabola is

either $y^2 = 4ax$ or $y^2 = -4ax$.

Since the focus (3, 0) lies on the right of the origin, the parabola is right hand parabola $y^2 = 4ax$ with $a = 3$. Hence the required equation is

$$y^2 = 4(3)x \quad \text{or} \quad y^2 = 12x.$$

10. Vertex (0, 0); focus (– 2, 0).

Sol. Since the focus (– 2, 0) lies on the x-axis, the x-axis itself is the axis of the parabola. Therefore, the equation of the parabola is

either $y^2 = 4ax$ or $y^2 = -4ax$.

Since the focus (– 2, 0) lies on the left of the origin, the parabola is left hand parabola $y^2 = -4ax$ with $a = 2$. Hence the required equation is

$$y^2 = -4(2)x \quad \text{or} \quad y^2 = -8x.$$

11. Vertex (0, 0); passing through (2, 3) and axis is along x-axis.

Sol. Since the axis of the parabola is along x-axis, its equation is of the form $y^2 = 4ax$ or $y^2 = -4ax$, where the sign depends on whether the parabola opens rightwards or leftwards. Since the parabola passes through (2, 3) which lies in the first quadrant, it must open rightwards. Thus the equation is of the form

$$y^2 = 4ax \quad ...(i)$$

Since it passes through (2, 3), we have

$$3^2 = 4a(2) \quad \Rightarrow \quad a = \frac{9}{8}$$

Putting $a = \frac{9}{8}$ in (*i*), the required equation is

$$y^2 = 4\left(\frac{9}{8}\right)x \quad \text{or} \quad y^2 = \frac{9}{2}x$$

or $\quad 2y^2 = 9x.$

12. Vertex (0, 0), passing through (5, 2) and symmetric with respect to *y*-axis.

Sol. Since the parabola is symmetric about *y*-axis and has its vertex at the origin, the equation is of the form $x^2 = 4ay$ or $x^2 = -4ay$, where the sign depends on whether the parabola opens upwards or downwards. But the parabola passes through (5, 2) which lies in the first quadrant, it must open upwards. Thus the equation is of the form

$$x^2 = 4ay \quad ...(i)$$

Since it passes through (5, 2), we have

$$5^2 = 4a(2) \quad \Rightarrow \quad a = \frac{25}{8}$$

Putting $a = \frac{25}{8}$ in (*i*), the required equation is

$$x^2 = 4\left(\frac{25}{8}\right)y \quad \text{or} \quad x^2 = \frac{25}{2}y$$

or $\quad 2x^2 = 25y.$

EXERCISE 11.3 (*Page No.: 255*)

In each of the Exercises 1 to 9, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

1. $\frac{x^2}{36} + \frac{y^2}{16} = 1.$

Sol. Since the denominator of $\frac{x^2}{36}$ is larger than the denominator of $\frac{y^2}{16}$, the major axis is along the x-axis.

Comparing the given equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$a^2 = 36 \text{ and } b^2 = 16 \quad \text{so that } a = 6 \text{ and } b = 4.$$

Also $c = \sqrt{a^2-b^2} = \sqrt{36-16} = \sqrt{20} = \sqrt{4\times 5} = 2\sqrt{5}$.

$\therefore$ The foci are $(\pm c, 0) = (\pm 2\sqrt{5}, 0)$

The vertices are $(\pm a, 0) = (\pm 6, 0)$

The length of major axis $= 2a = 2 \times 6 = 12$.

The length of minor axis $= 2b = 2 \times 4 = 8$.

Eccentricity $e = \frac{c}{a} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$.

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 16}{6} = \frac{16}{3}$.

2. $\frac{x^2}{4}+\frac{y^2}{25}=1$.

Sol. Since the denominator of $\frac{y^2}{25}$ is larger than the denominator of $\frac{x^2}{4}$, the major axis is along the y-axis.

Comparing the given equation with $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, we have

$a^2 = 25$ and $b^2 = 4$ so that $a = 5$ and $b = 2$.

Also $c = \sqrt{a^2-b^2} = \sqrt{25-4} = \sqrt{21}$.

$\therefore$ The foci are $(0, \pm c) = (0, \pm \sqrt{21})$

The vertices are $(0, \pm a) = (0, \pm 5)$.

The length of major axis $= 2a = 2 \times 5 = 10$.

The length of minor axis $= 2b = 2 \times 2 = 4$.

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{21}}{5}$.

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 4}{5} = \frac{8}{5}$.

3. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1.$

Sol. Since the denominator of $\dfrac{x^2}{16}$ is larger than the denominator of $\dfrac{y^2}{9}$, the major axis is along the x-axis.

Comparing the given equation with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we have

$$a^2 = 16 \text{ and } b^2 = 9 \text{ so that } a = 4 \text{ and } b = 3.$$

Also $c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}$.

$\therefore$ The foci are $(\pm c, 0) = (\pm \sqrt{7}, 0)$

The vertices are $(\pm a, 0) = (\pm 4, 0)$.

The length of major axis $= 2a = 2 \times 4 = 8$.

The length of minor axis $= 2b = 2 \times 3 = 6$.

Eccentricity $e = \dfrac{c}{a} = \dfrac{\sqrt{7}}{4}$.

Length of latus rectum $= \dfrac{2b^2}{a} = \dfrac{2\times 9}{4} = \dfrac{9}{2}$.

4. $\dfrac{x^2}{25} + \dfrac{y^2}{100} = 1.$

Sol. Since the denominator of $\dfrac{y^2}{100}$ is larger than the denominator of $\dfrac{x^2}{25}$, the major axis is along the y-axis.

Comparing the given equation with $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, we have

$$a^2 = 100 \text{ and } b^2 = 25 \text{ so that } a = 10 \text{ and } b = 5.$$

Also $c = \sqrt{a^2 - b^2} = \sqrt{100-25} = \sqrt{75}$

$= \sqrt{25\times 3} = 5\sqrt{3}$.

$\therefore$ The foci are $(0, \pm c) = (0, \pm 5\sqrt{3})$

The vertices are $(0, \pm a) = (0, \pm 10)$.

The length of major axis = $2a = 2 \times 10 = 20$.

The length of minor axis = $2b = 2 \times 5 = 10$.

Eccentricity $e = \frac{c}{a} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$.

Length of latus rectum = $\frac{2b^2}{a} = \frac{2 \times 25}{10} = 5$.

5. $\frac{x^2}{49} + \frac{y^2}{36} = 1$.

Sol. Since the denominator of $\frac{x^2}{49}$ is larger than the denominator of $\frac{y^2}{36}$, the major axis is along the x-axis.

Comparing the given equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$a^2 = 49 \text{ and } b^2 = 36 \text{ so that } a = 7 \text{ and } b = 6.$$

Also $c = \sqrt{a^2 - b^2} = \sqrt{49 - 36} = \sqrt{13}$.

$\therefore$ The foci are $(\pm c, 0) = (\pm \sqrt{13}, 0)$

The vertices are $(\pm a, 0) = (\pm 7, 0)$.

The length of major axis = $2a = 2 \times 7 = 14$.

The length of minor axis = $2b = 2 \times 6 = 12$.

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{13}}{7}$.

Length of latus rectum = $\frac{2b^2}{a} = \frac{2 \times 36}{7} = \frac{72}{7}$.

6. $\frac{x^2}{100} + \frac{y^2}{400} = 1$.

Sol. Since the denominator of $\frac{y^2}{400}$ is larger than the denominator of $\frac{x^2}{100}$, the major axis is along the y-axis.

Comparing the given equation with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we have

$$a^2 = 400 \text{ and } b^2 = 100 \text{ so that } a = 20 \text{ and } b = 10.$$

Also $c = \sqrt{a^2 - b^2} = \sqrt{400-100} = \sqrt{300} = \sqrt{100 \times 3} = 10\sqrt{3}$.

$\therefore$ The foci are $(0, \pm c) = (0, \pm 10\sqrt{3})$

The vertices are $(0, \pm a) = (0, \pm 20)$.

The length of major axis $= 2a = 2 \times 20 = 40$.

The length of minor axis $= 2b = 2 \times 10 = 20$.

Eccentricity $e = \frac{c}{a} = \frac{10\sqrt{3}}{20} = \frac{\sqrt{3}}{2}$.

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 100}{20} = 10$.

7. $\mathbf{36x^2 + 4y^2 = 144}$.

Sol. Dividing by 144 (to make the R.H.S. unity), the given equation becomes

$$\frac{x^2}{4} + \frac{y^2}{36} = 1$$

Since the denominator of $\frac{y^2}{36}$ is larger than the denominator of $\frac{x^2}{4}$, the major axis is along the y-axis.

Comparing with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we have

$$a^2 = 36 \text{ and } b^2 = 4 \text{ so that } a = 6 \text{ and } b = 2.$$

Also $c = \sqrt{a^2 - b^2} = \sqrt{36-4} = \sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$.

$\therefore$ The foci are $(0, \pm c) = (0, \pm 4\sqrt{2})$

The vertices are $(0, \pm a) = (0, \pm 6)$.

The length of major axis $= 2a = 2 \times 6 = 12$.

The length of minor axis $= 2b = 2 \times 2 = 4$.

Eccentricity $e = \frac{c}{a} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$.

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 4}{6} = \frac{4}{3}$.

8. $16x^2 + y^2 = 16$.

Sol. Dividing by 16, (to make the R.H.S. unity,) the given equation becomes

$$\frac{x^2}{1} + \frac{y^2}{16} = 1.$$

Since the denominator of $\frac{y^2}{16}$ is larger than the denominator of $\frac{x^2}{1}$, the major axis is along the y-axis.

Comparing with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we have

$$a^2 = 16 \text{ and } b^2 = 1 \text{ so that } a = 4 \text{ and } b = 1.$$

Also $c = \sqrt{a^2 - b^2} = \sqrt{16-1} = \sqrt{15}$.

$\therefore$ The foci are $(0, \pm c) = (0, \pm \sqrt{15})$

The vertices are $(0, \pm a) = (0, \pm 4)$.

The length of major axis $= 2a = 2 \times 4 = 8$.

The length of minor axis $= 2b = 2 \times 1 = 2$.

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{15}}{4}$.

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 1}{4} = \frac{1}{2}$.

9. $4x^2 + 9y^2 = 36$.

Sol. Dividing by 36, (to make the R.H.S. unity), the given equation becomes

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Since the denominator of $\frac{x^2}{9}$ is larger than the denominator of $\frac{y^2}{4}$, the major axis is along the x-axis.

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$a^2 = 9$ and $b^2 = 4$ so that $a = 3$ and $b = 2$.

Also $c = \sqrt{a^2 - b^2} = \sqrt{9-4} = \sqrt{5}$.

$\therefore$ The foci are $(\pm c, 0) = (\pm \sqrt{5}, 0)$

The vertices are $(\pm a, 0) = (\pm 3, 0)$.

The length of major axis $= 2a = 2 \times 3 = 6$.

The length of minor axis $= 2b = 2 \times 2 = 4$.

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{5}}{3}$.

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times4}{3} = \frac{8}{3}$.

In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions:

10. Vertices (± 5, 0), foci (± 4, 0).

Sol. Since the vertices and foci are on x-axis ($\because$ $y = 0$), therefore major axis is along x-axis and hence equation of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...}(i)$$

Vertices are $(\pm a, 0) = (\pm 5, 0)$ (given) $\therefore$ $a = 5$

$\therefore$ $a^2 = 25$

Foci are $(\pm c, 0) = (\pm 4, 0)$ (given) $\therefore$ $c = 4$

The relation $c^2 = a^2 - b^2$ gives $4^2 = 5^2 - b^2$

$\Rightarrow b^2 = 25 - 16 = 9$

Putting these values of a^2 and b^2 in (i),

Equation of ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \text{ or } \frac{9x^2 + 25y^2}{25\times9} = 1 \text{ or } 9x^2 + 25y^2 = 225.$$

11. Vertices (0, ± 13), foci (0, ± 5).

Sol. Since the vertices and foci are on y-axis ($\because$ $x = 0$), therefore, major axis is along y-axis and hence equation of ellipse is of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad ...(i)$$

Vertices are $(0, \pm a) = (0, \pm 13)$ $\Rightarrow$ $a = 13$

Foci are $(0, \pm c) = (0, \pm 5)$ $\Rightarrow$ $c = 5$

The relation $c^2 = a^2 - b^2$ gives $5^2 = (13)^2 - b^2$

$\Rightarrow$ $b^2 = 169 - 25 = 144.$

$\therefore$ From (*i*), the equation of ellipse is

$$\frac{x^2}{144} + \frac{y^2}{169} = 1.$$

12. Vertices (± 6, 0), foci (± 4, 0).

Sol. Since the vertices and foci are on x-axis ($\because$ $y = 0$), major axis is along x-axis and equation of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Vertices are $(\pm a, 0) = (\pm 6, 0)$ $\Rightarrow$ $a = 6$

Foci are $(\pm c, 0) = (\pm 4, 0)$ $\Rightarrow$ $c = 4$

The relation $c^2 = a^2 - b^2$ gives $4^2 = 6^2 - b^2$

$\Rightarrow$ $b^2 = 36 - 16 = 20.$

$\therefore$ From (*i*), the equation of ellipse is

$$\frac{x^2}{36} + \frac{y^2}{20} = 1.$$

13. Ends of major axis (± 3, 0), ends of minor axis (0, ± 2).

Sol. Since the major axis is along x-axis ($\because$ $y = 0$), the equation of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Ends of major axis are $(\pm a, 0) = (\pm 3, 0)$ $\Rightarrow$ $a = 3$

Ends of minor axis are $(0, \pm b) = (0, \pm 2)$ $\Rightarrow$ $b = 2$

$\therefore$ From (*i*), the equation of ellipse is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

14. Ends of major axis $(0, \pm \sqrt{5})$, ends of minor axis (± 1, 0).

Sol. Since the major axis is along y-axis ($\because$ $x = 0$), the equation of ellipse is of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad ...(i)$$

Ends of major axis are $(0, \pm a) = (0, \pm \sqrt{5}) \Rightarrow a = \sqrt{5}$

Ends of minor axis are $(\pm b, 0) = (\pm 1, 0) \Rightarrow b = 1$

$\therefore$ From (*i*), the equation of ellipse is

$$\frac{x^2}{1} + \frac{y^2}{5} = 1.$$

15. Length of major axis 26, foci (± 5, 0).

Sol. Since the foci are on x-axis ($\because y = 0$), the major axis is along x-axis and the equation of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

Length of major axis is $2a = 26 \Rightarrow a = 13$

Foci are $(\pm c, 0) = (\pm 5, 0) \Rightarrow c = 5$

The relation $c^2 = a^2 - b^2$ gives $5^2 = (13)^2 - b^2$

$\Rightarrow b^2 = 169 - 25 = 144$

$\therefore$ From (*i*), the equation of ellipse is

$$\frac{x^2}{169} + \frac{y^2}{144} = 1.$$

16. Length of minor axis 16, foci (0, ± 6).

Sol. Since the foci are on y-axis ($\because x = 0$), the major axis is along y-axis and the equation of ellipse is of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad ...(i)$$

Length of minor axis is $2b = 16 \Rightarrow b = 8$

Foci are $(0, \pm c) = (0, \pm 6) \Rightarrow c = 6$

The relation $c^2 = a^2 - b^2$ gives $6^2 = a^2 - 8^2$

$\Rightarrow a^2 = 36 + 64 = 100$

$\therefore$ From (*i*), the equation of ellipse is

$$\frac{x^2}{64} + \frac{y^2}{100} = 1.$$

17. Foci (± 3, 0), a = 4.

Sol. Since the foci are on x-axis ($\because y = 0$), the major axis is along x-axis and the equation of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

Foci are $(\pm c, 0) = (\pm 3, 0)$ $\Rightarrow$ $c = 3$

Also $a = 4$ (given)

The relation $c^2 = a^2 - b^2$ gives $3^2 = 4^2 - b^2$

$\Rightarrow$ $b^2 = 16 - 9 = 7$

$\therefore$ From (*i*), the equation of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{7} = 1.$$

18. $b = 3$, $c = 4$, centre at the origin, foci on the x-axis.

Sol. Because centre is at the origin and foci are on x-axis, therefore, equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

$b = 3$, $c = 4$ (given)

We know that for ellipse

$$a^2 = b^2 + c^2 = 3^2 + 4^2 = 25$$

Putting values of a^2 and b^2 in (*i*), required equation of ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

19. Centre at (0, 0), major axis on the y-axis and passes through the points (3, 2) and (1, 6).

Sol. Since the major axis is on the y-axis, the equation of ellipse is of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad ...(i)$$

The points (3, 2) and (1, 6) lie on (*i*)

$\therefore$
$$\frac{9}{b^2} + \frac{4}{a^2} = 1 \quad ...(ii)$$

$$\frac{1}{b^2} + \frac{36}{a^2} = 1 \quad ...(iii)$$

Multiplying (*iii*) by 9, we have

$$\frac{9}{b^2} + \frac{324}{a^2} = 9 \quad ...(iv)$$

Subtracting (*ii*) from (*iv*) (to eliminate b^2), we have

$$\frac{320}{a^2} = 8 \quad \Rightarrow \quad 8a^2 = 320 \Rightarrow \quad a^2 = 40.$$

Putting $a^2 = 40$ in (*iii*), we have

$$\frac{1}{b^2} + \frac{36}{40} = 1$$

$$\Rightarrow \qquad \frac{1}{b^2} = 1 - \frac{9}{10} = \frac{1}{10} \quad \Rightarrow \quad b^2 = 10$$

∴ From (*i*), the equation of ellipse is

$$\frac{x^2}{10} + \frac{y^2}{40} = 1.$$

20. Major axis on the x-axis and passes through the points (4, 3) and (6, 2).

Sol. Since the major axis is on the x-axis, the equation of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Since the points (4, 3) and (6, 2) lie on (*i*)

$$\therefore \qquad \frac{16}{a^2} + \frac{9}{b^2} = 1 \qquad ...(ii)$$

and $$\frac{36}{a^2} + \frac{4}{b^2} = 1 \qquad ...(iii)$$

Multiplying (*ii*) by 4, (*iii*) by 9 and subtracting (to eliminate b^2), we get

$$\frac{64}{a^2} - \frac{324}{a^2} = 4 - 9$$

$$\frac{64 - 324}{a^2} = -5 \Rightarrow \frac{-260}{a^2} = -5$$

$$\Rightarrow 5a^2 = 260 \quad \Rightarrow a^2 = \frac{260}{5} = 52$$

Putting $a^2 = 52$ in (*ii*), we have

$$\frac{16}{52} + \frac{9}{b^2} = 1$$

$$\Rightarrow \qquad \frac{9}{b^2} = 1 - \frac{4}{13} = \frac{9}{13} \quad \Rightarrow \quad 9b^2 = 9 \times 13 \Rightarrow b^2 = 13$$

$\therefore$ From (*i*), the equation of ellipse is

$$\frac{x^2}{52} + \frac{y^2}{13} = 1 \quad \text{or} \quad \frac{x^2+4y^2}{52} = 1 \quad \text{or} \quad x^2 + 4y^2 = 52.$$

EXERCISE 11.4 (*Page No.: 262*)

In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas.

1. $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

Sol. Comparing $\frac{x^2}{16} - \frac{y^2}{9} = 1$ with the standard equation

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$$a^2 = 16 \text{ and } b^2 = 9 \text{ so that } a = 4 \text{ and } b = 3$$

Also $\qquad c = \sqrt{a^2+b^2} = \sqrt{16+9} = 5.$

$\therefore$ The foci are $(\pm c, 0) = (\pm 5, 0)$

The vertices are $(\pm a, 0) = (\pm 4, 0)$

Eccentricity $\qquad e = \frac{c}{a} = \frac{5}{4}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{4} = \frac{9}{2}.$

2. $\frac{y^2}{9} - \frac{x^2}{27} = 1.$

Sol. Comparing $\frac{y^2}{9} - \frac{x^2}{27} = 1$ with the standard equation

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we have

$a^2 = 9$ and $b^2 = 27$ so that $a = 3$ and $b = \sqrt{27} = \sqrt{9\times 3} = 3\sqrt{3}$

Also $\qquad c = \sqrt{a^2+b^2} = \sqrt{9+27} = \sqrt{36} = 6.$

$\therefore$ The foci are $(0, \pm c) = (0, \pm 6)$

The vertices are $(0, \pm a) = (0, \pm 3)$

Eccentricity $e = \frac{c}{a} = \frac{6}{3} = 2$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 27}{3} = 18.$

3. $\mathbf{9y^2 - 4x^2 = 36.}$

Sol. Dividing both sides of the given equation by 36 (to make the R.H.S. unity), we get

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

Comparing it with the standard equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \text{ we have}$$

$a^2 = 4$ and $b^2 = 9$ so that $a = 2$ and $b = 3$.

Also $c = \sqrt{a^2 + b^2} = \sqrt{4+9} = \sqrt{13}$.

∴ The foci are $(0, \pm c) = (0, \pm \sqrt{13})$

The vertices are $(0, \pm a) = (0, \pm 2)$

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

Length of the latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{2} = 9.$

4. $\mathbf{16x^2 - 9y^2 = 576.}$

Sol. Dividing both sides of the given equation by 576 (to make the R.H.S. unity), we get

$$\frac{x^2}{36} - \frac{y^2}{64} = 1$$

Comparing it with the standard equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$a^2 = 36$ and $b^2 = 64$ so that $a = 6$ and $b = 8$.

Also $c = \sqrt{a^2 + b^2} = \sqrt{36+64} = \sqrt{100} = 10.$

$\therefore$ The foci are $(\pm c, 0) = (\pm 10, 0)$

The vertices are $(\pm a, 0) = (\pm 6, 0)$

Eccentricity $e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 64}{6} = \frac{64}{3}$.

5. $5y^2 - 9x^2 = 36$.

Sol. Dividing both sides of the given equation by 36 (to make the R.H.S. unity), we get

$$\frac{5y^2}{36} - \frac{x^2}{4} = 1 \quad \text{or} \quad \frac{y^2}{36/5} - \frac{x^2}{4} = 1$$

Comparing it with the standard equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \text{ we have}$$

$$a^2 = \frac{36}{5} \text{ and } b^2 = 4 \text{ so that } a = \frac{6}{\sqrt{5}} \text{ and } b = 2$$

Also $c = \sqrt{a^2 + b^2} = \sqrt{\frac{36}{5} + 4} = \sqrt{\frac{56}{5}} = \sqrt{\frac{4 \times 14}{5}} = \frac{2\sqrt{14}}{\sqrt{5}}$

$\therefore$ The foci are $(0, \pm c) = \left(0, \pm \frac{2\sqrt{14}}{\sqrt{5}}\right)$

The vertices are $(0, \pm a) = \left(0, \pm \frac{6}{\sqrt{5}}\right)$

Eccentricity $e = \frac{c}{a} = \frac{\frac{2\sqrt{14}}{\sqrt{5}}}{\frac{6}{\sqrt{5}}} = \frac{\sqrt{14}}{3}$

Length of the latus rectum $= \frac{2b^2}{a} = \frac{2 \times 4}{\frac{6}{\sqrt{5}}} = \frac{4\sqrt{5}}{3}$.

6. $49y^2 - 16x^2 = 784$.

Sol. Dividing both sides of the given equation by 784 (to make the R.H.S. unity), we get

$$\frac{y^2}{16} - \frac{x^2}{49} = 1$$

Comparing it with the standard equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \text{ we have}$$

$a^2 = 16$ and $b^2 = 49$ so that $a = 4$ and $b = 7$

Also $c = \sqrt{a^2 + b^2} = \sqrt{16 + 49} = \sqrt{65}$

$\therefore$ The foci are $(0, \pm c) = (0, \pm \sqrt{65})$

The vertices are $(0, \pm a) = (0, \pm 4)$

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{65}}{4}$

Length of the latus rectum $= \frac{2b^2}{a} = \frac{2 \times 49}{4} = \frac{49}{2}$.

In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions.

7. Vertices (± 2, 0), foci (± 3, 0).

Sol. Since the foci are on x-axis ($\because y = 0$), the equation of the hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...}(i)$$

Vertices are $(\pm a, 0) = (\pm 2, 0) \Rightarrow a = 2$

Foci are $(\pm c, 0) = (\pm 3, 0) \Rightarrow c = 3$

The relation $c^2 = a^2 + b^2$ gives $3^2 = 2^2 + b^2$

$\Rightarrow b^2 = 9 - 4 = 5$

$\therefore$ From (*i*), the equation of hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{5} = 1.$$

8. Vertices are (0, ± 5), foci (0, ± 8).

Sol. Since the foci are on y-axis ($\because x = 0$), the equation of the hyperbola is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{...}(i)$$

Vertices are $(0, \pm a) = (0, \pm 5)$ $\Rightarrow$ $a = 5$

Foci are $(0, \pm c) = (0, \pm 8)$ $\Rightarrow$ $c = 8$

The relation $c^2 = a^2 + b^2$ gives $8^2 = 5^2 + b^2$

$\Rightarrow$ $b^2 = 64 - 25 = 39$

$\therefore$ From (*i*), the equation of hyperbola is

$$\frac{y^2}{25} - \frac{x^2}{39} = 1.$$

9. Vertices (0, ± 3), foci (0, ± 5).

Sol. Since the foci are on the y-axis ($\because x = 0$), the equation of hyperbola is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad ...(i)$$

Vertices are $(0, \pm a) = (0, \pm 3)$ $\Rightarrow$ $a = 3$

Foci are $(0, \pm c) = (0, \pm 5)$ $\Rightarrow$ $c = 5$

The relation $c^2 = a^2 + b^2$ gives $5^2 = 3^2 + b^2$

$\Rightarrow$ $b^2 = 25 - 9 = 16$

$\therefore$ From (*i*), the equation of hyperbola is

$$\frac{y^2}{9} - \frac{x^2}{16} = 1.$$

10. Foci (± 5, 0), the transverse axis is of length 8.

Sol. Because the foci $(\pm 5, 0)$ lie on x-axis ($\because y = 0$), the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(*i*)

Foci are $(\pm 5, 0) = (\pm c, 0)$

Comparing $c = 5$

Transverse axis is of length 8 (= $2a$)

$\therefore$ $a = 4$ and hence $a^2 = 16$

The relation $c^2 = a^2 + b^2$ gives $5^2 = 4^2 + b^2$

$\Rightarrow 25 = 16 + b^2 \Rightarrow b^2 = 9$

Putting values of a^2 and b^2 in (*i*), equation of required hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

11. Foci (0, ± 13), the conjugate axis is of length 24.

Sol. Because the foci (0, ± 13) lie on y-axis ($\because x = 0$), the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. ...(*i*)

Foci are $(0, \pm 13) = (0, \pm c)$

$\therefore$ $c = 13$

conjugate axis is of length = 24 (given)

i.e., $2b = 24 \therefore b = 12 \quad \therefore b^2 = 144$

We know that for hyperbola, $c^2 = a^2 + b^2$

$\therefore$ $169 = a^2 + 144$ or $a^2 = 169 - 144 = 25$

Putting values of a^2 and b^2 in (*i*), equation of required hyperbola is

$$\frac{y^2}{25} - \frac{x^2}{144} = 1.$$

12. Foci ($\pm 3\sqrt{5}$, 0), the latus rectum is of length 8.

Sol. Since the foci $(\pm 3\sqrt{5}, 0)$ are on x-axis ($\because y = 0$), the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(*i*)

Foci are $(\pm c, 0) = (\pm 3\sqrt{5}, 0) \quad \therefore c = 3\sqrt{5}$

Length of latus rectum $= \frac{2b^2}{a} = 8$ or $b^2 = 4a$ ...(*ii*)

Putting c = $3\sqrt{5}$ and $b^2 = 4a$ from (*ii*) in the relation $c^2 = a^2 + b^2$, we have $(3\sqrt{5})^2 = a^2 + 4a$ or $45 = a^2 + 4a$

or $a^2 + 4a - 45 = 0$

or $(a + 9)(a - 5) = 0$ so that $a = -9, 5$

Since a (length of semi-transverse axis) cannot be negative, reject $a = -9$.

$\therefore a = 5$ and hence $a^2 = 25$.

and therefore from (*ii*) $b^2 = 4a = 4 \times 5 = 20$

$\therefore$ from (*i*), the equation of the required hyperbola is

$$\frac{x^2}{25} - \frac{y^2}{20} = 1 \text{ or } \frac{4x^2 - 5y^2}{100} = 1 \text{ or } 4x^2 - 5y^2 = 100.$$

13. Foci (± 4, 0), the latus rectum is of length 12.

Sol. Since the foci are on x-axis ($\because$ y = 0), the equation of the hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Foci are $(\pm c, 0) = (\pm 4, 0) \Rightarrow c = 4$

Length of latus rectum $= \frac{2b^2}{a} = 12 \Rightarrow 2b^2 = 12a$

$$\Rightarrow \qquad b^2 = 6a \qquad ...(ii)$$

Putting $c = 4$ and $b^2 = 4a$ from (*ii*),

the relation $c^2 = a^2 + b^2$ gives $16 = a^2 + 6a$

or $a^2 + 6a - 16 = 0$ or $(a + 8)(a - 2) = 0$

so that $a = -8, 2$.

Since a (length of semi-transverse axis) cannot be negative, reject $a = -8$ $\therefore$ $a = 2$ and hence $a^2 = 4$ and therefore from (*ii*), $b^2 = 6a = 6 \times 2 = 12$.

$\therefore$ From (*i*), the equation of required hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{12} = 1.$$

14. Vertices (± 7, 0), e = $\frac{4}{3}$.

Sol. Since the vertices are on x-axis ($\because$ y = 0), the equation of hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Vertices are $(\pm a, 0) = (\pm 7, 0) \Rightarrow a = 7 \Rightarrow a^2 = 49$

$$e = \frac{4}{3} \Rightarrow \frac{c}{a} = \frac{4}{3} \Rightarrow 3c = 4a = 4 \times 7 = 28 \Rightarrow c = \frac{28}{3}$$

From the relation $c^2 = a^2 + b^2$, we have $b^2 = c^2 - a^2$

$$= \left(\frac{28}{3}\right)^2 - 7^2 = \frac{784}{9} - 49$$

$$= \frac{784 - 441}{9} = \frac{343}{9}.$$

$\therefore$ From (*i*), the equation of required hyperbola is

$$\frac{x^2}{49} - \frac{y^2}{\frac{343}{9}} = 1 \quad \text{or} \quad \frac{x^2}{49} - \frac{9y^2}{343} = 1.$$

15. Foci $(0, \pm\sqrt{10})$, passing through (2, 3).

Sol. Since the foci are on y-axis ($\because x = 0$), the equation of the hyperbola is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad ...(i)$$

Foci are $(0, \pm c) = (0, \pm\sqrt{10})$ $\therefore$ $c = \sqrt{10}$

Since the point (2, 3) lies on the hyperbola, we have

$$\frac{9}{a^2} - \frac{4}{b^2} = 1 \quad \text{or} \quad \frac{9}{a^2} - \frac{4}{c^2 - a^2} = 1 \qquad (\because c^2 = a^2 + b^2$$

$$\Rightarrow b^2 = c^2 - a^2)$$

Putting $c = \sqrt{10}$,

or $$\frac{9}{a^2} - \frac{4}{10 - a^2} = 1 \Rightarrow \frac{9(10 - a^2) - 4a^2}{a^2(10 - a^2)} = 1$$

cross-multiplying, we have

$\Rightarrow 9(10 - a^2) - 4a^2 = a^2(10 - a^2)$

$\Rightarrow 90 - 9a^2 - 4a^2 = 10a^2 - a^4$

$\Rightarrow a^4 - 23a^2 + 90 = 0$

Put $a^2 = t$. Therefore $t^2 - 23t + 90 = 0$

$\Rightarrow t^2 - 18t - 5t + 90 = 0 \Rightarrow t(t - 18) - 5(t - 18) = 0$

$\Rightarrow (t - 18)(t - 5) = 0$

$\therefore$ Either $t - 18 = 0$ or $t - 5 = 0$

i.e., $t = 18$ or $t = 5$

i.e., $a^2 = 18$ or $a^2 = 5$ ($\because t = a^2$)

$\therefore$ $a^2 = 18, 5$

When $a^2 = 18$, we get $b^2 = c^2 - a^2 = 10 - 18 = -8 < 0$

$\therefore$ Rejecting $a^2 = 18$, we have $a^2 = 5$ so that

$b^2 = c^2 - a^2 = 10 - 5 = 5$

Putting these values of a^2 and b^2 in (*i*),

Equation of the required hyperbola is

$$\frac{y^2}{5} - \frac{x^2}{5} = 1 \quad \text{or} \quad y^2 - x^2 = 5.$$

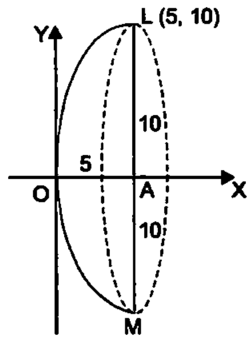
MISCELLANEOUS EXERCISE ON CHAPTER 11

(Page No.: 264)

1. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

Sol. Taking the origin as the vertex and the axis of the reflector as positive x-axis, the equation of the parabolic section is

rightward parabola $y^2 = 4ax$ ...(*i*)



Depth OA = 5 cm, diameter LM = 20 cm

so that AL = AM = 10 cm.

∴ Coordinate of point L are (OA, AL) = (5, 10).

Since point L lies on parabola (*i*), we have

$$(10)^2 = 4a\ (5) \quad \therefore \quad a = \frac{100}{20} = 5$$

Focus is $(a, 0) = (5, 0)$ *i.e.*, at A.

Hence, focus is at the mid-point of the given diameter.

2. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

Sol. Taking the vertex of the parabola as the origin O and axis along OY′, the equation of the parabolic arch is

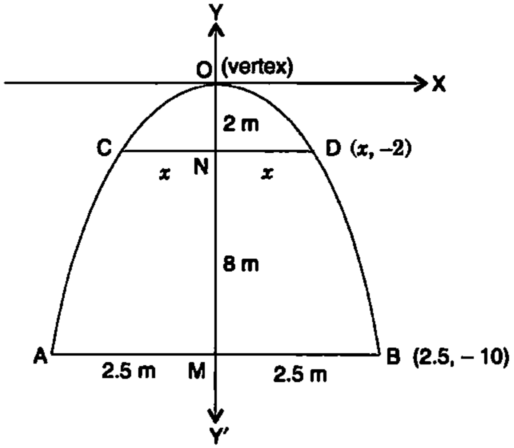
downward parabola $x^2 = -4ay$ [∵ upward parabola can't have a base] ...(*i*)

Since MB = $\frac{1}{2}$ base AB = $\frac{5}{2}$ = 2.5 m (For figure, see next page)

and OM = 10 m, the coordinates of B are (2.5, – 10).

B lies on (*i*) ⇒ $(2.5)^2 = -4a(-10)$

$$\Rightarrow \quad 6.25 = 40a \Rightarrow \quad a = \frac{6.25}{40}$$



Putting this value of a in (i),
equation (i) becomes

$$x^2 = -\frac{6.25}{10}y$$

or $x^2 = -0.625y$ ...(ii)

Let the width of the parabola at a distance 2m [= ON (say)] from vertex of parabola be CD.

If CD = $2x$, then coordinates of D are (x, – 2).

Since D(x, – 2) lies on (ii).

$\therefore$ $x^2 = -0.625 \times (-2)$

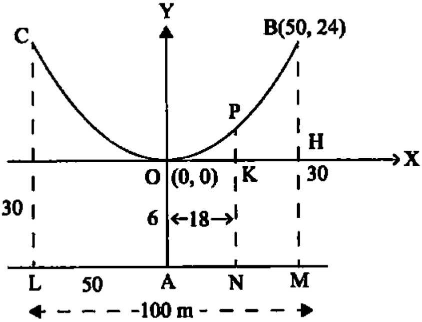
$$= 1.250 = \frac{1250}{1000} = \frac{125}{100} = \frac{5}{4} \qquad \Rightarrow \qquad x = \frac{\sqrt{5}}{2}$$

Hence, CD = width at depth 2 m = $2x = 2 \times \dfrac{\sqrt{5}}{2}$

$$= \sqrt{5} = 2.33 \text{ m (approximately)}.$$

3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

Sol. Let the cable of suspension bridge (hanging in the form of a parabola) be the parabolic arc COB with vertex at the lowest point O and axis vertical. Let LM = 100 m be the horizontal roadway such that CL = BM = (longest wire) 30 m and OA = (shortest wire) 6 m. [∵ O, the vertex of the parabola is the lowest point of this upward parabola]



We have to find the length of supporting wire NP attached to the roadway at N where AN = 18 m. Now AL = AM = 50m. (∵ LM = 100m)

Let the coordinate axes be chosen as shown in the figure. We know that equation of the upward parabola with vertex at O(0,0) and axis along positive y-axis is

$$x^2 = 4ay \quad \text{...}(i)$$

Since it passes through the point B whose co-ordinates are

$$x = OH = AM = 50 \text{ and}$$

$$y = BH = BM - HM = BM - OA = 30 - 6 = 24$$

Putting $x = 50$ and $y = 24$ in (i), we have

$$(50)^2 = 4a\,(24) \Rightarrow 2500 = 96a \Rightarrow a = \frac{2500}{96} = \frac{625}{24}$$

Putting the values of a in (i), the equation of the parabola is

$$x^2 = 4 \times \frac{625y}{24}$$

or
$$x^2 = \frac{625y}{6} \quad \text{...}(ii)$$

Through the point N draw a vertical line to meet the parabola at point P. Then co-ordinates of point P are

$$x = OK = AN = 18 \text{ and}$$

$$y = PK = PN - KN = PN - OA = PN - 6$$

Putting these values of x and y in (ii),

$$(18)^2 = \frac{625}{6}(PN-6) \Rightarrow 324 \times 6 = 625\,(PN-6)$$

$\therefore$ PN − 6 = $\frac{324\times 6}{625} = \frac{2044}{625} = \frac{2044\times 16}{625\times 16}$
(to make denominator = 10000)

$\Rightarrow$ PN − 6 = $\frac{32704}{10000}$ = 3.2704

$\therefore$ PN = 6 + 3.2704 = 9.2704

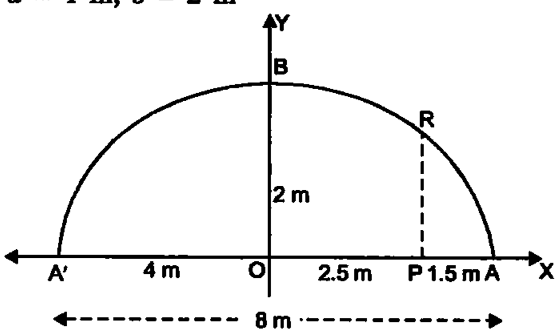
$\therefore$ Length of the supporting wire attached to the roadway at point N which is 18m from the middle point A = PN = 9.2704m.

4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

Sol. Let ABA′ be the given semi-elliptic arch such that width AA′ = 8 m and OB = 2 m.

The arch is a part of ellipse whose semi-major axis is $\frac{8}{2}$ = 4 m and semi-minor axis is OB = 2 m.

i.e., a = 4 m, b = 2 m



$\therefore$ Equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Putting values of a and b, we have

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \qquad ...(i)$$

We have to find the height PR at P where AP (distance from one end say A of the arc h APA′) = 1.5 m.

Now OP = OA − AP
= 4 − 1.5 = 2.5 m

For the point R, x = OP = 2.5 = $\frac{25}{10} = \frac{5}{2}$

Putting $x = \frac{5}{2}$ in (*i*), we get $\frac{\frac{25}{4}}{16} + \frac{y^2}{4} = 1$

or $\frac{y^2}{4} = 1 - \frac{25}{64}$ or $y^2 = \frac{64-25}{16} = \frac{39}{16}$

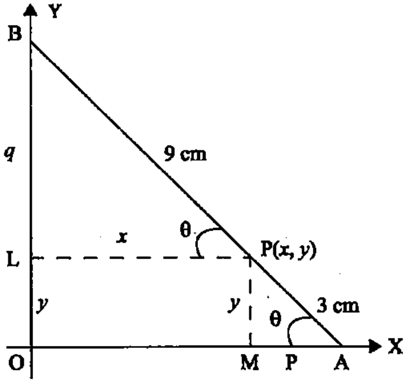
$\therefore \quad y = \frac{\sqrt{39}}{4} = \frac{6.24}{4} = 1.56$

Hence, the required height of arch = PR = ordinate (*i.e.*, y) of R = 1.56 m (approx.)

5. A rod of length 12 cm moves with its ends always touching the coordinates axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the *x*-axis.

Sol. Let AB be the rod of length 12 cm and P(x, y) the point on it such that AP = 3 cm. (given)

$\therefore \quad$ PB = AB – AP = 12 – 3 = 9 cm



From P, draw PL and PM perpendicular on y-axis and x-axis respectively.

$\therefore \quad$ PL $= x$ and PM $= y$.

Let ∠ OAB = θ.

$\therefore$ ∠ LPB is also = θ | corresponding angles

In right angled ΔAMP, $\sin\theta = \frac{MP}{AP} = \frac{y}{3}$ ...(*i*)

In right angled ΔPLB, $\cos\theta = \frac{PL}{PB} = \frac{x}{9}$ ...(*ii*)

Putting these values of cos θ and sin θ from (*ii*) and (*i*) in the relation $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

or $$\frac{x^2}{81} + \frac{y^2}{9} = 1$$

which is the required locus of point *P*.

It is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Hence, the locus of P is an ellipse.

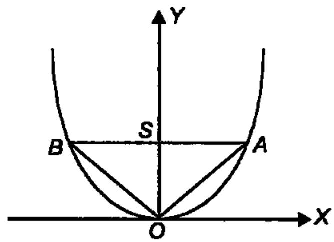
6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

Sol. Equation of parabola is $x^2 = 12y$.

Comparing with upward parabola $x^2 = 4ay$, we have

$$4a = 12 \quad \therefore \quad a = 3$$

In the figure, S is the focus, O is the vertex and AB ($\perp$ OS) is the latus rectum. (By definition of latus rectum)



Latus rectum AB = $4a$ = 12

Since the focus S is $(0, a) = (0, 3)$

$\therefore$ OS = 3

Area of ΔOAB = $\frac{1}{2}$ AB $\times$ OS | $\frac{1}{2}$ base $\times$ height

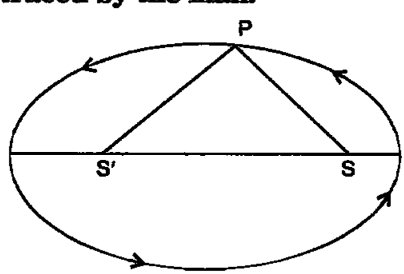
$= \frac{1}{2} \times 12 \times 3 = 18$ sq. units.

7. A man running a race-course notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the path traced by the man.

Sol. Let S and S′ be the two flag posts and P, the position of the man at any instant.

Since PS + PS′ = 10. (given),

$\therefore$ by definition of ellipse, the path traced by the man is an ellipse having its foci



at the two flag posts and length of major axes = 10.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

Because length of major axis = $2a$

$\therefore \quad 2a = 10 \quad \Rightarrow \quad a = 5$

Also SS′ = distance between foci = $(2c)$ = 8 (given)

$\therefore \quad 2c = 8 \quad \Rightarrow \quad c = 4$

Using the relation $c^2 = a^2 - b^2$, we have

$4^2 = 5^2 - b^2 \Rightarrow b^2 = 25 - 16 = 9$

Putting values of a^2 and b^2 in (i),

the equation of the (ellipse) path is $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

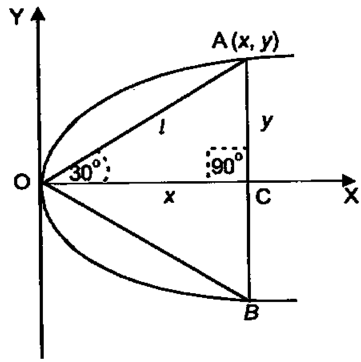
8. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Sol. The given parabola is

$$y^2 = 4ax \quad ...(i)$$

Its vertex is the origin O.

Let OAB be the equilateral triangle inscribed in the parabola. Let each side of triangle be of length l.



By symmetry, AB ⊥ OX and

$$\angle BOC = \angle AOC = 30°$$

In ΔOCA, $\cos 30° = \frac{OC}{OA} = \frac{x}{l}$

or $\frac{\sqrt{3}}{2} = \frac{x}{l}$ $\therefore$ $x = l\frac{\sqrt{3}}{2}$

and $\sin 30° = \frac{AC}{OA} = \frac{y}{l}$ or $\frac{1}{2} = \frac{y}{l}$

$\therefore$ $y = \frac{l}{2}$

$\therefore$ Coordinates of point A are $(x, y) = \left(\frac{l\sqrt{3}}{2}, \frac{l}{2}\right)$.

Since A lies on parabola (*i*), therefore, coordinates of point A satisfy eqn. (*i*)

$\therefore$ $\left(\frac{l}{2}\right)^2 = 4a \cdot \frac{\sqrt{3}}{2}l$

or $\frac{l^2}{4} = 2a\sqrt{3}\,l$ or $l^2 = 8a\sqrt{3}\,l$

Dividing both sides by l,

$\Rightarrow$ $l = 8\sqrt{3}\,a$

Hence, the length of side of equilateral triangle = $8\sqrt{3}\,a$.