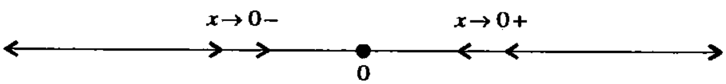
13 Limits and Derivatives

Lesson at a Glance

Section I
Limits

1. Meaning of 'x approaches zero' or 'x tends to zero'.
x can approach zero from right or from left.



$x \to 0^+ \Rightarrow x$ is close to zero and $x > 0$.

$x \to 0^- \Rightarrow x$ is close to zero and $x < 0$.

These two expressions combined are written as $x \to 0$.

Clearly, $x \to 0 \Rightarrow x \neq 0$ but close to zero.

Similarly, $x \to a$ implies $x \to a^+$ and $x \to a^-$

i.e., x is close to a and $x > a$ and x is close to a and $x < a$ but $x \neq a$.

2. Limit of a function. Consider the function $y = \frac{x^2 - 1}{x - 1}$.

The value of the function at $x = 1$ is of the form $\frac{0}{0}$ which is meaningless. In this case, we cannot divide the numerator by denominator since $x - 1$ is zero.

Now, suppose x is not actually equally to 1 but very nearly equal to 1, then $x - 1$ is not equal to zero. Hence in this case, we can divide the numerator by denominator.

$\therefore \frac{x^2 - 1}{x - 1} = x + 1$ and approximate value of $x + 1$ as $x \to 1$ is $1 + 1 = 2$.

In fact $\lim_{x \to c} f(x) = l$

$\Rightarrow$ **The approximate value of $y = f(x)$ is l when x is very close to c**

(*i.e.*, x takes up values slightly less than c and slightly greater than c but $x \neq c$.)

3. Some very important formulae on limits

1. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1}$, where n is a rational number.

2. (*a*) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{x}{\sin x} = 1$

   (*b*) $\lim_{x \to 0} \frac{\tan x}{x} = 1$ and $\lim_{x \to 0} \frac{x}{\tan x} = 1$

where x is the angle measured in radians both for formulae (*a*) and (*b*).

4. Some very useful tips for doing problems on limits:

1. Rationalise
2. L.C.M.
3. Take common
4. Form factors
5. Cancel
6. $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$
7. $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
8. $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$
9. For tan C – tan D or cot C – cot D; change these T-ratios into sin and cos; then take L.C.M. and then apply sin A cos B – cos A sin B = sin (A – B).
10. To evaluate $\lim_{x \to a} f(x)$; we can also put $x = a + h$ so that as $x \to a$, $a + h \to a$, *i.e.*, $h \to 0$.

$\therefore \lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$.

Remark. All the tips listed in above result 4 help us in getting rid of negative sign in the problems on limits which is our most important objective in problems on limits.

5. Algebra of limits. Let f and g be two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

[*i.e.*, limit of sum = sum of limits]

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

[*i.e.*, limit of difference = difference of limits]

3. $\lim_{x \to a} [f(x) \cdot g(x)] = [\lim_{x \to a} f(x)] [\lim_{x \to a} g(x)]$

[*i.e.*, limit of product = product of limits]

4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

[*i.e.*, limits of quotient = quotient of limits, provided limit of the denominator is non-zero.]

5. $\lim_{x \to a} [Cf(x)] = C \cdot \lim_{x \to a} f(x)$, where C is a constant.

6. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

7. $\lim_{x \to a} C = C$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} f(x) = f(a)$, where $f(x)$ is a polynomial function.

10. If $f(x)$ is a rational function (*i.e.*, quotient of polynomials) given by $f(x) = \frac{g(x)}{h(x)}$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}.$$ Provided $h(a) \neq 0$

6. One sided limits

1. To evaluate Right Hand Limit = $\lim_{x \to a^+} f(x)$

Put $x = a + h$, $h > 0$ so that $h \to 0^+$ as $x \to a^+$

$$\therefore \quad \lim_{x \to a^+} f(x) = \lim_{h \to 0^+} f(a + h)$$

2. To evaluate Left Hand Limit = $\lim_{x \to a^-} f(x)$

Put $x = a - h$, $h > 0$ so that $h \to 0^+$ as $x \to a^-$

$\therefore \quad \lim_{x \to a^-} f(x) = \lim_{h \to 0^+} f(a - h)$

$\lim_{x \to a} f(x) = l$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$ (say).

If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then we say that limit does not exist.

Remark. Both for right hand limit and left hand limit, $h \to 0^+$.

7. The following types of problems on limits can be done only by finding both right hand limit and left hand limit:

1. Limit of modulus functions.

Definition of modulus function

$|x| = x$ if $x \geq 0$

and $|x| = -x$ if $x < 0$

2. Limits of bracket function or greatest integer function $[x]$.

3. Limits of functions with partitioned domain

i.e., $f(x) = \begin{cases} g(x) & \text{if} \quad x \geq a \\ h(x) & \text{if} \quad x < a \end{cases}$.

Remark 1. In fact, any problem on limits can be done by finding both left hand limit and right hand limit but for the above three type of limits, we have to find both left hand limit and right hand limits separately.

2. If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$ (say), then we say that $\lim_{x \to a} f(x)$ exists.

3. If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$; then we say that $\lim_{x \to a} f(x)$ does not exist.

8. **Results on bracket function $[x]$ where $[x]$ denotes the greatest integer $\leq x$**

As $h \to 0^+$,

1. If c is an integer, then $[c - h] = c - 1$ and $[c + h] = c$.

2. If c is a real number not an integer, then $[c - h] = [c + h] = [c]$.

Section II

9. Differentiability (or derivability) at a point $x = a$

$f(x)$ is said to be derivable at a point $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists (*i.e.*, is unique and finite).

Then, this limit is denoted by $f'(a)$.

10. To find $\frac{dy}{dx}$ or $f'(x)$ from first principles or by definition or by $a - b$ initio method

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

11. Formulae on differentiation (*i.e.*, to find $\frac{dy}{dx}$ or $f'(x)$)

1. $\frac{d}{dx}(c) = 0$ where c is a constant.

2. (*a*) $\frac{d}{dx}(x^n) = n\,x^{n-1}$ where n is a fixed number, integer or rational.

   (*b*) $\frac{d}{dx}(u^n) = n\,u^{n-1}\frac{du}{dx}$ where u is a function of x.

3. $\frac{d}{dx}(cu) = c\frac{d}{dx}(u)$ where c is a constant and u is a function of x.

12. Product rule. (*a*) $\frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$ where u and v are functions of x.

(*b*) $\frac{d}{dx}(uvw) = \frac{du}{dx} \cdot vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}$ where u, v, w are functions of x.

The above formula is also true for derivative of product of four functions also and in fact true for the product of any finite number of functions.

13. $\frac{d}{dx}(u \pm v \pm w \pm \ldots) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v) \pm \frac{d}{dx}(w) \pm \ldots$ where $u, v, w, \ldots$ are functions of x.

14. Quotient rule. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}.v - u\frac{dv}{dx}}{v^2}$ where u and v are functions of x.

15. (*a*) $\frac{d}{dx}(\sin x) = \cos x$

(*b*) $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$ where u is a function of x.

16. (*a*) $\frac{d}{dx}(\cos x) = -\sin x$ (*b*) $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

17. (*a*) $\frac{d}{dx}(\tan x) = \sec^2 x$ (*b*) $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$

18. (*a*) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$ (*b*) $\frac{d}{dx}(\cot u) = -\text{cosec}^2 u \frac{du}{dx}$

19. (*a*) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(*b*) $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$

20. (*a*) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(*b*) $\frac{d}{dx}(\text{cosec } u) = -\text{cosec } u \cot u \frac{d}{dx}$

Note: It can be observed from formulae 15 to 20 that **derivatives of T-functions beginning with letter C *i.e.*, cos x, cot x and cosec x are negative.**

Note: Chain rule of differentiation

The formulae listed in 2(*b*), 15(*b*), 16(*b*), 17(*b*) 18(*b*), 19(*b*), and 20(*b*) are called formulae on chain rule of differentiation

EXERCISE 13.1 (*Page No.: 301–303*)

Evaluate the following limits in Exercise 1 to 22.

1. $\lim_{x \to 3} x + 3.$

Sol. We know that the limit of a polynomial function is the value of the function at the prescribed point. *i.e.*, if $f(x)$ is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$, obtained by writing a for x in the function.

$\therefore \quad \lim_{x \to 3} x + 3 = 3 + 3 = 6.$

2. $\lim_{x \to \pi} \left(x - \frac{22}{7}\right)$.

Sol. $\lim_{x \to \pi} \left(x - \frac{22}{7}\right) = \left(\pi - \frac{22}{7}\right)$

Remark. $\pi \neq \frac{22}{7}$ since π is **irrational** whereas $\frac{22}{7}$ is **rational**. However, $\frac{22}{7}$ is an **approximate** value of π. $\frac{355}{113}$ is another **approximate** value of π.

3. $\lim_{r \to 1} \pi r^2$.

Sol. $\lim_{r \to 1} \pi r^2 = \pi \times 1^2 = \pi.$

4. $\lim_{x \to 4} \frac{4x+3}{x-2}$.

Sol. $\lim_{x \to 4} \frac{4x+3}{x-2} = \frac{4(4)+3}{4-2} = \frac{19}{2}$.

5. $\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$.

Sol. $\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1} = \frac{(-1)^{10}+(-1)^5+1}{(-1)-1} = \frac{1-1+1}{-2} = -\frac{1}{2}$.

6. $\lim_{x \to 0} \frac{(x+1)^5-1}{x}$.

On putting $x = 0$, we get $\frac{1^5-1}{0} = \frac{0}{0}$ which is an indeterminate form.

Sol. Put $x + 1 = y$, *i.e.*, $x = y - 1$ so that $y \to 1$ as $x \to 0$.

$\therefore \quad \lim_{x \to 0} \frac{(x+1)^5-1}{x} = \lim_{y \to 1} \frac{y^5-1^5}{y-1}$

$\left[\text{Form } \lim_{x \to a} \frac{x^n-a^n}{x-a}, \text{Here } n = 5, a = 1\right]$

$= 5 \times 1^{5-1}$ $[na^{n-1}]$

$= 5.$

7. $\lim\limits_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 - 4}.$

Sol. $\lim\limits_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 - 4}$ $\left(\text{Putting } x = 2, \text{ we get the Form } \dfrac{0}{0}\right)$

Forming Factors

$[3x^2 - x - 10 = 3x^2 - 6x + 5x - 10 = 3x(x - 2) + 5(x - 2)$

$= (x - 2)(3x + 5)]$

$= \lim\limits_{x \to 2} \dfrac{(x-2)(3x+5)}{(x-2)(x+2)}$ [**Cancelling** $(x - 2) \neq 0$]

$= \lim\limits_{x \to 2} \dfrac{3x+5}{x+2} = \dfrac{3 \times 2 + 5}{2+2} = \dfrac{11}{4}.$

8. $\lim\limits_{x \to 3} \dfrac{x^4 - 81}{2x^2 - 5x - 3}.$

Sol. $\lim\limits_{x \to 3} \dfrac{x^4 - 81}{2x^2 - 5x - 3}$ $\left(\text{Putting } x = 3, \text{ we get the Form } \dfrac{0}{0}\right)$

Let us form factors of both Numerator and Denominator,

$[x^4 - 81 = (x^2 - 9)(x^2 + 9) = (x - 3)(x + 3)(x^2 + 9)$

and $2x^2 - 5x - 3 = 2x^2 - 6x + x - 3 = 2x(x - 3) + (x - 3)$

$= (x - 3)(2x + 1)]$

$= \lim\limits_{x \to 3} \dfrac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$ [**Cancelling** $(x - 3) \neq 0$]

$= \lim\limits_{x \to 3} \dfrac{(x+3)(x^2+9)}{2x+1} = \dfrac{(3+3)(3^2+9)}{2(3)+1} = \dfrac{6 \times 18}{7} = \dfrac{108}{7}.$

9. $\lim\limits_{x \to 0} \dfrac{ax+b}{cx+1}.$

Sol. $\lim\limits_{x \to 0} \dfrac{ax+b}{cx+1} = \dfrac{a(0)+b}{c(0)+1} = b.$

10. $\lim\limits_{z \to 1} \dfrac{z^{1/3} - 1}{z^{1/6} - 1}.$

Sol. On putting $z = 1$, we get the form $\left(\dfrac{0}{0}\right)$

Put $z^{1/6} = y$ so that $y \to 1$ as $z \to 1$. Then

$$\lim_{z \to 1} \frac{z^{1/3} - 1}{z^{1/6} - 1} = \lim_{y \to 1} \frac{y^2 - 1}{y - 1} \qquad [z^{1/3} = z^{2/6} = (z^{1/6})^2 = y^2]$$

$\left(\text{This is again } \frac{0}{0} \text{ form}\right)$

$$= \lim_{y \to 1} \frac{(y+1)(y-1)}{y-1}$$

Cancelling $(y - 1)$, $\quad = \lim_{y \to 1} (y + 1) = 1 + 1 = 2.$

Second solution:

$$\lim_{z \to 1} \frac{z^{1/3} - 1}{z^{1/6} - 1} = \lim_{z \to 1} \frac{z^{1/3} - 1^{1/3}}{z^{1/6} - 1^{1/6}}$$

Dividing both numerator and denominator by $(z-1)$,

$$= \lim_{z \to 1} \left(\frac{\frac{z^{1/3} - 1^{1/3}}{z - 1}}{\frac{z^{1/6} - 1^{1/6}}{z - 1}} \right) = \frac{\frac{1}{3}(1)^{1/3-1}}{\frac{1}{6}(1)^{1/6-1}} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{\frac{1}{3}}{\frac{1}{6}} = \frac{1}{3} \times \frac{6}{1} = 2$$

11. $\lim\limits_{x \to 1} \dfrac{ax^2 + bx + c}{cx^2 + bx + a}$, $a + b + c \neq 0$.

Sol. $\lim\limits_{x \to 1} \dfrac{ax^2 + bx + c}{cx^2 + bx + a} = \dfrac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a} = \dfrac{a + b + c}{c + b + a} = 1.$

12. $\lim\limits_{x \to -2} \dfrac{\frac{1}{x} + \frac{1}{2}}{x + 2}$.

Sol. $\lim\limits_{x \to -2} \dfrac{\frac{1}{x} + \frac{1}{2}}{x + 2}$ $\qquad \left(\text{Putting } x = -2, \text{ we get the Form } \frac{0}{0}\right)$

Taking L.C.M. to write the given function as a rational function.

$$= \lim_{x \to -2} \frac{x + 2}{2x(x + 2)}$$

$$= \lim_{x \to -2} \frac{1}{2x} \quad \text{[Cancelling } (x+2) \neq 0]$$

$$= \frac{1}{2(-2)} = -\frac{1}{4}.$$

13. $\lim_{x \to 0} \frac{\sin ax}{bx}$.

Sol. $\lim_{x \to 0} \frac{\sin ax}{bx} = \lim_{x \to 0} \left(\frac{a}{b} \cdot \frac{\sin ax}{ax}\right) = \frac{a}{b} \lim_{ax \to 0} \left(\frac{\sin ax}{ax}\right)$

[$\because$ as $x \to 0$, $ax \to 0$]

$$= \frac{a}{b} \times 1 = \frac{a}{b}.$$

14. $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$, $a, b \neq 0$.

Sol. $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \lim_{x \to 0} \left(\frac{a}{b} \cdot \frac{\sin ax}{ax} \cdot \frac{bx}{\sin bx}\right)$

$$= \frac{a}{b}\left(\lim_{ax \to 0} \frac{\sin ax}{ax} + \lim_{bx \to 0} \frac{\sin bx}{bx}\right) \quad \left(\therefore \frac{c}{d} = \frac{1}{\left(\frac{d}{c}\right)}\right)$$

[$\because$ as $x \to 0$, $ax \to 0$ and $bx \to 0$]

$$= \frac{a}{b}(1 + 1) = \frac{a}{b}.$$

15. $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$.

Sol. Put $\pi - x = t$ so that $t \to 0$ as $x \to \pi$.

$$\therefore \lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \frac{1}{\pi} \lim_{t \to 0} \frac{\sin t}{t} = \frac{1}{\pi} \times 1 = \frac{1}{\pi}.$$

16. $\lim_{x \to 0} \frac{\cos x}{\pi - x}$.

Sol. $\lim_{x \to 0} \frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$.

17. $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$.

Sol. $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{2\cos^2 x - 1 - 1}{\cos x - 1}$

$$= \lim_{x \to 0} \frac{2(\cos^2 x - 1)}{(\cos x - 1)}$$

Forming factors of numerator,

$$= \lim_{x \to 0} \frac{2(\cos x + 1)(\cos x - 1)}{\cos x - 1}$$

$$= \lim_{x \to 0} 2(\cos x + 1)$$

$$= 2(\cos 0 + 1) = 2\,(1 + 1) = 4$$

18. $\lim_{x \to 0} \dfrac{ax + x \cos x}{b \sin x}$.

Sol. Dividing numerator and denominator by x

$$\lim_{x \to 0} \frac{ax + x\cos x}{b \sin x} = \lim_{x \to 0} \frac{a + \cos x}{b\left(\frac{\sin x}{x}\right)} = \frac{\lim_{x \to 0} (a + \cos x)}{\lim_{x \to 0} b\left(\frac{\sin x}{x}\right)}$$

$$= \frac{a+1}{b(1)} = \frac{a+1}{b}.$$

19. $\lim_{x \to 0} x \sec x$.

Sol. $\lim_{x \to 0} x \sec x = \lim_{x \to 0} \dfrac{x}{\cos x} = \dfrac{0}{\cos 0} = \dfrac{0}{1} = 0.$

20. $\lim_{x \to 0} \dfrac{\sin ax + bx}{ax + \sin bx}$.

Sol. Dividing numerator and denominator by x

$$\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx} = \lim_{x \to 0} \frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}}$$

$$= \lim_{x \to 0} \frac{a\left(\frac{\sin ax}{ax}\right) + b}{a + b\left(\frac{\sin bx}{bx}\right)}$$

$$= \frac{\lim_{ax \to 0} \left[a\left(\frac{\sin ax}{ax}\right) + b\right]}{\lim_{bx \to 0} \left[a + b\left(\frac{\sin bx}{bx}\right)\right]}$$

[$\because$ as $x \to 0$, $ax \to 0$ and $bx \to 0$]

$$= \frac{a(1) + b}{a + b(1)} = \frac{a+b}{a+b} = 1.$$

21. $\lim_{x \to 0} (\text{cosec } x - \cot x)$.

Sol. $\lim_{x \to 0} (\text{cosec } x - \cot x) = \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$

(on changing all T-ratios in terms of sin x and cos x)

Taking L.C.M; $= \lim_{x \to 0} \frac{1 - \cos x}{\sin x}$

Rationalising the numerator, $= \lim_{x \to 0} \left(\frac{1-\cos x}{\sin x} \times \frac{1+\cos x}{1+\cos x}\right)$

$= \lim_{x \to 0} \frac{1-\cos^2 x}{\sin x(1+\cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{\sin x(1+\cos x)}$

Cancelling sin x, $= \lim_{x \to 0} \frac{\sin x}{(1+\cos x)} = \frac{\sin 0}{(1+\cos 0)} = \frac{0}{(1+1)}$

$= \frac{0}{2} = 0$

22. $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$.

Sol. Put $x = \frac{\pi}{2} + t$ so that $t = x - \frac{\pi}{2}$ and as $x \to \frac{\pi}{2}$, $t \to 0$

$\therefore \quad \lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} = \lim_{t \to 0} \frac{\tan(\pi + 2t)}{t}$

$= \lim_{t \to 0} \frac{\tan 2t}{t} = \lim_{t \to 0} 2 . \frac{\tan 2t}{2t}$

[$\because$ $\tan(\pi + \theta) = \tan \theta$]

$= 2 \lim_{2t \to 0} \frac{\tan 2t}{2t} = 2 \times 1 = 2.$

23. Find $\lim_{x \to 0} f(x)$ **and** $\lim_{x \to 1} f(x)$**, where** $f(x)$

$$= \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}.$$

Sol. In the neighbourhood of 0, $f(x)$ is defined differently. Therefore, we shall find both left hand limit and right hand limit.

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x + 3)$

[$\because$ when $x \to 0^-$, $x < 0$ and $f(x) = 2x + 3$ (given)]

$= 2 \times 0 + 3 = 3$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3(x + 1)$$

[$\because$ when $x \to 0^+$, $x > 0$ and $f(x) = 3(x + 1)$ (given)]

$= 3(0 + 1) = 3$

$\Rightarrow \quad \lim_{x \to 0^-} f(x) = 3 = \lim_{x \to 0^+} f(x)$

$\therefore \quad \lim_{x \to 0} f(x)$ exists and $= 3$

In the neighbourhood of 1, $f(x) = 3(x + 1)$

[$\because x \to 1^-$ or $x \to 1^+ \Rightarrow x > 0$]

$\therefore \quad \lim_{x \to 1} f(x) = \lim_{x \to 1} 3(x + 1) = 3(1 + 1) = 6.$

24. Find $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$.

Sol. Here $f(x)$ is defined differently in the neighbourhood of 1. Therefore, we shall find both left hand limit and right hand limit.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - 1)$$

[$\because$ when $x \to 1^-$, $x < 1$ and $f(x) = x^2 - 1$]

$= 1^2 - 1 = 0$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x^2 - 1)$$

[$\because$ when $x \to 1^+$, $x > 1$ and $f(x) = -x^2 - 1$]

$= -1^2 - 1 = -2$

Since $\lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$, therefore, $\lim_{x \to 1} f(x)$ does not exist.

25. Evaluate $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

Sol. We know that $|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$

We shall find both left hand limit and right hand limit.

Now $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$

$[x \to 0^- \Rightarrow x < 0 \Rightarrow |x| = -x]$

Again $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} (1) = 1$

$[x \to 0^+ \Rightarrow x > 0 \Rightarrow |x| = x]$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, therefore, $\lim_{x \to 0} f(x)$ does not exist.

26. Find $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Sol. L.H.L. $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{|x|}$

$= \lim_{x \to 0^-} \frac{x}{-x}$ $[\because x \to 0^- \Rightarrow x < 0 \therefore |x| = -x]$

$= \lim_{x \to 0^-} (-1) = -1$

R.H.L. $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{|x|}$

$= \lim_{x \to 0^+} \frac{x}{x}$ $[\because x \to 0^+ \Rightarrow x > 0 \therefore |x| = x]$

$= \lim_{x \to 0^+} 1 = 1$

Since L.H.L. $\neq$ R.H.L., therefore, $\lim_{x \to 0} f(x)$ does not exist.

27. Find $\lim_{x \to 5} f(x)$, where $f(x) = |x| - 5$.

Sol. L.H.L. $= \lim_{x \to 5^-} (|x| - 5)$

$= \lim_{x \to 5^-} (x - 5)$ $[\because x \to 5^-$ x is slightly less than 5 $\Rightarrow x > 0 \therefore |x| = x]$

$= 5 - 5 = 0$

R.H.L. = $\lim_{x \to 5^+} (|x| - 5)$

$= \lim_{x \to 5^+} (x - 5)$ [$\because x \to 5^+ \Rightarrow x > 0 \therefore |x| = x$]

$= 5 - 5 = 0$

Since L.H.L. = 0 = R.H.L.

$\therefore$ $\lim_{x \to 5} f(x)$ exists and = 0.

28. Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and if $\lim_{x \to 1} f(x) = f(1)$ **what are possible values of** a **and** b**?**

Sol. Here $f(x)$ is defined differently in the neighbourhood of 1. Therefore, we shall find both left hand limit and right hand limit.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (a + bx) = a + b \times 1 = a + b$$

Again $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (b - ax) = b - a \times 1 = b - a$

Also $f(1) = 4$

$\because$ $\lim_{x \to 1} f(x) = f(1)$ (given),

$\therefore$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 4$

$\therefore$ $\lim_{x \to 1^-} f(x) = 4$ and $\lim_{x \to 1^+} f(x) = 4$

$\Rightarrow$ $a + b = 4$ and $b - a = 4$

Adding, $2b = 8$ $\therefore$ $b = 4$

Putting $b = 4$ in $a + b = 4$, we have $a + 4 = 4$ or $a = 0$.

29. Let $a_1, a_2, ..., a_n$ **be fixed real numbers and define a function**

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n).$$

What is $\lim_{x \to a_i} f(x)$**? For some** $a \neq a_1, a_2, \dots a_n$**, compute** $\lim_{x \to a} f(x)$**.**

Sol. $f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$ ...(i)

Let us find $\lim\limits_{x \to a_i} f(x)$ for $i = 1$

i.e., $\lim\limits_{x \to a_1} f(x) = \lim\limits_{x \to a_1} (x - a_1)(x - a_2) \dots (x - a_n)$

Putting $x = a_1$

$= (a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n)$

$= 0 (a_1 - a_2) \dots (a_1 - a_n) = 0$

Again $\lim\limits_{x \to a_i} f(x) = \lim\limits_{x \to a_i} (x - a_1)(x - a_2) \dots (x - a_i)$
$\dots (x - a_n)$

Putting $x = a_i$

$= (a_i - a_1)(a_i - a_2) \dots (a_i - a_i) \dots (a_i - a_n)$

$= (a_i - a_1)(a_i - a_2) \dots 0 \dots (a_i - a_n) = 0$

for all $i = 1, 2, 3, \dots, n$.

Again $\lim\limits_{x \to a} f(x)$ for some $a \neq a_1, a_2, \dots, a_n$

$\lim\limits_{x \to a} (x - a_1)(x - a_2) \dots (x - a_n)$

Putting $x = a$

$= (a - a_1)(a - a_2) \dots (a - a_n)$.

30. If $f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$

For what value(s) of a does $\lim\limits_{x \to a} f(x)$ exist?

Sol. We know that $|x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$

$\therefore \quad f(x) = \begin{cases} -x+1, & x<0 \\ 0, & x=0 \\ x-1, & x>0 \end{cases}$

Since $a \in \mathbf{R}$, three cases arise.

Case 1. When $a < 0$, $f(x) = -x + 1$

$\therefore \quad \lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} (-x + 1) = -a + 1$

$\Rightarrow \lim_{x \to a} f(x)$ exists for all $a < 0$.

Case 2. When $a > 0$, $f(x) = x - 1$

$\therefore \lim_{x \to a} f(x) = \lim_{x \to a} (x - 1) = a - 1$

$\Rightarrow \lim_{x \to a} f(x)$ exists for all $a > 0$.

Case 3. When $a = 0$.

$$\lim_{x \to a^-} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x + 1) = -0 + 1 = 1.$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x - 1) = 0 - 1 = -1.$$

Since $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ for $a = 0$, $\lim_{x \to a} f(x)$ does not exist.

Hence $\lim_{x \to a} f(x)$ exists for all $a \neq 0$.

31. If the function $f(x)$ satisfies $\lim_{x \to 1} \dfrac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \to 1} f(x)$.

Sol. Given: $$\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$$

$$\therefore \quad \frac{\lim_{x \to 1} (f(x) - 2)}{\lim_{x \to 1} (x^2 - 1)} = \pi$$

But $\lim_{x \to 1} (x^2 - 1) = 1^2 - 1 = 1 - 1 = 0$, therefore we must have $\lim_{x \to 1} (f(x) - 2) = 0$,

because if $\lim_{x \to 1} (f(x) - 2)$ is non-zero, then the given limit becomes $\dfrac{\text{non-zero}}{0} = \infty$ which does not exist and hence can't be π (given).

$\therefore \lim_{x \to 1} f(x) - \lim_{x \to 1} 2 = 0$

$\Rightarrow \lim_{x \to 1} f(x) = \lim_{x \to 1} 2 = 2.$ $\qquad \therefore \lim_{x \to 1} f(x) = 2.$

[**Remark :** If $\lim_{x \to a} \frac{g(x)}{h(x)} = a$ real number l and $\lim_{x \to a} h(x) = h(a) = 0$, then $\lim_{x \to a} g(x)$ must be 0.]

32. If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1. \\ nx^3 + m, & x > 1 \end{cases}$

For what integers m and n does both $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ exist?

Sol. Here $f(x)$ is defined differently in the neighbourhood of 0 as well as 1.

Therefore, we shall find both left hand limit and right hand limit both for $x = 0$ and $x = 1$.

Now $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (mx^2 + n) = m \times 0^2 + n = n$

Again $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (nx + m) = n \times 0 + m = m$

$\because$ $\lim_{x \to 0} f(x)$ exist (given),

$\therefore$ $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ $\therefore$ $n = m$ ...(*i*)

Also $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (nx + m) = n \times 1 + m = n + m$

Again $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (nx^3 + m) = n \times 1^3 + m = n + m$

Here $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = n + m$, therefore, $\lim_{x \to 1} f(x)$ exists for all values of m and n and $\lim_{x \to 1} f(x) = m + n$

From (*i*), we conclude that m and n must be equal integers.

EXERCISE 13.2 (*Page No.: 312–313*)

1. Find the derivative of $x^2 - 2$ at $x = 10$.

Sol. Here $f(x) = x^2 - 2$, $(x=)a = 10$.

$\because f(a + h) = f(10 + h) = (10 + h)^2 - 2$

$= 100 + h^2 + 20h - 2 = h^2 + 20h + 98$

and $f(a) = f(10) = (10)^2 - 2 = 100 - 2 = 98$

We know that $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$

$\therefore f'(10) = \lim\limits_{h\to 0} \dfrac{f(10+h)-f(10)}{h}$

$= \lim\limits_{h\to 0} \dfrac{h^2+20h+98-98}{h} = \lim\limits_{h\to 0} \dfrac{h^2+20h}{h}$

$= \lim\limits_{h\to 0} \dfrac{h(h+20)}{h}$

cancellling h, $= \lim\limits_{h\to 0} (h + 20) = 0 + 20 = 20$

2. Find the derivative of $99x$ at $x = 100$.

Sol. Here, $f(x) = 99x$, $(x =)\ a = 100$

$f(a+h) = f(100 + h) = 99\,(100 + h), f(a) = f(100) = 99\,(100)$

$f'(100) = \lim\limits_{h\to 0} \dfrac{f(100+h)-f(100)}{h}$

$\left|\because f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}\right.$

$= \lim\limits_{h\to 0} \dfrac{99(100+h)-99(100)}{h}$

$= \lim\limits_{h\to 0} \dfrac{9900+99h-9900}{h} = \lim\limits_{h\to 0} \dfrac{99h}{h}$

cancelling h, $= \lim\limits_{h\to 0}\ 99 = 99$

Hence the derivative of $99x$ at $x = 100$ is 99.

3. Find the derivative of x at $x = 1$.

Sol. Here $f(x) = x$, $a = 1$.

$f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{(1+h)-1}{h} = \lim\limits_{h\to 0} \dfrac{h}{h}$

$= \lim\limits_{h\to 0}\ 1 = 1.$

4. Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ ***(ii)*** $(x - 1)(x - 2)$

(iii) $\dfrac{1}{x^2}$ ***(iv)*** $\dfrac{x+1}{x-1}$.

Sol. (*i*) Let $f(x) = x^3 - 27$.

Changing x to $x + h$, $f(x + h) = (x + h)^3 - 27$.

We know that $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - 27 - x^3 + 27}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + h^3 + 3xh(x+h) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{h^3 + 3x^2h + 3xh^2}{h} = \lim_{h \to 0} \frac{h(h^2 + 3x^2 + 3xh)}{h}$$

$$= \lim_{h \to 0} (h^2 + 3x^2 + 3xh) = 0 + 3x^2 + 0 = 3x^2$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 3x^2.$$

Hence $\dfrac{d}{dx}(x^3 - 27) = 3x^2$.

(*ii*) Let $f(x) = (x - 1)(x - 2) = x^2 - 3x + 2$.

Changing x to $x + h$, $f(x + h) = (x + h)^2 - 3(x + h) + 2$.

We know that $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{[(x+h)^2 - 3(x+h) + 2] - (x^2 - 3x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + h^2 + 2hx - 3x - 3h + 2 - x^2 + 3x - 2)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2hx - 3h}{h} = \lim_{h \to 0} \frac{h(h + 2x - 3)}{h}$$

$$= \lim_{h \to 0} (h + 2x - 3) = 0 + 2x - 3 = 2x - 3$$

$$\Rightarrow \frac{d}{dx}(f(x)) = 2x - 3. \text{ Hence } \frac{d}{dx}((x-1)(x-2)) = 2x - 3.$$

(*iii*) Let $f(x) = \dfrac{1}{x^2}$.

Changing x to $x + h$, $f(x + h) = \dfrac{1}{(x+h)^2}$.

We know that $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$$

Taking L.C.M.

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{x^2 (x+h)^2}\right] = \lim_{h \to 0} \frac{x^2 - (x^2 + h^2 + 2xh)}{hx^2 (x+h)^2}$$

$$= \lim_{h \to 0} \frac{-2xh - h^2}{hx^2 (x+h)^2} = \lim_{h \to 0} \frac{-h(2x+h)}{hx^2 (x+h)^2}$$

$$= \lim_{h \to 0} \frac{-(2x+h)}{x^2 (x+h)^2} = \frac{-(2x+0)}{x^2 (x+0)^2} = \frac{-2x}{x^2 (x)^2} = \frac{-2}{x^3}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \frac{-2}{x^3} \qquad \text{Hence } \frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}.$$

(*iv*) Here $f(x) = \dfrac{x+1}{x-1}$

Changing x to $x + h$, $f(x + h) = \dfrac{x+h+1}{x+h-1}$

We know that $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x+h)+1}{(x+h)-1} - \frac{x+1}{x-1}\right]$$

Taking L.C.M,

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x+1+h)(x-1) - (x+1)(x-1+h)}{(x+h-1)(x-1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{[(x^2 - x + x - 1 + hx - h - x^2 + x - hx - x + 1 - h}{(x+h-1)(x-1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-2h}{(x+h-1)(x-1)} \right]$$

cancelling h, $= \lim_{h \to 0} \frac{-2}{(x+h-1)(x-1)}$

$$= \frac{-2}{(x+0-1)(x-1)} = \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{x+1}{x-1}\right) = -\frac{2}{(x-1)^2}.$$

5. For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1,$$

prove that $f'(1) = 100 f'(0)$.

Sol. Given $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$

$$\Rightarrow \quad f'(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0$$

$\left[\because \frac{d}{dx}x^n = nx^{n-1}, \frac{d}{dx}(c) = 0, \frac{d}{dx}(c\,f(x)) = c\frac{d}{dx}f(x)\right]$

or $\quad f'(x) = x^{99} + x^{98} + \dots + x + 1 \quad$ ...(*i*)

Putting $x = 0$ and $x = 1$ in (*i*), we have

$f'(0) = 0 + 0 + \dots + 0 + 1 = 1 \quad$ ...(*ii*)

and $\quad f'(1) = 1^{99} + 1^{98} + \dots + 1 + 1 \quad$ (100 terms)

$= 1 + 1 + \dots + 1 + 1 = 100$

$= 100 \times 1$

$\therefore \quad f'(1) = 100 f'(0) \quad$ [By (*ii*)]

6. Find the derivatives of $x^n + ax^{n-1} + a^2x^{n-2} + \dots a^{n-1}x + a^n$ for some fixed real number a.

Sol. $f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$

$$\Rightarrow \quad f'(x) = \frac{d}{dx}(x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n)$$

$$= \frac{d}{dx}(x^n) + a\frac{d}{dx}(x^{n-1}) + a^2\frac{d}{dx}(x^{n-2}) + \dots + a^{n-1}\frac{d}{dx}(x) + \frac{d}{dx}(a^n)$$

$$= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1}(1) + 0$$

$$= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1}.$$

7. For some constants a and b, find the derivative of

(i) $(x-a)(x-b)$ (ii) $(ax^2+b)^2$ (iii) $\frac{x-a}{x-b}$.

Sol. (i) Let $f(x) = (x-a)(x-b)$; | ***uv* form**

Applying product rule,

then $f'(x) = \frac{d}{dx}(x-a) \times (x-b) + (x-a) \times \frac{d}{dx}(x-b)$

$= 1 \times (x-b) + (x-a) \times 1 = x - b + x - a$

$= 2x - (a+b).$

(ii) Let $f(x) = (ax^2+b)^2 = a^2x^4 + 2abx^2 + b^2$, then

$$f'(x) = a^2\frac{d}{dx}(x^4) + 2ab\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

$= a^2(4x^3) + 2ab(2x) + 0$

[$\because$ b is constant (given) $\Rightarrow b^2$ is also constant. For example $3^2 = 9$ = constant]

$= 4a^2x^3 + 4abx = 4ax(ax^2+b).$

(iii) Here $f(x) = \frac{x-a}{x-b}$

Applying quotient rule, then

$$f'(x) = \frac{\frac{d}{dx}(x-a).(x-b) - (x-a).\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{1.(x-b)-(x-a).1}{(x-b)^2} = \frac{(x-b)-(x-a)}{(x-a)^2}$$

$$= \frac{x-b-x+a}{(x-b)^2} = \frac{a-b}{(x-b)^2}.$$

8. Find the derivative of $\frac{x^n - a^n}{x-a}$ for some constant a.

Sol. Let $f(x) = \frac{x^n - a^n}{x-a}$, then by applying quotient rule

$$f'(x) = \frac{\frac{d}{dx}(x^n - a^n).(x-a) - (x^n - a^n).\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$= \frac{nx^{n-1}.(x-a) - (x^n - a^n).1}{(x-a)^2}$$

$$\left[\because a^n \text{ is a constant, } \therefore \frac{d}{dx}a^n = 0\right]$$

$$= \frac{nx^n - nax^{n-1} - x^n + a^n}{(x-a)^2}$$

$$(\because x^{n-1}.\, x = x^{n-1}.\, x^1 = x^{n-1+1} = x^n)$$

$$= \frac{(n-1)x^n - nax^{n-1} + a^n}{(x-a)^2}.$$

9. Find the derivative of

(i) $2x - \frac{3}{4}$ **(ii)** $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$ **(iv)** $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$ **(vi)** $\frac{2}{x+1} - \frac{x^2}{3x-1}$.

Sol. (i) Let $f(x) = 2x - \frac{3}{4}$, then

$$f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$$

$$= \frac{d}{dx}(2x) - \frac{d}{dx}\left(\frac{3}{4}\right)$$

$$= 2\frac{d}{dx}(x) - 0 = 2(1) = 2.$$

(ii) Here $f(x) = (5x^3 + 3x - 1)(x - 1)$

Applying product rule of differentiation

$$f'(x) = \frac{d}{dx}(5x^3 + 3x - 1) \times (x - 1) + (5x^3 + 3x - 1) \times \frac{d}{dx}(x - 1)$$

$$= (15x^2 + 3)(x - 1) + (5x^3 + 3x - 1) \times 1$$

$$= 15x^3 - 15x^2 + 3x - 3 + 5x^3 + 3x - 1$$
$$= 20x^3 - 15x^2 + 6x - 4.$$

(*iii*) Here $f(x) = x^{-3}(5 + 3x)$

$$= 5x^{-3} + 3x^{-2}, \text{ then}$$

$$f'(x) = \frac{d}{dx}(5x^{-3} + 3x^{-2}) = 5\frac{d(x^{-3})}{dx} + 3\frac{d(x^{-2})}{dx}$$
$$= 5(-3x^{-4}) + 3(-2x^{-3})$$
$$= -\frac{15}{x^4} - \frac{6}{x^3} = -\frac{15+6x}{x^4}$$
$$= -\frac{3}{x^4}(5 + 2x).$$

(*iv*) Let $f(x) = x^5(3 - 6x^{-9}) = 3x^5 - 6x^{-4}$, then

$$f'(x) = 3\frac{d}{dx}(x^5) - 6\frac{d}{dx}(x^{-4})$$
$$= 3(5x^4) - 6(-4x^{-5}) = 15x^4 + \frac{24}{x^5}.$$

(*v*) Let $f(x) = x^{-4}(3 - 4x^{-5}) = 3x^{-4} - 4x^{-9}$, then

$$f'(x) = 3\frac{d}{dx}(x^{-4}) - 4\frac{d}{dx}(x^{-9})$$
$$= 3(-4x^{-5}) - 4(-9x^{-10}) = -\frac{12}{x^5} + \frac{36}{x^{10}}$$
$$= \frac{12}{x^5}\left(\frac{3}{x^5} - 1\right).$$

(*vi*) Let $f(x) = \dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

$$\therefore f'(x) = 2\frac{d}{dx}\left(\frac{1}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

Applying quotient rule,

$$f'(x) = 2\left[\frac{\frac{d}{dx}(1).(x+1) - 1\frac{d}{dx}(x+1)}{(x+1)^2}\right]$$

$$- \left[\frac{\frac{d}{dx}(x^2).(3x-1) - x^2 \frac{d}{dx}(3x-1)}{(3x-1)^2} \right]$$

$$= 2\left[\frac{0(x+1)-1.1}{(x+1)^2}\right] - \left[\frac{2x(3x-1)-x^2.3}{(3x-1)^2}\right]$$

$$= 2\left(\frac{-1}{(x+1)^2}\right) - \left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \frac{(3x^2-2x)}{(3x-1)^2}.$$

10. Find the derivative of cos x from first principle.

Sol. Let $f(x) = \cos x$.

Changing x to $x + h$, $f(x + h) = \cos(x + h)$

We know that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{2.\frac{h}{2}}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \lim_{h \to 0} -\sin\left(x+\frac{h}{2}\right).\frac{\sin h/2}{h/2}$$

$$= -\lim_{h \to 0} \sin\left(x+\frac{h}{2}\right).\lim_{\frac{h}{2} \to 0}\frac{\sin h/2}{h/2}$$

$$= -\sin(x+0).1 = -\sin x$$

$$\left[\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$\Rightarrow \frac{d}{dx}(f(x)) = -\sin x$ Hence $\frac{d}{dx}(\cos x) = -\sin x$.

11. Find the derivative of the following functions:

(i) $\sin x \cos x$ **(ii)** $\sec x$

(iii) $5 \sec x + 4 \cos x$ **(iv)** $\operatorname{cosec} x$

(v) $3 \cot x + 5 \operatorname{cosec} x$ **(vi)** $5 \sin x - 6 \cos x + 7$

(vii) $2 \tan x - 7 \sec x$.

Sol. (i) Let $f(x) = \sin x \cos x$; | *uv* **form**

Applying Product rule,

$$\text{then } f'(x) = \frac{d}{dx}(\sin x \cos x)$$

$$= \frac{d}{dx}(\sin x) \times \cos x + \sin x \times \frac{d}{dx}(\cos x)$$

$$= \cos x \, . \cos x + \sin x(-\sin x)$$

$$= \cos^2 x - \sin^2 x = \cos 2x.$$

(ii) Here $f(x) = \sec x = \dfrac{1}{\cos x}$

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}(1) \times \cos x - 1 \times \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{0 \times \cos x - (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x.$$

(iii) Here $f(x) = 5 \sec x + 4 \cos x$

$$\therefore \quad f'(x) = 5 \frac{d}{dx}(\sec x) + 4 \frac{d}{dx}(\cos x)$$

$$= 5(\sec x \tan x) + 4(-\sin x)$$

$$= 5 \sec x \tan x - 4 \sin x.$$

(iv) Here $f(x) = \operatorname{cosec} x = \dfrac{1}{\sin x}$

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}(1) \times \sin x - 1 \times \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{0 \times \sin x - \cos x}{\sin^2 x} = - \frac{\cos x}{\sin^2 x}$$

$$= - \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = - \text{cosec } x \cot x.$$

(v) Here $f(x) = 3 \cot x + 5 \text{ cosec } x$

$\therefore \quad f'(x) = 3(- \text{cosec}^2 x) + 5(- \text{cosec } x \cot x)$

$= - 3 \text{ cosec}^2 x - 5 \text{ cosec } x \cot x.$

(vi) Here $f(x) = 5 \sin x - 6 \cos x + 7$

$\therefore \quad f'(x) = 5(\cos x) - 6(- \sin x) + 0$

$= 5 \cos x + 6 \sin x.$

(vii) Here $f(x) = 2 \tan x - 7 \sec x$

$\therefore \quad f'(x) = 2 \sec^2 x - 7 \sec x \tan x.$

MISCELLANEOUS EXERCISE ON CHAPTER 13

(Page No.: 317–318)

1. Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin (x + 1)$ (iv) $\cos \left(x - \frac{\pi}{8}\right)$.

Sol. (i) Here $f(x) = -x$

Changing x to $x + h$, $f(x + h) = -(x + h)$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h} = \lim_{h \to 0} \frac{(-x-h+x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h}$$

cancelling h, $= \lim_{h \to 0} (-1) = -1.$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = -\frac{1}{x}$.

Changing x to $x + h$, $f(x + h) = \frac{-1}{x+h}$.

We know that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{-\frac{1}{x+h} - \left(-\frac{1}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[-\frac{1}{x+h} + \frac{1}{x}\right]$$

Taking L.C.M.

or $\quad f'(x) = \lim_{h \to 0} \frac{-x + x + h}{h(x+h)(x)}$

$$= \lim_{h \to 0} \frac{h}{h(x+h)(x)}$$

cancelling h, $= \lim_{h \to 0} \frac{1}{(x+h)(x)} = \frac{1}{(x+0)x} = \frac{1}{(x)(x)} = \frac{1}{x^2}$

$\Rightarrow \frac{d}{dx}(f(x)) = \frac{1}{x^2}$ Hence $\frac{d}{dx}((-x)^{-1}) = \frac{1}{x^2}$.

(*iii*) Here $f(x) = \sin(x + 1)$

changing x to $x + h$, $f(x + h) = \sin(x + h + 1)$

$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sin(x+h+1) - \sin(x+1)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\frac{(x+h+1)+(x+1)}{2}\sin\frac{(x+h+1)-(x+1)}{2}}{h}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+2+h}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(x+1+\frac{h}{2}\right)\sin\frac{h}{2}}{2.\frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\left(x+1+\frac{h}{2}\right) . \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\left(x+1+\frac{h}{2}\right) \times \lim_{\frac{h}{2} \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}$$

$$= \cos(x + 1 + 0) \times 1 = \cos(x + 1).$$

(*iv*) Let $f(x) = \cos\left(x-\frac{\pi}{8}\right)$

Changing x to $x + h$; $f(x + h) = \cos\left(x+h-\frac{\pi}{8}\right)$.

We know that $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \frac{\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{x+h-\frac{\pi}{8}+x-\frac{\pi}{8}}{2}\right)\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)}{h}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}}{2.\frac{h}{2}}$$

$$= \lim_{h \to 0} \left[-\sin\left(x+\frac{h}{2}-\frac{\pi}{8}\right).\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]$$

$$= -\lim_{h \to 0} \sin\left(x+\frac{h}{2}-\frac{\pi}{8}\right).\lim_{\frac{h}{2} \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\sin\left(x+0-\frac{\pi}{8}\right).1 = -\sin(x-\frac{\pi}{8}) \quad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$\Rightarrow \frac{d}{dx}(f(x)) = -\sin\left(x-\frac{\pi}{8}\right)$$

Hence $\frac{d}{dx}\left[\cos\left(x-\frac{\pi}{8}\right)\right] = -\sin\left(x-\frac{\pi}{8}\right)$.

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, *q*, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

2. $(x + a)$.

Sol. Here $f(x) = x + a$

$\Rightarrow \quad f'(x) = 1 + 0 = 1.$

3. $(px + q)\left(\frac{r}{x}+s\right)$.

Sol. Let $f(x) = (px + q)\left(\frac{r}{x}+s\right) = pr + psx + \frac{qr}{x} + qs$, then

$$f'(x) = \frac{d}{dx}(pr) + ps\frac{d}{dx}(x) + qr\frac{d}{dx}(x^{-1}) + \frac{d}{dx}(qs)$$
$$= 0 + ps\,(1) + qr\,(-1x^{-2}) + 0$$
$$= ps - \frac{qr}{x^2}.$$

4. $(ax + b)(cx + d)^2$.

Sol. Here $f(x) = (ax + b)(cx + d)^2$.

Applying product rule of differentiation

$$f'(x) = \frac{d}{dx}(ax + b) \times (cx + d)^2 + (ax + b) \times \frac{d}{dx}(cx+d)^2$$
$$= (a \times 1 + 0) \times (cx + d)^2 + (ax + b) \times 2(cx + d) \times \frac{d}{dx}(cx + d)$$

[**By chain rule:** here $\frac{d}{dx}u^n = nu^{n-1}\frac{d}{dx}u$]

$$= a(cx + d)^2 + 2(ax + b)(cx + d) \times c \quad \left|\because \frac{d}{dx}(cx+d) = c\times 1+0 = c\right.$$
$$= a(cx + d)^2 + 2c(ax + b)(cx + d).$$

5. $\frac{ax+b}{cx+d}$.

Sol. Here $f(x) = \frac{ax+b}{cx+d}$.

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}(ax+b) \times (cx+d) - (ax+b) \times \frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$= \frac{a(cx+d)-(ax+b)c}{(cx+d)^2} = \frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2}.$$

6. $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$.

Sol. Let $f(x) = \dfrac{1+\frac{1}{x}}{1-\frac{1}{x}} = \dfrac{\frac{x+1}{x}}{\frac{x-1}{x}} = \dfrac{x+1}{x-1}$

Applying quotient rule of differentiation,

$$\therefore \quad f'(x) = \frac{\frac{d}{dx}(x+1)\,.\,(x-1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{1\,.(x-1)-(x+1)\,.1}{(x-1)^2} = \frac{-2}{(x-1)^2}.$$

7. $\dfrac{1}{ax^2+bx+c}$.

Sol. Let $f(x) = \dfrac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x) = \frac{\frac{d}{dx}(1)\,.\,(ax^2+bx+c)-1\,.\,\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

$$= \frac{(0)(ax^2+bx+c)-(2ax+b)}{(ax^2+bx+c)^2}$$

$$= -\frac{2ax+b}{(ax^2+bx+c)^2}$$

8. $\dfrac{ax+b}{px^2+qx+r}$.

Sol. Here $f(x) = \dfrac{ax+b}{px^2+qx+r}$.

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}(ax+b)\times(px^2+qx+r)-(ax+b)\times\frac{d}{dx}(px^2+qx+r)}{(px^2+qx+r)^2}$$

$$= \frac{a(px^2+qx+r)-(ax+b)(2px+q)}{(px^2+qx+r)^2}$$

$$= \frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{(px^2+qx+r)^2}$$

$$= \frac{-apx^2-2bpx+ar-bq}{(px^2+qx+r)^2}.$$

9. $\dfrac{px^2+qx+r}{ax+b}$.

Sol. Here $f(x) = \dfrac{px^2+qx+r}{ax+b}$.

Applying quotient rule of differentiation

$$f'(x) = \frac{\frac{d}{dx}(px^2+qx+r)\times(ax+b)-(px^2+qx+r)\times\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$= \frac{(2px+q)(ax+b)-(px^2+qx+r)(a)}{(ax+b)^2}$$

$$= \frac{2apx^2+2bpx+aqx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$= \frac{apx^2+2bpx+bq-ar}{(ax+b)^2}.$$

10. $\dfrac{a}{x^4} - \dfrac{b}{x^2} + \cos x$.

Sol. Let $f(x) = \dfrac{a}{x^4} - \dfrac{b}{x^2} + \cos x = ax^{-4} - bx^{-2} + \cos x$, then

$$f'(x) = a\frac{d}{dx}(x^{-4}) - b\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(\cos x)$$

$$= a(-4x^{-5}) - b(-2x^{-3}) + (-\sin x)$$

$$= -\frac{4a}{x^5} + \frac{2b}{x^3} - \sin x.$$

11. $4\sqrt{x} - 2.$

Sol. Let $f(x) = 4\sqrt{x} - 2$, then

$$f'(x) = \frac{d}{dx}(4\sqrt{x}) - \frac{d}{dx}(2)$$

$$= 4\frac{d}{dx}(x^{1/2}) - 0 = 4\left(\frac{1}{2}x^{1/2-1}\right)\left[\because \frac{d}{dx}x^n = nx^{n-1}\right]$$

$$= 2x^{-\frac{1}{2}} = \frac{2}{x^{\frac{1}{2}}} = \frac{2}{\sqrt{x}}.$$

12. $(ax + b)^n.$

Sol. Let $f(x) = (ax + b)^n$

Differentiating both sides w.r.t x,

$$f'(x) = \frac{d}{dx}(ax + b)^n$$

$$= n(ax + b)^{n-1}\frac{d}{dx}(ax + b)$$

[**By chain rule:** Here $\frac{d}{dx}u^n = nu^{n-1}\frac{d}{dx}u$ where u is a function of x]

$\Rightarrow f'(x) = n(ax + b)^{n-1}[a(1) + 0] = n(ax + b)^{n-1}a$

$= na(ax + b)^{n-1}$

13. $(ax + b)^n (cx + d)^m.$

Sol. Let $f(x) = (ax + b)^n(cx + d)^m$, then

$$f'(x) = \frac{d}{dx}[(ax + b)^n (cx + d)^m]$$

Applying product rule,

$$= \frac{d}{dx}(ax + b)^n \cdot (cx + d)^m + (ax + b)^n \cdot \frac{d}{dx}(cx + d)^m$$

$$= n(ax + b)^{n-1}\frac{d}{dx}(ax + b).(cx + d)^m + (ax + b)^n m(cx + d)^{m-1}$$

$$\frac{d}{dx}(cx + d)$$

[**By chain rule:** here $\frac{d}{dx} u^n = nu^{n-1} \frac{d}{dx} u$]

$= na(ax + b)^{n-1}(cx + d)^m + (ax + b)^n \cdot mc(cx + d)^{m-1}$

$= na(ax + b)^{n-1}(cx + d)^{m-1} (cx + d) + (ax + b)^{n-1} (ax + b)\ m\ c\ (cx + d)^{m-1}$

[$\because$ $(cx + d)^m = (cx + d)^{m-1+1} = (cx + d)^{m-1} (cx + d)$.

similarly $(ax + b)^n = (ax + b)^{n-1} (ax + b)$]

Taking $(ax + b)^{n-1} (cx + d)^{m-1}$ common

$= (ax + b)^{n-1} (cx + d)^{m-1} [na(cx + d) + mc(ax + b)]$

$= (ax + b)^{n-1} (cx + d)^{m-1} [nacx + nad + macx + mbc]$

$= (ax + b)^{n-1} (cx + d)^{m-1} [ac(n + m)x + nad + mbc]$.

14. **sin(*x* + *a*).**

Sol. Here $f(x) = \sin (x + a) = \sin x \cos a + \cos x \sin a$.

$$\therefore\ f'(x) = \frac{d}{dx} (\sin x \cos a) + \frac{d}{dx} (\cos x \sin a)$$

$$= \cos a \frac{d}{dx} (\sin x) + \sin a \frac{d}{dx} (\cos x)$$

$$= \cos a \cos x + \sin a(-\sin x)$$

$$= \cos x \cos a - \sin x \sin a$$

$$= \cos (x + a).$$

Alternatively:

$$f'(x) = \frac{d}{dx} [\sin (x + a)] = \cos (x + a) \cdot \frac{d}{dx} (x + a)$$

$$= \cos (x + a) \times (1 + 0) = \cos (x + a) \times 1 = \cos (x + a)$$

15. **cosec *x* cot *x*.**

Sol. Let $f(x) = \text{cosec } x \cot x$; | ***uv* form**

Applying product rule,

$$\text{then } f'(x) = \frac{d}{dx}(\text{cosec } x) \times \cot x + \text{cosec } x \times \frac{d}{dx}(\cot x)$$

$$= (-\text{cosec } x \cot x) \cot x + \text{cosec } x\ (-\text{cosec}^2 x)$$

$$= -\text{cosec } x \cot^2 x - \text{cosec}^3 x$$

$$= -\text{cosec } x\ (\cot^2 x + \text{cosec}^2 x).$$

16. $\dfrac{\cos x}{1 + \sin x}$.

Sol. Here $f(x) = \dfrac{\cos x}{1 + \sin x}$.

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}(\cos x) \times (1+\sin x) - \cos x \times \frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x \times (1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{1+\sin x}.$$

17. $\dfrac{\sin x + \cos x}{\sin x - \cos x}$.

Sol. Let $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$, then by quotient rule,

$$f'(x) = \frac{\frac{d}{dx}(\sin x + \cos x).(\sin x - \cos x) - (\sin x + \cos x).\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{\sin x \cos x - \cos^2 x - \sin^2 x + \sin x \cos x - \sin x \cos x - \sin^2 x - \cos^2 x - \sin x \cos x}{(\sin x - \cos x)^2}$$

$$= \frac{-2\sin^2 x - 2\cos^2 x}{(\sin x - \cos x)^2} = \frac{-2(\sin^2 x + \cos^2 x)}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2} \quad [\because \sin^2 x + \cos^2 x = 1]$$

18. $\dfrac{\sec x - 1}{\sec x + 1}$.

Sol. Here $f(x) = \dfrac{\sec x - 1}{\sec x + 1}$

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}(\sec x - 1) \times (\sec x + 1) - (\sec x - 1) \times \frac{d}{dx}(\sec x + 1)}{(\sec x + 1)^2}$$

$$= \frac{\sec x \tan x(\sec x + 1) - (\sec x - 1)(\sec x \tan x)}{(\sec x + 1)^2}$$

$$= \frac{\sec^2 x \tan x + \sec x \tan x - \sec^2 x \tan x + \sec x \tan x}{(\sec x + 1)^2}$$

$$= \frac{2 \sec x \tan x}{(\sec x + 1)^2}.$$

19. $\sin^n x$.

Sol. Let $f(x) = \sin^n x = (\sin x)^n$

Applying chain rule of differentiation,

$$\left[\text{Here } \frac{d}{dx} u^n = n u^{n-1} \frac{d}{dx} u \text{ where } u = \sin x \text{ is a function of } x\right]$$

$$f'(x) = n (\sin x)^{n-1} \frac{d}{dx}(\sin x)$$

$$= n \sin^{n-1} x \cos x$$

20. $\dfrac{a + b \sin x}{c + d \cos x}$.

Sol. Here $f(x) = \dfrac{a + b \sin x}{c + d \cos x}$

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}(a + b \sin x) \times (c + d \cos x) - (a + b \sin x) \times \frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2}$$

$$= \frac{b \cos x(c + d \cos x) - (a + b \sin x)(- d \sin x)}{(c + d \cos x)^2}$$

$$= \frac{bc \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c + d \cos x)^2}$$

$$= \frac{bc \cos x + ad \sin x + bd(\cos^2 x + \sin^2 x)}{(c + d \cos x)^2}$$

$$= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2}.$$

21. $\frac{\sin (x + a)}{\cos x}$.

Sol. Here $f(x) = \frac{\sin (x + a)}{\cos x}$.

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}[\sin (x + a)] \times \cos x - \sin (x + a) \times \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos (x + a) \frac{d}{dx}(x + a)(\text{By chain rule}) \times \cos x - \sin (x + a)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos (x + a)(1) \cos x + \sin (x + a) \sin x}{\cos^2 x}$$

Using cos A cos B + sin A sin B = cos (A – B),

$$= \frac{\cos [(x + a) - x]}{\cos^2 x} = \frac{\cos a}{\cos^2 x}.$$

22. $x^4(5 \sin x - 3 \cos x)$.

Sol. Here $f(x) = x^4 (5 \sin x - 3 \cos x)$

Applying product rule of differentiation,

$$f'(x) = \frac{d}{dx}(x^4) \times (5 \sin x - 3 \cos x) + x^4 \times \frac{d}{dx}(5 \sin x - 3 \cos x)$$

$$= 4x^3 (5 \sin x - 3 \cos x) + x^4 (5 \cos x + 3 \sin x)$$

$$= x^3 (20 \sin x - 12 \cos x + 5x \cos x + 3 x \sin x).$$

23. $(x^2 + 1) \cos x$.

Sol. Let $f(x) = (x^2 + 1) \cos x$,

Applying product rule of differentiation,

$$\text{then } f'(x) = \frac{d}{dx}(x^2+1) \times \cos x + (x^2+1) \times \frac{d}{dx}(\cos x)$$
$$= 2x \cos x + (x^2+1)(-\sin x)$$
$$= 2x \cos x - (x^2+1) \sin x.$$

24. $(ax^2 + \sin x)(p + q \cos x)$.

Sol. Here $f(x) = (ax^2 + \sin x)(p + q \cos x)$

Applying product rule of differentiation,

$$f'(x) = \frac{d}{dx}(ax^2 + \sin x) \times (p + q \cos x) + (ax^2 + \sin x) \times \frac{d}{dx}(p + q \cos x)$$
$$= (2ax + \cos x)(p + q \cos x) + (ax^2 + \sin x)(-q \sin x)$$
$$= (2ax + \cos x)(p + q \cos x) - q \sin x(ax^2 + \sin x).$$

25. $(x + \cos x)(x - \tan x)$.

Sol. Let $f(x) = (x + \cos x)(x - \tan x)$; | ***uv* form**

Applying product rule of differentiation,

$$\text{then } f'(x) = \frac{d}{dx}(x + \cos x)(x - \tan x) + (x + \cos x) \times \frac{d}{dx}(x - \tan x)$$
$$= (1 - \sin x)(x - \tan x) + (x + \cos x)(1 - \sec^2 x).$$

26. $\dfrac{4x + 5\sin x}{3x + 7\cos x}$.

Sol. Let $f(x) = \dfrac{4x + 5\sin x}{3x + 7\cos x}$, then by quotient rule,

$$f'(x) = \frac{\frac{d}{dx}(4x + 5\sin x).(3x + 7\cos x) - (4x + 5\sin x)\frac{d}{dx}(3x + 7\cos x)}{(3x + 7\cos x)^2}$$
$$= \frac{(4 + 5\cos x)(3x + 7\cos x) - (4x + 5\sin x)(3 - 7\sin x)}{(3x + 7\cos x)^2}$$
$$= \frac{(12x + 28\cos x + 15x\cos x + 35\cos^2 x) - (12x - 28x\sin x + 15\sin x - 35\sin^2 x)}{(3x + 7\cos x)^2}$$

$$= \frac{12x + 28\cos x + 15x\cos x + 35\cos^2 x - 12x + 28x\sin x - 15\sin x + 35\sin^2 x}{(3x + 7\cos x)^2}$$

$$= \frac{28(\cos x + x\sin x) + 15(x\cos x - \sin x) + 35(\cos^2 x + \sin^2 x)}{(3x + 7\cos x)^2}.$$

$$= \frac{28(\cos x + x\sin x) + 15(x\cos x - \sin x) + 35}{(3x + 7\cos x)^2}.$$

27. $\dfrac{x^2 \cos\left(\dfrac{\pi}{4}\right)}{\sin x}$.

Sol. Let $f(x) = \dfrac{x^2 \cos\dfrac{\pi}{4}}{\sin x} = \dfrac{1}{\sqrt{2}} \dfrac{x^2}{\sin x}$

$$\therefore \quad f'(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{2}}\frac{x^2}{\sin x}\right)$$

$$= \frac{1}{\sqrt{2}} \frac{d}{dx} \frac{x^2}{\sin x}$$

Applying quotient rule,

$$= \frac{1}{\sqrt{2}}\left[\frac{\frac{d}{dx}(x^2) \cdot \sin x - x^2 \cdot \frac{d}{dx}\sin x}{\sin^2 x}\right]$$

$$= \frac{2x\sin x - x^2\cos x}{\sqrt{2}\sin^2 x}$$

$$= \frac{x(2\sin x - x\cos x)}{\sqrt{2}\sin^2 x}.$$

28. $\dfrac{x}{1 + \tan x}$.

Sol. Here $f(x) = \dfrac{x}{1 + \tan x}$.

Applying quotient rule of differentiation,

$$f'(x) = \frac{\frac{d}{dx}(x) \times (1 + \tan x) - x \times \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$= \frac{1(1 + \tan x) - x\sec^2 x}{(1 + \tan x)^2} = \frac{1 + \tan x - x\sec^2 x}{(1 + \tan x)^2}.$$

29. $(x + \sec x)(x - \tan x)$.

Sol. Here $f(x) = (x + \sec x)(x - \tan x)$

Applying product rule of differentiation,

$$f'(x) = \frac{d}{dx}(x + \sec x) \times (x - \tan x) + (x + \sec x) \times \frac{d}{dx}(x - \tan x)$$

$$= (1 + \sec x \tan x)(x - \tan x) + (x + \sec x)(1 - \sec^2 x).$$

30. $\dfrac{x}{\sin^n x}$.

Sol. Let $f(x) = \dfrac{x}{\sin^n x}$, then by quotient rule,

$$f'(x) = \frac{\frac{d}{dx}(x) . \sin^n x - x . \frac{d}{dx}(\sin^n x)}{(\sin^n x)^2}$$

$$= \frac{1 . \sin^n x - x . n \sin^{n-1} x \cos x}{\sin^{2n} x}$$

$$\left[\because \text{by chain rule, } \frac{d}{dx}\sin^n x = \frac{d}{dx}(\sin x)^n = \frac{d}{dx}u^n (\text{where } u = \sin x) = n u^{n-1} \frac{d}{dx} n \sin^{n-1} x \frac{d}{dx}\sin x = n \sin^{n-1} x \cos x\right]$$

$$= \frac{\sin^{n-1} x(\sin x - nx \cos x)}{\sin^{2n} x}$$

$[\because \sin^n x = (\sin x)^n = (\sin x)^{n-1+1} = (\sin x)^{n-1} \sin x = \sin^{n-1} x \sin x]$

$$= \frac{\sin x - nx \cos x}{\sin^{n+1} x}.$$

$(\because 2n - (n - 1) = 2n - n + 1 = n + 1)$