4 Motion in a Plane

Lesson at a Glance

- Motion in a plane is called as motion in two dimensions *e.g.*, projectile motion, circular motion etc. For the analysis of such motion our reference will be made of an origin and two co-ordinate axes X and Y.

• Scalar and Vector Quantities

Scalar Quantities. The physical quantities which are completely specified by their magnitude or size alone are called scalar quantities.

Examples. Length, mass, density, speed, work, etc.

Vector Quantities. Vector quantities are those physical quantities which are characterised by both magnitude and direction.

Examples. Velocity, displacement, acceleration, force, momentum, torque etc.

• Unit Vector

A unit vector is a vector of unit magnitude and points in a particular direction. It is used to specify the direction only. Unit vector is represented by putting a cap (∧) over the quantity.

The unit vector in the direction of $\vec{A}$ is denoted by $\hat{A}$ and defined by

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A} \text{ or } \vec{A} = A\hat{A}$$

• Equal Vectors

Vectors $\vec{A}$ and $\vec{B}$ are said to be equal if $|\vec{A}| = |\vec{B}|$ as well as their directions are same.

• Zero Vector

A vector with zero magnitude and an arbitrary direction is called a zero vector. It is represented by $\vec{0}$ and also known as null vector.

• Negative of a Vector

The vector whose magnitude is same as that of $\vec{a}$ but the direction is opposite to that of $\vec{a}$ is called the negative of $\vec{a}$ and is written as $-\vec{a}$.

- **Parallel Vectors**

$\vec{A}$ and $\vec{B}$ are said to be parallel vectors if they have same direction, and may or may not have equal magnitude $(\vec{A} \parallel \vec{B})$. If the directions are opposite, then $\vec{A}$ is anti-parallel to $\vec{B}$.

- **Coplanar Vectors**

Vectors are said to be coplanar if they lie in the same plane or they are parallel to the same plane, otherwise they are said to be non-coplanar vectors.

- **Displacement Vector**

The displacement vector is a vector which gives the position of a point with reference to a point other than the origin of the co-ordinate system.

Displacement vector $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$.

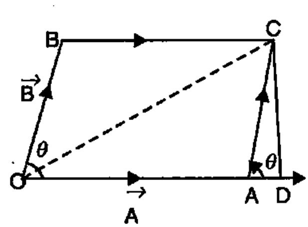
- **Parallelogram Law of Vector Addition**

If two vectors, acting simultaneously at a point, can be represented both in magnitude and direction by the two adjacent sides of a parallelogram drawn from a point, then the resultant is represented completely both in magnitude and direction by the diagonal of the parallelogram passing through that point.

If $\vec{A}$ and $\vec{B}$ be two adjacent sides of a parallelogram, inclined at angle θ, then the magnitude of resultant vector is given as

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

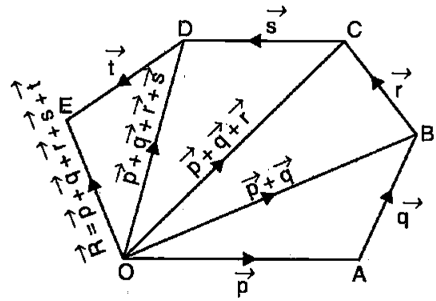
Direction of resultant $\vec{R}$. Let α be the angle made by resultant $\vec{R}$ with vector $\vec{A}$. Then



$$\alpha = \tan^{-1}\frac{B\sin\theta}{A + B\cos\theta}$$

• Polygon Law of Vector Addition

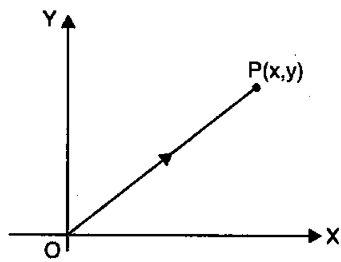
If a number of vectors are represented both in magnitude and direction by the sides of a polygon taken in the same order, then the resultant vector is represented both in magnitude and direction by the closing side of the polygon taken in the opposite order.



Resultant $\vec{R} = \vec{p}+\vec{q}+\vec{r}+\vec{s}+\vec{t}$

• Position Vector

Position vector is a vector to represent any position of a body. The straight line joining the origin and the point represents the position vector. It is represented by both magnitude and direction.



It is represented by $\vec{r} = \overrightarrow{OP} = x\hat{i} + y\hat{i}$ where $\hat{i}$ and $\hat{j}$ are the unit vectors along x and y axis respectively.

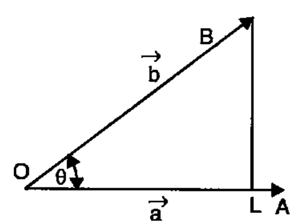
If position vector $\vec{r}$ is in three dimensions, then it is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along x, y and z co-ordinates respectively.

$$\left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$$

• Multiplication of Vectors

(*i*) **Scalar product (Dot product).** Scalar product of two vectors is defined as the product of the magnitude of two vectors with cosine of smaller angle between them.

It is always a scalar, so it is called as scalar product.



$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

Geometrically, $\vec{a}\cdot\vec{b}$ = (Mod of $\vec{a}$) (Projection of b on $\vec{a}$)

(*ii*) **Vector product (Cross product).** The cross or vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as,

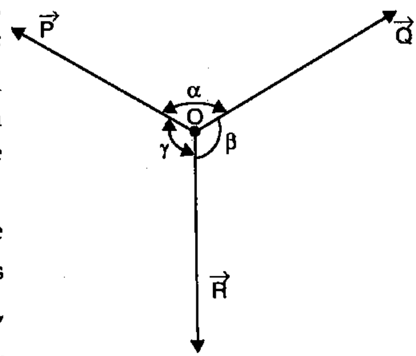
$$\vec{A} \times \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \sin\theta\, \hat{n}, \quad \text{where}$$

θ – angle between $\vec{A}$ and $\vec{B}$ taken in anti-clockwise direction.

$\hat{n}$ – unit vector in the direction perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

• Lami's Theorem

Lami's theorem states, "If a particle under the simultaneous action of three forces is in equilibrium, then each force has a constant ratio with the sine of the angle between the other two forces."



If three forces $\vec{P}, \vec{Q}$ and $\vec{R}$ are acting on a particle O in directions given by angles α, β and γ, then, the particle O is in equilibrium, when

$$\frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha}$$

• Angular Displacement

Angular displacement of the object moving around a circular path is defined as the angle traced out by the radius vector at the centre of the circular path in a given time.

$$\theta \text{ (angle)} = \text{arc/radius}$$

$\theta \rightarrow$ the magnitude of angular displacement. It is expressed in radians (rad).

• Angular Velocity

Angular velocity of an object in circular motion is defined as the time rate of change of its angular displacement.

It is denoted by ω and is measured in radians per second ($rad.s^{-1}$).

$$\omega = \frac{\text{angular displacement}}{\text{Time}} = \frac{\theta}{t} = \frac{d\theta}{dt}$$

• Angular Acceleration

Angular acceleration of an object in circular motion is defined as the time rate of change of its angular velocity.
It is denoted by 'α' and measured in rad s^{-2}.

$$\alpha = \frac{\text{angular velocity change}}{\text{time taken}} = \frac{d\omega}{dt}$$

• Uniform Circular Motion

When a body moves in a circular path with a constant speed, then the motion of the body is known as uniform circular motion.

The time taken by the object to complete one revolution on its circular path is called time period.
For circular motion, the number of revolutions completed per unit time is known as the frequency (ν). Unit of frequency is 1 Hertz (1 Hz).
It is found that

$$\nu \cdot T = 1 \quad \text{or} \quad \nu = \frac{1}{T}$$

• The relation between angular velocity, frequency and time period is given by

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi\nu$$

• Centripetal Acceleration

To maintain a particle in its uniform circular motion a radially inward acceleration should be continuously maintained. It is known as the centripetal acceleration.

$$a_c = \frac{v^2}{r} = r\omega^2 = \frac{r \cdot 4\pi^2}{T^2} = r \cdot 4\pi^2 \cdot \omega^2$$

TEXTBOOK QUESTIONS SOLVED

4.1. *State, for each of the following physical quantities, if it is a scalar or a vector:*

volume, mass, speed, acceleration, density, number of moles, velocity, angular frequency, displacement, angular velocity.

Sol. **Scalars:** Volume, mass, speed, density, number of moles, angular frequency.

Vectors: Acceleration, velocity, displacement, angular velocity.

4.2. *Pick out the two scalar quantities in the following list:*
force, angular momentum, work, current, linear momentum, electric field, average velocity, magnetic moment, relative velocity.

Sol. Work and current are the scalar quantities in the given list.

4.3. *Pick out the only vector quantity in the following list:*
Temperature, pressure, impulse, time, power, total path length, energy, gravitational potential, coefficient of friction, charge.

Sol. Impulse.

4.4. *State with reasons, whether the following algebraic operations with scalar and vector physical quantities are meaningful:*
(a) adding any two scalars, (b) adding a scalar to a vector of the same dimensions, (c) multiplying any vector by any scalar, (d) multiplying any two scalars, (e) adding any two vectors, (f) adding a component of a vector to the same vector.

Sol. (*a*) No, because only the scalars of same dimensions can be added.

(*b*) No, because a scalar cannot be added to a vector.

(*c*) Yes, multiplying a vector with a scalar gives the scalar (number) times the vector quantity which makes sense and one gets a bigger vector. For example, when acceleration $\vec{A}$ is multiplied by mass *m*, we get a force $\vec{F} = m\vec{A}$.

(*d*) Yes, two scalars multiplied yield a meaningful result, for example multiplication of rise in temperature of water and its mass gives the amount of heat absorbed by that mass of water.

(*e*) No, because the two vectors of same dimensions can be added.

(*f*) Yes, because both are vectors of the same dimensions.

4.5. *Read each statement below carefully and state with reasons, if it is true or false:*

(a) The magnitude of a vector is always a scalar.

(b) Each component of a vector is always a scalar.

(c) The total path length is always equal to the magnitude of the displacement vector of a particle.

(d) The average speed of a particle (defined as total path length divided by the time taken to cover the path) is either greater or equal to the magnitude of average velocity of the particle over the same interval of time.

(e) Three vectors not lying in a plane can never add up to give a null vector.

Sol. (*a*) True, magnitude of the velocity of a body moving in a straight line may be equal to the speed of the body.

(*b*) False, each component of a vector is always a vector, not scalar.

(*c*) False, total path length can also be more than the magnitude of displacement vector of a particle.

(*d*) True, because the total path length is either greater than or equal to the magnitude of the displacement vector.

(*e*) True, this is because the resultant of two vectors will not lie in the plane of third vector and hence cannot cancel its effect to give null vector.

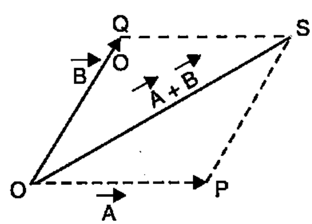
4.6. *Establish the following inequalities geometrically or otherwise:*

(*a*) $\left|\vec{A}+\vec{B}\right| \le \left|\vec{A}\right|+\left|\vec{B}\right|$ (*b*) $\left|\vec{A}+\vec{B}\right| \ge \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right|$

(*c*) $\left|\vec{A}-\vec{B}\right| \le \left|\vec{A}\right|+\left|\vec{B}\right|$ (*d*) $\left|\vec{A}-\vec{B}\right| \ge \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right|$

When does the equality sign above apply?

Sol. Consider two vectors $\vec{A}$ and $\vec{B}$ be represented by the sides $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ of a parallelogram OPSQ. According to parallelogram law of vector addition; $(\vec{A}+\vec{B})$ will be represented by $\overrightarrow{OS}$ as shown in Fig. Thus



$$OP = \left|\vec{A}\right|, \quad OQ = PS = \left|\vec{B}\right|$$

and $$OS = \left|\vec{A}+\vec{B}\right|$$

(*a*) **To prove** $\left|\vec{A}+\vec{B}\right| \le \left|\vec{A}\right|+\left|\vec{B}\right|$

We know that the length of one side of a triangle is always less than the sum of the lengths of the other two sides. Hence from Δ OPS, we have

$OS < OP + PS$ or $OS < OP + OQ$

or $$\left|\vec{A}+\vec{B}\right| < \left|\vec{A}\right|+\left|\vec{B}\right| \quad ...(i)$$

If the two vectors $\vec{A}$ and $\vec{B}$ are acting along the same straight line and in the same direction

then $$\left|\vec{A}+\vec{B}\right| = \left|\vec{A}\right|+\left|\vec{B}\right| \quad ...(ii)$$

Combining the conditions mentioned in (*i*) and (*ii*) we have

$$\left|\vec{A}+\vec{B}\right| \le \left|\vec{A}\right|+\left|\vec{B}\right|$$

(*b*) **To prove** $\left|\vec{A}+\vec{B}\right| \ge \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right|$

From Δ OPS, we have $OS + PS > OP$ or $OS > |OP - PS|$ or $OS > |OP - OQ|$ ...(*iii*) ($\because PS = OQ$)
The modulus of $(OP - PS)$ has been taken because the L.H.S. is always positive but the R.H.S. may be negative if $OP < PS$. Thus from (*iii*) we have.

$$\left|\vec{A}+\vec{B}\right| > \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right| \quad ...(iv)$$

If the two vectors $\vec{A}$ and $\vec{B}$ are acting along a straight line in opposite directions, then

$$\left|\vec{A}+\vec{B}\right| = \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right| \quad ...(v)$$

Combining the conditions mentioned in (*iv*) and (*v*) we get

$$\left|\vec{A}+\vec{B}\right| \ge \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right|$$

(*c*) **To prove** $\left|\vec{A}-\vec{B}\right| \le \left|\vec{A}\right|+\left|\vec{B}\right|$

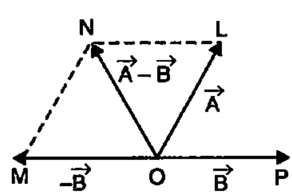
In fig. $\overrightarrow{OL}$ and $\overrightarrow{OM}$ represents vectors $\vec{A}$ and $\vec{B}$ respectively. Here $\overrightarrow{ON}$ represents $\vec{A}-\vec{B}$.

Consider the Δ OMN,

$$ON < MN + OM$$

or $$\left|\vec{A}-\vec{B}\right| < \left|\vec{A}\right|+\left|-\vec{B}\right|$$

or $$\left|\vec{A}-\vec{B}\right| < \left|\vec{A}\right|+\left|\vec{B}\right| \quad ...(vi)$$



When $\vec{A}$ and $\vec{B}$ are along the same straight line, but point in the opposite direction, then

$$\left|\vec{A}-\vec{B}\right| = \left|\vec{A}\right|+\left|\vec{B}\right| \quad ...(vii)$$

Combining equation (*vi*) and (*vii*), we get

$$\left|\vec{A}-\vec{B}\right| \le \left|\vec{A}\right|+\left|\vec{B}\right|$$

(*d*) **To prove** $\left|\vec{A}-\vec{B}\right| \ge \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right|$

Let us consider the Δ OMN,

$$ON + OM > MN \quad \text{or} \quad ON > |MN - OM|$$

Since $MN = OL \quad \therefore \quad ON > |OL - OM|$

or $$\left|\vec{A}-\vec{B}\right| > \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right| \quad \text{...(viii)}$$

When $\vec{A}$ and $\vec{B}$ are along the same straight line and point in the same direction, then

$$\left|\vec{A}-\vec{B}\right| = \left|\vec{A}\right|-\left|\vec{B}\right| \quad \text{...(ix)}$$

Combining equations (*viii*) and (*ix*), we get

$$\left|\vec{A}-\vec{B}\right| \ge \left|\left|\vec{A}\right|-\left|\vec{B}\right|\right|$$

4.7. *Given $\vec{a}+\vec{b}+\vec{c}+\vec{d} = \vec{0}$, which of the following statements are correct:*

(a) $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ must each be a null vector,

(b) The magnitude of $(\vec{a}+\vec{c})$ equals the magnitude of $(\vec{b}+\vec{d})$.

(c) The magnitude of $\vec{a}$ can never be greater than the sum of the magnitudes of $\vec{b},\vec{c}$ and $\vec{d}$.

(d) $\vec{b}+\vec{c}$ must lie in the plane of $\vec{a}$ and $\vec{d}$ if $\vec{a}$ and $\vec{d}$ are not collinear, and in the line of $\vec{a}$ and $\vec{d}$, if they are collinear?

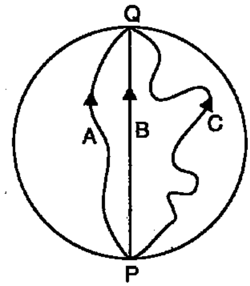
Sol. (*a*) This statement is not correct. Each need not be a null vector. Even when $\vec{a} = -\vec{b}$ and $\vec{c} = -\vec{d}$, they can form a null vector.

(*b*) This statement is correct. When $\left|\vec{a}+\vec{c}\right| = \left|\vec{b}+\vec{d}\right|$, the addition may be a null vector, if $\vec{a}+\vec{c} = -(\vec{b}+\vec{d})$ and are collinear.

(*c*) This statement is correct. Let $\left|\vec{a}\right| \not> \left|\vec{b}+\vec{c}+\vec{d}\right|$. If it is true the vector sum cannot be zero. Even if $\vec{b},\vec{c},\vec{d}$ form a triangle, the vector sum $\vec{b}+\vec{c}+\vec{d} = 0$ but then $\vec{a}+\vec{b}+\vec{c}+\vec{d}$ is not zero.

(*d*) This statement is correct. If $\vec{b}+\vec{c}$ do not lie in the plane of $\vec{a}+\vec{d}$, the vector sum $(\vec{a}+\vec{b})+(\vec{c}+\vec{d})$ is not zero because the addends will have different magnitude and different direction.

4.8. *Three girls skating on a circular ice ground of radius 200 m start from a point P on the edge of the ground and reach a point Q diametrically opposite to P following different paths as shown in Fig. What is the magnitude of the displacement vector for each? For which girl is this equal to the actual length of path skate?*



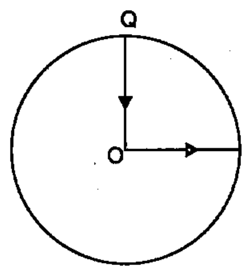
Sol. Displacement for each girl = $\overrightarrow{PQ}$.

$\therefore$ Magnitude of the displacement for each girl

= PQ = diameter of circular ice ground

= 2 × 200 = 400 m.

4.9. *A cyclist starts from the centre O of a circular park of radius 1 km, reaches the edge P of the park, then cycles along the circumference, and returns to the centre along QO as shown in Fig. If the round trip takes 10 min, what is the (a) net displacement, (b) average velocity, and (c) average speed of the cyclist?*



Sol. (*a*) Since both the initial and final positions are the same therefore the net displacement is zero.

(*b*) Average velocity is the ratio of net displacement and total time taken. Since the net displacement is zero therefore the average velocity is also zero.

(*c*) Average speed = $\dfrac{\text{distance covered}}{\text{time taken}}$

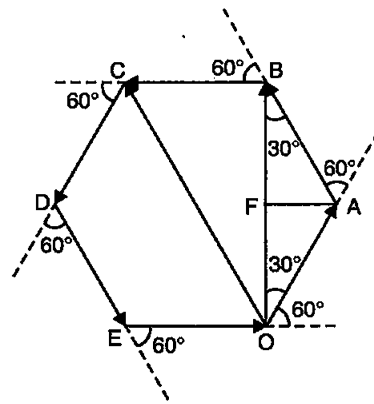
$$= \frac{OP + \text{Actual distance } PQ + QO}{10 \text{ minute}}$$

$$= \frac{1 \text{ km} + \frac{1}{4} \times 2\pi \times 1 \text{ km} + 1 \text{ km}}{10/60\, h}$$

$$= 6\left(2 + \frac{22}{14}\right) \text{km h}^{-1} = 6 \times \frac{50}{14} \text{ km h}^{-1}$$

$= 21.43$ km h^{-1}.

4.10. *On an open ground, a motorist follows a track that turns to his left by an angle of 60° after every 500 m. Starting from a given turn, specify* *the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of the displacement with the total path length covered by the motorist in each case.*



Sol. (*i*) The path followed by the motorist will be a closed hexagonal path.

Suppose the motorist starts his journey from the point *O*. He takes the turn at the point C.

Displacement = $\overrightarrow{OC}$

Here $OC = \sqrt{(OB)^2 + (BC)^2} = \sqrt{(OF + FB)^2 + (BC)^2}$

$= \sqrt{(500 \cos 30° + 500 \cos 30°)^2 + (500)^2}$

$= \sqrt{\left(2 \times 500 \times \frac{\sqrt{3}}{2}\right)^2 + (500)^2}$

$= 500\sqrt{4} = 1000 \text{ m} = 1 \text{ km}$

Total path length = 500 m + 500 m + 500 m = 1500 m = 1.5 km

$$\frac{\text{Magnitude of displacement}}{\text{Total path length}} = \frac{1}{1.5} = \frac{2}{3} = 0.67$$

(*ii*) The motorist will take the sixth turn at O.

Displacement is zero. So, displacement vector is a null vector.

Path length is 3000 m, *i.e.*, 3 km.

Ratio of magnitude of displacement and path length is zero.

(*iii*) The motorist will take the 8th turn at *B*.

Magnitude of displacement = 2 × 500 cos 30°

$$= 500\sqrt{3}\text{m} = \frac{\sqrt{3}}{2}\text{km} = 0.866 \text{ km}$$

Path length = 8 × 500 m = 4 km

Ratio of magnitude of displacement and path length is $\frac{\sqrt{3}/2}{4}$

i.e., $\frac{\sqrt{3}}{8} = 0.22$

4.11. *A passenger arriving in a new town wishes to go from the station to a hotel located 10 km away on a straight road from the station. A dishonest cabman takes him along a circuitous path 23 km long and reaches the hotel in 28 min. What is (a) the average speed of the taxi, (b) the magnitude of average velocity? Are the two equal?*

Sol. Here, actual path length travelled, s = 23 km; Displacement = 10 km;

Time taken, $t = 28 \text{ min} = \frac{28}{60} \text{ h}$

(*a*) Average speed of taxi

$$= \frac{\text{actual path length}}{\text{time taken}} = \frac{23}{28/60} \text{ km/h} = 49.3 \text{ km/h}$$

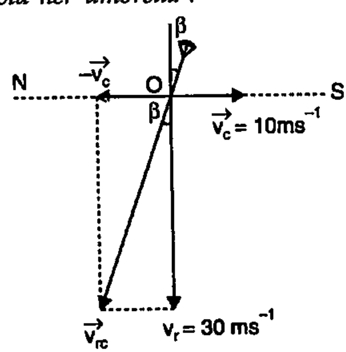
(*b*) Magnitude of average velocity

$$= \frac{\text{displacement}}{\text{time taken}} = \frac{10}{28/60} \text{ km/h} = 21.4 \text{ km/h}$$

The average speed is not equal to the magnitude of average velocity. The two are equal for the motion of taxi along a straight path in one direction.

4.12. *Rain is falling vertically with a speed of 30 m s^{-1}. A woman rides a bicycle with a speed of 10 m s^{-1} in the north to south direction. What is the direction in which she should hold her umbrella ?*

Sol. The situation has been demonstrated in the figure below. Here $\vec{v}_r$ = 30 ms^{-1} is the rain velocity in vertically downward direction and $\vec{v}_c$ = 10 ms^{-1} is the velocity of cyclist woman in horizontal plane from north N to south S.



∴ Relative velocity of rain w.r.t. cyclist $\vec{v}_{rc}$ subtends an angle β with vertical such that

$$\tan \beta = \frac{|\vec{v}_c|}{|\vec{v}_r|} = \frac{10}{30} = \frac{1}{3}$$

$$\therefore \qquad \beta = \tan^{-1}\left(\frac{1}{3}\right) = 18^\circ 26'$$

Hence, the woman should hold her umbrella at 18° 26′ south of vertical.

4.13. *A man can swim with a speed of 4.0 km h^{-1} in still water. How long does he take to cross a river 1.0 km wide if the river flows steadily at 3.0 km h^{-1} and he makes his strokes normal to the river current? How far down the river does he go when he reaches the other bank?*

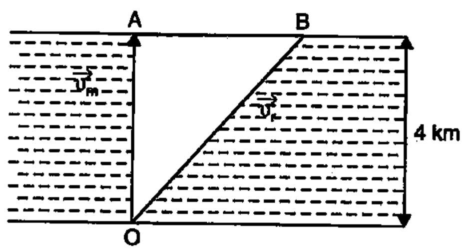
Sol. Here, $\vec{V_m}$ = 4 km h^{-1}; $\vec{v_r}$ = 3 km h^{-1};

OA = 1 km

Let t = time taken by man to reach the other bank.

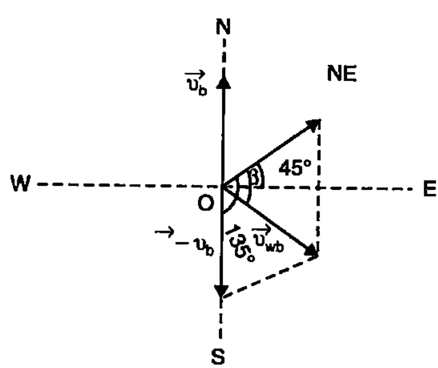
then $t = \frac{OA}{v_m} = \frac{1}{4} = 0.25$ h

Distance, $AB = v_r \times t = 3 \times 0.25 = 0.75$ km.



4.14. *In a harbour, wind is blowing at the speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along the N-E direction. If the boat starts moving at a speed of 51 km/h to the north, what is the direction of the flag on the mast of the boat?*

Sol. When the boat is anchored in the harbour, the flag flutters along the N-E direction. It shows that the velocity of wind is along the north-east direction. When the boat starts moving, the flag will flutter along the direction of relative velocity of wind w.r.t. boat. Let $\vec{v}_{wb}$ be the relative velocity of wind w.r.t. boat and β be the angle between $\vec{v}_{wb}$ and $\vec{v}_w$. (see fig.)



Now, $\vec{v}_{wb} = \vec{v}_w + \left(-\vec{v}_b\right)$

Here, $\left|\vec{v}_w\right| = 72$ km/h

$\left|-\vec{v}_b\right| = 51$ km/h

Angle between $\vec{v}_w$ and $-\vec{v}_b$ is 135° *i.e.*, $\theta = 135°$. Then

$$\tan\beta = \frac{51\sin 135°}{72 + 51\cos 135°} = \frac{51\sin 45°}{72 + 51(-\cos 45°)}$$

$$= \frac{51 \times \left(1/\sqrt{2}\right)}{72 - 51\left(1/\sqrt{2}\right)} = 1.0039$$

$\therefore$ $\beta = \tan^{-1}(1.0039) = 45.1°$

Angle w.r.t. east direction = 45.1° − 45° = 0.1°

It means the flag will flutter almost due east.

4.15. *The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 m s⁻¹ can go without hitting the ceiling of the hall?*

Sol. Maximum height $h_{max} = 25$ m; Horizontal range, R =?

Velocity of projection,

$$v = 40 \text{ ms}^{-1}$$

We know that $h_{max} = \dfrac{v^2\sin^2\theta}{2g}$

or $\sin^2\theta = \dfrac{25 \times 2 \times 9.8}{40 \times 40} = 0.30625$

or $\sin\theta = 0.5534$

$\theta = \sin^{-1}(0.5534) = 33.6°$

Again, $R = \dfrac{v^2\sin 2\theta}{g} = \dfrac{40 \times 40 \sin 67.2°}{9.8}$

or $R = \dfrac{1600}{9.8} \times 0.9219 \text{ m} = 150.5 \text{ m}.$

4.16. *A cricketer can throw a ball to a maximum horizontal distance of 100 m. How much high above the ground can the cricketer throw the same ball?*

Sol. $R_{max} = 100$ m;

Since $R_{max} = \dfrac{v^2}{g} \Rightarrow 100 = \dfrac{v^2}{g}$

Using equation of motion

$$v^2 - u^2 = 2as$$

Here, $v = 0, \quad a = -g, \quad s = R_{max} = 100$ m

$\therefore \quad (0)^2 - u^2 = 2(-g) \times s$

$\Rightarrow \quad s = \frac{1}{2}\frac{u^2}{g}$

Since $u = v$

$\therefore \quad s = \frac{1}{2}\frac{v^2}{g} = \frac{1}{2} \times 100 = 50 \text{ m}.$

4.17. *A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, what is the magnitude and direction of acceleration of the stone?*

Sol. Here, $r = 80 \text{ cm} = 0.8 \text{ m};$

$$\nu = \frac{14}{25} \text{ rev/s}$$

$\therefore \quad \omega = 2\pi\nu = 2 \times \frac{22}{7} \times \frac{14}{25} \text{ rad/s} = \frac{88}{25} \text{ rad} \cdot \text{s}^{-1}$

The centripetal acceleration,

$$a = \omega^2 r = \left(\frac{88}{25}\right)^2 \times 0.80 = 9.90 \text{ ms}^{-2}$$

The direction of centripetal acceleration is along the string directed towards the centre of circular path.

4.18. *An aircraft executes a horizontal loop of radius 1.00 km with a steady speed of 900 km/h. Compare its centripetal acceleration with the acceleration due to gravity.*

Sol. Here $r = 1 \text{ km} = 10^3 \text{ m},$

$$v = 900 \text{ km h}^{-1} = 900 \times \frac{5}{18} = 250 \text{ ms}^{-1}$$

Centripetal acceleration $= a_c = \frac{v^2}{r} = \frac{(250)^2}{10^3} = 62.5 \text{ ms}^{-2}$

Now, $\frac{a_c}{g} = \frac{62.5}{9.8} = 6.38.$

4.19. *Read each statement below carefully and state, with reasons, if it is true or false:*

(a) The net acceleration of a particle in circular motion is always along the radius of the circle towards the centre.

(*b*) *The velocity vector of a particle at a point is always along the tangent to the path of the particle at that point.*

(*c*) *The acceleration vector of a particle in uniform circular motion averaged over one cycle is a null vector.*

Sol. (*a*) False, the net acceleration of a particle in circular motion is along the radius of the circle towards the centre only in uniform circular motion.

(*b*) True, because while leaving the circular path, the particle moves tangentially to the circular path.

(*c*) True, the direction of acceleration vector in a uniform circular motion is directed towards the centre of circular path. It is constantly changing with time. The resultant of all these vectors will be a zero vector.

4.20. *The position of a particle is given by*

$$r = 3.0t\ \hat{i}\ - 2.0t^2\ \hat{j}\ + 4.0\ \hat{k}\ m$$

where t is in seconds and the coefficients have the proper units for **r** *to be in metres.*

(*a*) *Find the* $\vec{v}$ *and* $\vec{a}$ *of the particle.*

(*b*) *What is the magnitude and direction of velocity of the particle at* $t = 2.0$ *s?*

Sol. Here $\qquad \vec{r}(t) = \left(3.0t\,\hat{i} - 2.0t^2\,\hat{j} + 4.0\,\hat{k}\right)$ m

(*a*) $\therefore \qquad \vec{v}(t) = \frac{\vec{dr}}{dt} = \left(3.0\,\hat{i} - 4.0t\,\hat{j}\right)$ m/s

and $\qquad \vec{a}(t) = \frac{\vec{dv}}{dt} = \left(-4.0\,\hat{j}\right)$ m/s^2

(*b*) Magnitude of velocity at $t = 2.0$ s,

$$v_{(t=2s)} = \sqrt{(3.0)^2 + (-4.0\times 2)^2} = \sqrt{9+64} = \sqrt{73}$$

$$= 8.54 \text{ m s}^{-1}$$

This velocity will subtend an angle β from x-axis, where

$$\tan\beta = \frac{(-4.0\times 2)}{(3.0)} = -2.667. = -2.6667.$$

$\therefore \qquad \beta = \tan^{-1}(-2.6667) = -69.44° = 69.44°$

from negative x-axis.

4.21. *A particle starts from the origin at t = 0 s with a velocity of 10.0 $\hat{j}$ m/s and moves in the x-y plane with a constant acceleration of $(8.0\hat{i} + 2.0\hat{j})$ m s^{-2}.*

(a) At what time is the x-coordinate of the particle 16 m? What is the y-coordinate of the particle at that time?

(b) What is the speed of the particle at the time?

Sol. It is given that $\vec{r}_{(t=0s)} = 0, \vec{v}_{(0)} = 10.0\,\hat{j}$ m/s

and $\vec{a}(t) = (8.0\,\hat{i} + 2.0\,\hat{j})$ m s^{-2}

(*a*) It means $x_0 = 0$, $u_x = 0$, $a_x = 8.0$ m s^{-2} and $x = 16$ m
Using relation

$$s = x - x_0 = u_x t + \frac{1}{2}a_x t^2, \text{ we have}$$

$$16 - 0 = 0 + \frac{1}{2} \times 8.0 \times t^2 \Rightarrow t = 2 \text{ s}$$

$$\therefore \quad y = y_0 + u_y t + \frac{1}{2}a_y t^2$$

$$= 0 + 10.0 \times 2 + \frac{1}{2} \times 2.0 \times (2)^2$$

$$= 20 + 4 = 24 \text{ m}$$

(*b*) Velocity of particle at $t = 2$ s along *x*-axis

$$v_x = u_x + a_x t = 0 + 8.0 \times 2 = 16.0 \text{ m/s}$$

and along *y*-axis

$$v_y = u_y + a_y t = 10.0 + 2.0 \times 2 = 14.0 \text{ m/s}$$

∴ Speed of particle at $t = 2$ s

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(16.0)^2 + (14.0)^2} = 21.26 \text{ m s}^{-1}.$$

4.22. *$\hat{i}$ and $\hat{j}$ are unit vectors along x and y-axis respectively. What is the magnitude and direction of the vectors $\hat{i} + \hat{j}$, and $\hat{i} - \hat{j}$? What are the components of a vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the directions of $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$? [You may use graphical method]*

Sol. (*i*) $$\hat{i} + \hat{j} = \sqrt{(1)^2 + (1)^2 + 2 \times 1 \times 1 \times \cos 90^\circ}$$

$$= \sqrt{2} = 1.414 \text{ units}$$

$$\tan\theta = \frac{1}{1} = 1, \quad \therefore \quad \theta = 45°$$

So the vector $\hat{i} + \hat{j}$ makes an angle of 45° with x-axis.

(*ii*) $$\left|\hat{i} - \hat{j}\right| = \sqrt{(1)^2 + (2)^2 - 2\times1\times1\times\cos 90°}$$

$$= \sqrt{2} = 1.414 \text{ units}$$

The vector $\hat{i} - \hat{j}$ makes an angle of –45° with x-axis.

(*iii*) Let us now determine the component of $\vec{A} = 2\hat{i} + 3\hat{j}$ in the direction of $\hat{i} + \hat{j}$.

Let $\vec{B} = \hat{i} + \hat{j}$

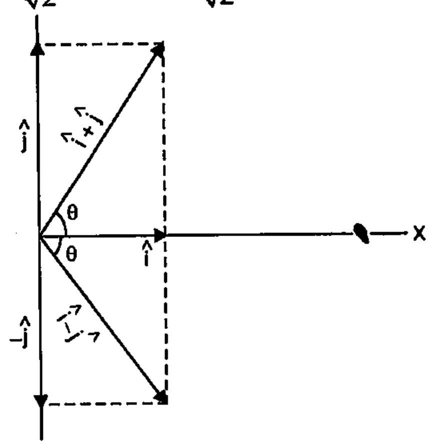
$$\vec{A}\cdot\vec{B} = AB\cos\theta = (A\cos\theta)\,B$$

So the component of $\vec{A}$ in the direction of $\vec{B} = \dfrac{\vec{A}\cdot\vec{B}}{B}$

$$= \frac{(2\hat{i}+3\hat{j})\cdot(\hat{i}+\hat{j})}{\sqrt{(1)^2+(1)^2}} = \frac{2\hat{i}\cdot\hat{i}+2\hat{i}\cdot\hat{j}+3\hat{j}\cdot\hat{i}+3\hat{j}\cdot\hat{j}}{\sqrt{2}} = \frac{5}{\sqrt{2}} \text{ units}$$

(*iv*) Component of $\vec{A}$ in the direction of $\hat{i} - \hat{j}$

$$= \frac{(2\hat{i}+3\hat{j})\cdot(\hat{i}-\hat{j})}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \text{ units}.$$



4.23. *For any arbitrary motion in space, which of the following relations are true:*

(a) $v_{average} = (1/2)\,[v(t_1) + v(t_2)]$

(b) $v_{average} = [r(t_2) - r(t_1)]/(t_2 - t_1)$

(c) $v(t) = v(0) + a\,t$

(d) $r(t) = r(0) + v(0)\,t + (1/2)\,a\,t^2$

(e) $a_{average} = [v(t_2) - v(t_1)]/(t_2 - t_1)$

(The 'average' stands for average of the quantity over the time interval t_1 to t_2)

Sol. (*b*)and (*e*) are true; others are false because relations (*a*), (*c*) and (*d*) hold only for uniform acceleration.

4.24. *Read each statement below carefully and state, with reasons and examples, if it is true or false:*

A scalar quantity is one that

(a) is conserved in a process

(b) can never take negative values

(c) must be dimensionless

(d) does not vary from one point to another in space

(e) has the same value for observers with different orientations of axes.

Sol. (*a*) False, because kinetic energy is a scalar but does not remain conserved in an inelastic collision.

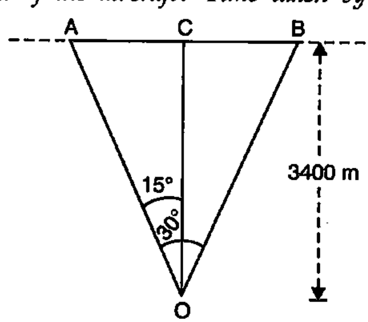
(*b*) False, because potential energy in a gravitational field may have negative values.

(*c*) False, because mass, length, time, speed, work etc., all have dimensions.

(*d*) False, because speed, energy etc., vary from point to point in space.

(*e*) True, because a scalar quantity will have the same value for observers with different orientations of axes since a scalar has no direction of its own.

4.25. *An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10 s apart is 30°, what is the speed of the aircraft? Time taken by aircraft from A to B is 10 s.*



Sol. In Fig, *O* is the observation point at the ground. *A* and *B* are the positions of air craft for which ∠AOB = 30°. Draw a perpendicular *OC* on *AB*. Here *OC* = 3400 m and ∠AOC = ∠COB = 15°.

In ΔAOC, $AC = OC \tan 15° = 3400 \times 0.2679 = 910.86$ m.

$AB = AC + CB = AC + AC = 2\ AC = 2 \times 910.86$ m

Speed of the aircraft,

$$v = \frac{\text{distance } AB}{\text{time}} = \frac{2 \times 910.86}{10} = 182.17 \text{ ms}^{-1} = \mathbf{182.2 \text{ ms}^{-1}}.$$

4.26. *A vector has magnitude and direction.*

(i) Does it have a location in the space?

(ii) Can it vary with time?

(iii) Will two equal vectors $\vec{a}$ and $\vec{b}$ at different locations in space necessarily have identical physical effects? Give examples in support of your answer.

Sol. (*i*) Besides having magnitude and direction, each vector has also a location in space.

(*ii*) A vector can vary with time. As an example, velocity and acceleration vectors may vary with time.

(*iii*) Two equal vectors $\vec{a}$ and $\vec{b}$ having different locations may not have same physical effect. As an example, two balls thrown with the same force, one from earth and the other from moon will attain different 'maximum heights'.

4.27. *A vector has both magnitude and direction. Does that mean anything that has magnitude and direction is necessarily a vector? The rotation of a body can be specified by the direction of the axis of rotation and the angle of rotation about the axis. Does that make any rotation a vector?*

Sol. No. Finite rotation of a body about an axis is not a vector because finite rotations do not obey the laws of vector addition.

4.28. *Can you associate vectors with (a) the length of a wire bent into a loop (b) a plane area (c) a sphere? Explain.*

Sol. (*a*) We cannot associate a vector with the length of a wire bent into a loop. This is because the length of the loop does not have a definite direction.

(*b*) We can associate a vector with a plane area. Such a vector is called area vector and its direction is represented by a normal drawn outward to the area.

(*c*) The area of a sphere does not point in any difinite direction. However, we can associate a null vector with the area of the sphere. We cannot associate a vector with the volume of a sphere.

4.29. *A bullet fired at an angle of 30° with the horizontal hits the ground 3 km away. By adjusting its angle of projection, can one hope to hit a target 5 km away? Assume the muzzle speed to the fixed, and neglect air resistance.*

Sol. Here R = 3 km = 3000 m, $\theta = 30°$, $g = 9.8$ m s^{-2}.

As $$R = \frac{u^2 \sin 2\theta}{g}$$

$$\Rightarrow \quad 3000 = \frac{u^2 \sin 2 \times 30°}{9.8} = \frac{u^2 \sin 60}{9.8}$$

$$\Rightarrow \quad u^2 = \frac{3000 \times 9.8}{\sqrt{3}/2} = 3464 \times 9.8$$

Also, $$R' = \frac{u^2 \sin 2\theta'}{g} \Rightarrow 5000 = \frac{3464 \times 9.8 \times \sin 2\theta}{9.8}$$

i.e., $$\sin 2\theta' = \frac{5000}{3464} = 1.44$$

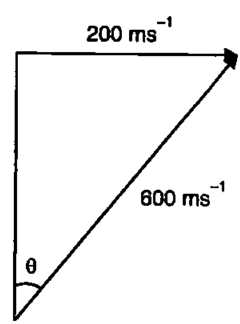
which is impossible because sine of an angle cannot be more than 1. Thus this target cannot be hoped to be hit.

4.30. *A fighter plane flying horizontally at an altitude of 1.5 km with speed 720 km h^{-1} passes directly overhead an anti-aircraft gun. At what angle from the vertical should the gun be fired for the shell with muzzle speed 600 m s^{-1} to hit the plane? At what minimum altitude should the pilot fly the plane to avoid being hit? (Take g = 10 m s^{-2})?*

Sol. Velocity of plane,

$$v_p = 720 \times \frac{5}{18} \text{ ms}^{-1} = 200 \text{ ms}^{-1}$$

Velocity of shell = 600 ms^{-1};



$$\sin \theta = \frac{200}{600} = \frac{1}{3}$$

or $$\theta = \sin^{-1}\left(\frac{1}{3}\right) = 19.47°$$

This angle is with the vertical.

Let h be the required minimum height.

Using equation

$$v^2 - u^2 = 2as, \quad \text{we get}$$

$$(0)^2 - (600 \cos \theta)^2 = -2 \times 10 \times h$$

or, $h = \dfrac{600 \times 600\left(1-\sin^2\theta\right)}{20}$

$$= 30 \times 600\left(1-\frac{1}{9}\right) = \frac{8}{9} \times 30 \times 600 \text{ m} = 16 \text{ km}.$$

4.31. *A cyclist is riding with a speed of 27 km/h. As he approaches a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at the constant rate of 0.50 m/s every second. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn?*

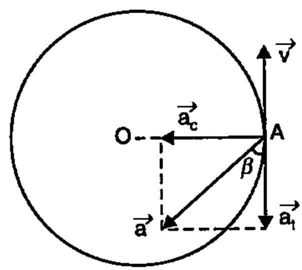
Sol. Here $v = 27 \text{ km/h} = 27 \times \frac{5}{18} \text{ m/s} = 7.5 \text{ m/s}, r = 80 \text{ m}$

and tangential acceleration $a_t = -0.50 \text{ m/s}^2$

∴ Centripetal acceleration

$$a_c = \frac{v^2}{r} = \frac{(7.5)^2}{80} = 0.70 \text{ ms}^{-2}$$

(radially inwards).



Thus, as shown in fig. above, two accelerations are acting in mutually perpendicular directions. If $\vec{a}$ be the resultant acceleration, then

$$\left|\vec{a}\right| = \sqrt{a_t^2 + a_c^2} = \sqrt{(0.5)^2 + (0.7)^2} = 0.86 \text{ ms}^{-2}$$

and $\tan\beta = \frac{a_c}{a_t} = \frac{0.7}{0.5} = 1.4$

⇒ $\beta = \tan^{-1}(1.4) = 54.5°$ from the direction of negative of the velocity.

4.32. *(a) Show that for a projectile the angle between the velocity and the x-axis as a function of time is given by*

$$\theta(t) = \tan^{-1}\left(\frac{v_{oy} - gt}{v_{ox}}\right)$$

(b) Shows that the projection angle θ_0 for a projectile launched from the origin is given by

$$\theta_0 = \tan^{-1}\left(\frac{4h_m}{R}\right)$$

where the symbols have their usual meaning.

Sol. (*a*) Let the projectile be fired at an angle θ with *x*-axis.

As θ depends on *t*, θ (*t*), at any instant

$$\tan\theta(t) = \frac{v_y}{v_x} = \frac{v_{oy} - gt}{v_{ox}}$$

(Since $v_y = v_{oy} - gt$ and $v_x = v_{ox}$)

$$\Rightarrow \quad \theta(t) = \tan^{-1}\left(\frac{v_{oy} - gt}{v_{ox}}\right)$$

(*b*) Since,

$$h_{max} = \frac{u^2 \sin^2\theta}{2g}$$

and

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\Rightarrow \quad \frac{h_{max}}{R} = \frac{u^2 \sin^2\theta/2g}{u^2 \sin 2\theta/g} = \frac{\tan\theta}{4}$$

(As $\sin 2\theta = 2\sin\theta\cos\theta$)

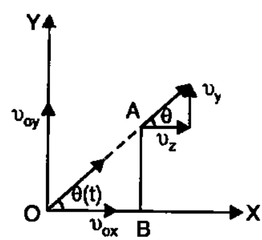
$$\Rightarrow \quad \frac{\tan\theta}{4} = \frac{h_{max}}{R}$$

or

$$\tan\theta = \frac{4\,h_{max}}{R}$$

or

$$\theta = \tan^{-1}\left(\frac{4\,h_{max}}{R}\right).$$



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