7 System of Particles and Rotational Motion

Lesson at a Glance

• A rigid body is a body with a perfectly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.

• Centre of Mass

For a system of particles, the centre of mass is defined as that point where the entire mass of the system is imagined to be concentrated, for consideration of its translational motion.

If there are two particles of masses m_1 and m_2 having position vectors $\vec{r}_1$ and $\vec{r}_2$, then the position vector of the centre of mass is given by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

• Vector Product or Cross Product of two vectors

The vector product or cross product of two vectors $\vec{A}$ and $\vec{B}$ is another vector $\vec{C}$, whose magnitude is equal to the product of the magnitudes of the two vectors and sine of the smaller angle between them.

If θ is the smaller angle between $\vec{A}$ and $\vec{B}$, then

$$\vec{A} \times \vec{B} = \vec{C} = AB \sin\theta\, \hat{C}$$

• The angular velocity of a body or a particle is defined as the ratio of the angular displacement of the body or the particle to the time interval during which this displacement occurs.

$$\omega = \frac{d\theta}{dt}$$

• The angular acceleration of a body is defined as the ratio of the change in the angular velocity to the time interval.

$$\text{Angular acceleration} = \frac{\text{Change in angular velocity}}{\text{time taken}}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

The unit of angular acceleration is rad s^{-2} and dimensional formula is $[M^0L^0L^{-2}]$.

• Torque

Torque is the moment of force.

Torque or moment of force = force × perpendicular distance

$$\vec{\tau} = \vec{r} \times \vec{F} = rF \sin\theta \; \hat{n}$$

where θ is smaller angle between $\vec{r}$ and $\vec{F}$; $\hat{n}$ is unit vector along $\vec{r}$.
It is measured in Nm and has dimensions of $[ML^2T^{-2}]$.

• Angular Momentum

The angular momentum (or moment of momentum) about an axis of rotation is a vector quantity, whose magnitude is equal to the product of the magnitude of momentum and the perpendicular distance of the line of action of momentum from the axis of rotation and its direction is perpendicular to the plane containing the momentum and the perpendicular distance.

It is given by

$$\vec{L} = \vec{r} \times \vec{p}$$

SI unit of angular momentum is kg m^2s^{-1} and its dimensional formula is $[M^1L^2T^{-1}]$.

• Axis of Rotation

A rigid body is said to be rotating if every point mass that makes it up, describes a circular path of a different radius but the same angular speed. The circular paths of all the point masses have a common centre. A line passing through this common centre is the axis of rotation.

• Couple

Two equal and opposite forces acting on a body but having different lines of action, form a couple. The net force due to a couple is zero, but they exert a torque and produce rotational motion.

• Moment of Inertia

The rotational inertia of a rigid body is referred to as its moment of inertia.

The moment of inertia of a body about an axis is defined as the sum of the products of the masses of the particles constituting the body and the square of their respective perpendicular distance from the axis.

It is given by $$I = m_1r_1^2 + m_2r_2^2 + \dots\dots\dots + m_nr_n^2 = \sum_{i=1}^{n} m_ir_i^2,$$

where m_i is the mass and r_i the distance of the i^{th} particle of the rigid body from the axis of rotation.

It is measured in kg m^2 and has the dimension of $[ML^2]$.

• Radius of Gyration

The distance of a point in a body from the axis of rotation, at which if whole of the mass of the body were supposed to be concentrated, its moment of inertia about the axis of rotation would be the same as that determined by the actual distribution of mass of the body is called radius of gyration.

If we consider that the whole mass of the body is concentrated at a distance K from the axis of rotation, then moment of inertia I can be expressed as

$$I = MK^2$$

where M is the total mass of the body and K is the radius of gyration. It is given as

$$K = \sqrt{\left(\frac{r_1^2 + r_2^2 + ... + r_n^2}{n}\right)}$$

• Theorem of Parallel Axes

According to this theorem, the moment of inertia I of a body about any axis is equal to its moment of inertia about a parallel axis through centre of mass, I_{cm}, plus Ma^2 where M is the mass of the body and 'a' is the perpendicular distance between the axes, *i.e.*,

$$I = I_{cm} + Ma^2$$

• Theorem of Perpendicular Axes

According to this theorem, the moment of inertia I of the body about a perpendicular axis is equal to the sum of moments of inertia of the body about two axes at right angles to each other in the plane of the body and intersecting at a point where the perpendicular axis passes, *i.e.*,

$$I = I_x + I_y$$

• A body in rotatory motion possesses rotational kinetic energy given by:

$$\text{Rotational } K.E. = \frac{1}{2} I \omega^2.$$

• Rolling Motion

The combination of rotational motion and the translational motion of a rigid body is known as rolling motion.

• Law of Conservation of Angular Momentum

According to the law of conservation of angular momentum, if there is no external couple acting, the total angular momentum of a rigid body or a system of particles is conserved.

If the moment of inertia of the body changes from I_1 to I_2 due to the change of the distribution of mass of the body, then angular velocity of the body changes from $\vec{\omega}_1$ to $\vec{\omega}_2$, such that

$$I_1 \vec{\omega}_1 = I_2\vec{\omega}_1 \quad \text{or} \quad I_1\omega_1 = I_2\omega_2.$$

TEXTBOOK QUESTIONS SOLVED

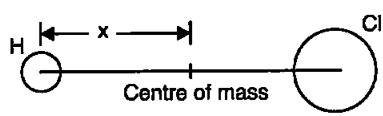
7.1. *Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?*

Ans. In all the four cases, as the mass density is uniform, centre of mass is located at their respective geometrical centres.

No, it is not necessary that the centre of mass of a body should lie on the body. For example, in case of a circular ring, centre of mass is at the centre of the ring, where there is no mass.

7.2. *In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.*

Ans. Let us choose the nucleus of the hydrogen atom as the origin for measuring distance.



Mass of hydrogen atom, m_1 = 1 unit (say)

Since cholorine atom is 35.5 times as massive as hydrogen atom,

$\therefore$ mass of cholorine atom, m_2 = 35.5 units

Now, $x_1 = 0$ and $x_2 = 1.27$ Å $= 1.27 \times 10^{-10}$ m

Distance of centre of mass of HCl molecule from the origin is given by

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{1 \times 0 + 35.5 \times 1.27 \times 10^{-10}}{1 + 35.5} \text{ m}$$

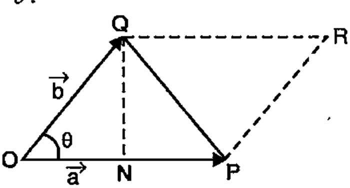
$$= \frac{35.5 \times 1.27}{36.5} \times 10^{-10} \text{m} = 1.235 \times 10^{-10}\text{m} = 1.235 \text{ Å}$$

7.3. *A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?*

Ans. When the child gets up and runs about on the trolley, the speed of the centre of mass of the trolley and child remains unchanged irrespective of the manner of motion of child. It is because here child and trolley constitute one single system and forces involved are purely internal forces. As there is no external force, there is no change in momentum of the system and velocity remains unchanged.

7.4. *Show that the area of the triangle contained between the vectors $\vec{a}$ and $\vec{b}$ is one half of the magnitude of $\vec{a} \times \vec{b}$.*

Ans. Let $\vec{a}$ be represented $\overrightarrow{OP}$ and $\vec{b}$ be represented by $\overrightarrow{OQ}$. Let $\angle POQ = \theta$, Fig.



Complete the || gm $OPRQ$. Join PQ.

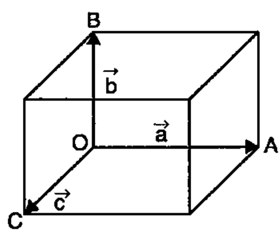
Draw $QN \perp OP$

In ΔOQN,
$$\sin\theta = \frac{QN}{OQ} = \frac{QN}{b}$$
$$QN = b\sin\theta$$

Now, by definition, $|\vec{a} \times \vec{b}| = ab\sin\theta = (OP)\,(QN)$
$$= \frac{2(OP)\,(QN)}{2} = 2 \times \text{area of } \Delta OPQ$$

$\therefore$ area of $\Delta OPQ = \frac{1}{2}|\vec{a} \times \vec{b}|$, which was to be proved.

7.5. *Show that $\vec{a}.(\vec{b} \times \vec{c})$ is equal in magnitude to the volume of the parallelopiped formed on the three vectors, $\vec{a}$, $\vec{b}$ and $\vec{c}$.*



Ans. Let *a* parallelopiped be formed on the three vectors.

$$\overrightarrow{OA} = \vec{a}, \quad \overrightarrow{OB} = \vec{b}$$

and $\overrightarrow{OC} = \vec{c}$

Now, $\vec{b} \times \vec{c} = bc \sin 90° \ \hat{n} = bc \ \hat{n}$

where $\hat{n}$ is unit vector along $\overrightarrow{OA}$ perpendicular to the plane containing $\vec{b}$ and $\vec{c}$.

Now $\vec{a}.(\vec{b} \times \vec{c}) = \vec{a}.\ bc\ \hat{n}$

$= (a)\ (bc) \cos 0° = a\ b\ c$

which is equal in magnitude to the volume of the parallelopiped.

7.6. *Find the components along the x, y, z-axes of the angular momentum $\vec{l}$ of a particle, whose position vector is $\vec{r}$ with components x, y, z and momentum is $\vec{p}$ with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane the angular momentum has only a z-component.*

Ans. We know that angular momentum $\vec{l}$ of a particle having position vector $\vec{r}$ and momentum $\vec{p}$ is given by

$$\vec{l} = \vec{r} \times \vec{p}$$

But $\vec{r} = [x\hat{i} + y\hat{j} + z\hat{k}]$,

where x, y, z are the components of

$$\vec{r} \text{ and } \vec{p} = [p_x\hat{i} + p_y\hat{j} + p_z\hat{k}]$$

$\therefore$ $\vec{l} = \vec{r} \times \vec{p} = [x\hat{i} + y\hat{j} + z\hat{k}] \times [p_x\hat{i} + p_y\hat{j} + p_z\hat{k}]$

or $$(l_x\hat{i} + l_y\hat{j} + l_z\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= (yp_z - zp_y)\,\hat{i} + (zp_x - xp_z)\,\hat{j} + (xp_y - yp_x)\,\hat{k}$$

From this relation, we conclude that

$$l_x = yp_z - zp_y, \quad l_y = zp_x - xp_z$$

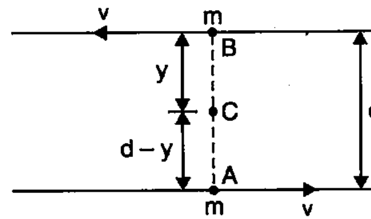
and $$l_z = xp_y - yp_x$$

If the given particle moves only in the $x - y$ plane, then $z = 0$ and $p_z = 0$ and hence,

$\vec{l} = (xp_y - yp_x)\,\hat{k}$, which is only the z-comonent of $\vec{l}$.

It means that for a particle moving only in the $x - y$ plane, the angular momentum has only the z-component.

7.7. *Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two particle system is the same whatever be the point about which the angular momentum is taken.*



Ans. Angular momentum about A,

$$L_A = mv \times 0 + mv \times d$$
$$= mvd$$

Angular momentum about B,

$$L_B = mv \times d + mv \times 0$$
$$= mvd$$

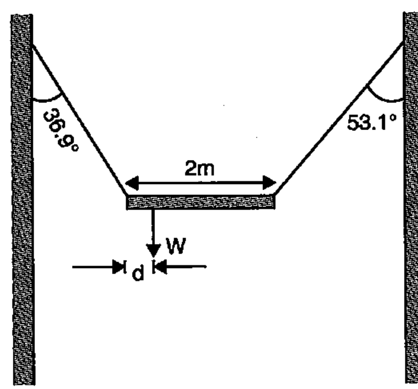
Angular momentum about C,

$$L_c = mv \times y + mv \times (d - y) = mvd$$

In all the three cases, the direction of angular momentum is the same.

$$\therefore \quad \vec{L}_A = \vec{L}_B = \vec{L}_C.$$

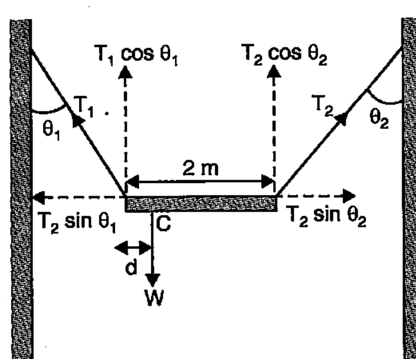
7.8. *A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.*



Ans. As it is clear from Fig.,

$$\theta_1 = 36.9°, \theta_2 = 53.1°.$$

If T_1, T_2 are the tensions in the two strings, then for equilibrium along the horizontal,



$$T_1 \sin \theta_1 = T_2 \sin \theta_2$$

$$\text{or} \quad \frac{T_1}{T_2} = \frac{\sin \theta_2}{\sin \theta_1}$$

$$= \frac{\sin 53.1°}{\sin 36.9°}$$

$$= \frac{0.7404}{0.5477}$$

$$= 1.3523$$

Let d be the distance of centre of gravity C of the bar from the left end.

For rotational equilibrium about C,

$$T_1 \cos \theta_1 \times d = T_2 \cos \theta_2 \, (2 - d)$$

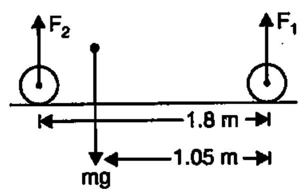
$$T_1 \cos 36.9° \times d = T_2 \cos 53.1° \, (2 - d)$$

$$T_1 \times 0.8366 \, d = T_2 \times 0.6718 \, (2 - d)$$

Put $T_1 = 13523 \, T_2$ and solve to get

$$d = 0.745 \text{ m}$$

7.9. *A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.*



Ans. Let F_1 and F_2 be the forces exerted by the level ground on front wheels and back wheels respectively.

Considering rotational equilibrium about the front wheels,

$$F_2 \times 1.8 = mg \times 1.05$$

or

$$F_2 = \frac{1.05}{1.8} \times 1800 \times 9.8 \text{ N} = 10290 \text{ N}$$

Force on each back wheel is

$$\frac{10290}{2} \text{ N or } 5145 \text{ N.}$$

Considering rotational equilibrium about the back wheels.

$$F_1 \times 1.8 = mg\,(1.8 - 1.05) = 0.75 \times 1800 \times 9.8$$

or

$$F_1 = \frac{0.75 \times 1800 \times 9.8}{1.8} = 7350 \text{ N.}$$

Force on each front wheel is $\frac{7350}{2}$ N or 3675 N.

7.10. (a) *Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be 2 $MR^2/5$, where M is the mass of the sphere and R is the radius of the sphere.*

(b) *Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $\frac{1}{4}MR^2$, find the moment of inertia about an axis normal to the disc passing through a point on its edge.*

Ans. (*a*) Moment of inertia of sphere about any diameter $= \frac{2}{5}MR^2$

Applying theorem of parallel axes,

Moment of inertia of sphere about a tangent to the sphere

$$= \frac{2}{5}MR^2 + M(R)^2 = \frac{7}{5}MR^2.$$

(*b*) We are given, moment of inertia of the disc about any of its diameters $= \frac{1}{4}MR^2$

(*i*) Using theorem of perpendicular axes, moment of inertia of the disc about an axis passing through its centre and normal to the disc $= 2 \times \frac{1}{4}MR^2 = \frac{1}{2}MR^2$.

(*ii*) Using theorem axes, moment of inertia of the disc passing through a point on its edge and normal to the dics

$$= \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2.$$

7.11. *Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?*

Ans. Let M be the mass and R the radius of the hollow cylinder, and also of the solid sphere. Their moments of inertia about the respective axes are

$$I_1 = MR^2 \quad \text{and} \quad I_2 = \frac{2}{5}MR^2$$

Let τ be the magnitude of the torque applied to the cylinder and the sphere, producing angular accelerations α_1 and α_2 respectively. Then

$$\tau = I_1\alpha_1 = I_2\alpha_2$$

or
$$\frac{\alpha_1}{\alpha_2} = \frac{I_2}{I_1} = \frac{(2/5)MR^2}{MR^2} = \frac{2}{5}$$

or
$$\alpha_2 = \frac{5}{2}\alpha_1.$$

The angular acceleration α_2 produced in the sphere is larger. Hence, the sphere will acquire larger angular speed after a given time.

7.12. *A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?*

Ans. $M = 20$ kg

Angular speed, $\omega = 100$ rad s^{-1}; $R = 0.25$ m

Moment of inertia of the cylinder about its axis

$$= \frac{1}{2}MR^2 = \frac{1}{2} \times 20 \times (0.25)^2 \text{ kg m}^2 = 0.625 \text{ kg m}^2$$

Rotational kinetic energy,

$$E_r = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.625 \times (100)^2 \text{ J} = 3125 \text{ J}$$

Angular momentum, $L = I\omega = 0.625 \times 100$ Js $= 62.5$ Js.

7.13. *(a) A child stands at the centre of a turntable with his arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that the turntable rotates without friction.*

(b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account of this increase in kinetic energy?

Ans. (*a*) Suppose, initial moment of inertia of the child is I_1. Then final moment of inertia, $I_2 = \frac{2}{5}I_1$

Also, $\nu_1 = 40$ rev min^{-1}

By using the principle of conservation of angular momentum, we get

$$I_1\omega_1 = I_2\omega_2 \quad \text{or} \quad I_1(2\pi\nu_1) = I_2(2\pi\nu_2)$$

or
$$\nu_2 = \frac{I_1\nu_1}{I_2} = \frac{I_1 \times 40}{\frac{2}{5} \times I_1} = 100 \text{ rev min}^{-1}$$

(*b*)
$$\frac{\text{Final K.E. of rotation}}{\text{Initial K.E. of rotation}} = \frac{\frac{1}{2}I_2\omega_2^2}{\frac{1}{2}I_1\omega_1^2} = \frac{\frac{1}{2}I_2(2\pi\nu_2)^2}{\frac{1}{2}I_1(2\pi\nu_1)^2} = \frac{I_2\nu_2^2}{I_1\nu_1^2}$$

$$= \frac{\frac{2}{5}I_1 \times (100)^2}{\frac{2}{5}I_1 \times (40)^2} = 2.5$$

Clearly, final $(K.E.)_{rot}$ becomes more because the child uses his internal energy when he folds his hands to increase the kinetic energy.

7.14. *A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.*

Ans. Here, $M = 3$ kg, $R = 40$ cm $= 0.4$ m

Moment of inertia of the hollow cylinder about its axis.

$$I = MR^2 = 3(0.4)^2 = 0.48 \text{ kg m}^2$$

Force applied $F = 30$ N

∴ Torque, $\tau = F \times R = 30 \times 0.4 = 12$ N–m.

If α is angular acceleration produced, then from $\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{12}{0.48} = 25 \text{ rad s}^{-2}$$

Linear acceleration, $a = R\alpha = 0.4 \times 25 = 10 \text{ ms}^{-2}$.

7.15. *To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 Nm. What is the power required by the engine?*

Note: Uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque. Assume that the engine is 100% efficient.

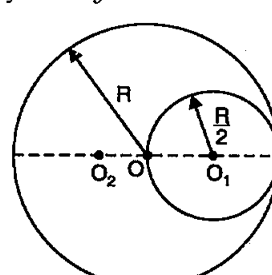
Ans. Here, $\omega = 200 \text{ rad s}^{-1}$; Torque, $\tau = 180$ N-m

Since, Power, P = Torque (τ) × angular speed (ω)

$= 180 \times 200 = 36000$ watt

$= 36$ KW.

7.16. *From a uniform disk of radius R, a circular hole of radius R/2 is cut out. The centre of the hole is at R/2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body.*



Ans. Let from a bigger uniform disc of radius R with centre O a smaller circular hole of radius $\frac{R}{2}$ with its centre at O_1 (where $OO_1 = \frac{R}{2}$) is cut out. Let centre of gravity or the centre of mass of remaining flat body be at O_2, where

$OO_2 = x$. If σ be mass per unit area, then mass of whole disc $M_1 = \pi R^2 \sigma$ and mass of cut out part

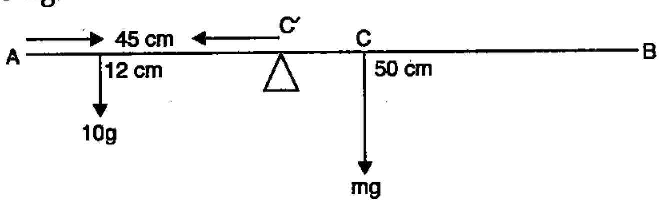
$$M_2 = \pi\left(\frac{R}{2}\right)^2 \sigma = \frac{1}{4}\pi R^2 \sigma = \frac{M_1}{4}$$

$$\therefore \quad x = \frac{M_1 \times (0) - M_2(OO_1)}{M_1 - M_2} = \frac{0 - \frac{M_1}{4} \times \frac{R}{2}}{M_1 - \frac{M_1}{4}} = -\frac{R}{6}$$

i.e., O_2 is at a distance $\frac{R}{6}$ from centre of disc on diametrically opposite side to centre of hole.

7.17. *A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?*

Ans. Let m be the mass of the stick concentrated at C, the 50 cm mark, see fig.



For equilibrium about C′, the 45 cm mark,

$$10g\,(45 - 12) = mg\,(50 - 45)$$
$$10g \times 33 = mg \times 5$$
$$\Rightarrow \quad m = \frac{10 \times 33}{5}$$
$$\text{or} \quad m = 66 \text{ grams.}$$

7.18. *A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination. (a) Will it reach the bottom with the same speed in each case? (b) Will it take longer to roll down one plane than the other? (c) If so, which one and why?*

Ans. (*a*) Using law of conservation of energy,

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

$$\text{or} \quad \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\frac{v^2}{R^2} = mgh$$

or $$\frac{7}{10}v^2 = gh \quad \text{or} \quad v = \sqrt{\frac{10gh}{7}}$$

Since h is same for both the inclined planes therefore **v is the same.**

(*b*) $$l = \frac{1}{2}\left(\frac{g\sin\theta}{1+\frac{K^2}{R^2}}\right)t^2 = \frac{g\sin\theta}{2\left(1+\frac{2}{5}\right)}t^2 = \frac{5g\sin\theta}{14}t^2$$

or $$t = \sqrt{\frac{14\,l}{5g\sin\theta}}$$

Now, $\sin\theta = \frac{h}{l}$ or $l = \frac{h}{\sin\theta}$

$\therefore$ $$t = \frac{1}{\sin\theta}\sqrt{\frac{14\,h}{5g}}$$

Lesser the value of θ, more will be t.

(*c*) Clearly, the solid sphere will take longer to roll down the plane with smaller inclination.

7.19. *A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it?*

Ans. Here, $R = 2$ m, $M = 100$ kg

$v = 20$ cm/s $= 0.2$ m/s

Total energy of the hoop $= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$= \frac{1}{2}Mv^2 + \frac{1}{2}(MR^2)\omega^2$

$= \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = Mv^2$

Work required to stop the hoop = total energy of the hoop

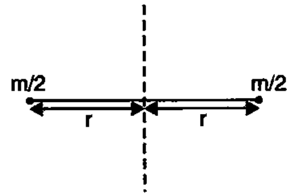
$W = Mv^2 = 100\ (0.2)^2 = 4$ Joule.

7.20. *The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-45} kg m^2 about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.*

Ans. Here, $m = 5.30 \times 10^{-26}$ kg

$I = 1.94 \times 10^{-46}$ kg m^2

$v = 500$ m/s



If $\frac{m}{2}$ is mass of each atom of oxygen and $2r$ is distance between the two atoms as shown in Fig, then

$$I = \frac{m}{2}r^2 + \frac{m}{2}r^2 = mr^2$$

$$r = \sqrt{\frac{I}{m}} = \sqrt{\frac{1.94 \times 10^{-46}}{5.30 \times 10^{-26}}} = 0.61 \times 10^{-10} \text{ m}$$

As K.E. of rotation $= \frac{2}{3}$ K.E. of translation

$$\therefore \quad \frac{1}{2}I\omega^2 = \frac{2}{3} \times \frac{1}{2}m\omega^2$$

$$\frac{1}{2}(mr^2)\,\omega^2 = \frac{1}{2}mv^2$$

$$\omega = \sqrt{\frac{2}{3}}\frac{v}{r} = \sqrt{\frac{2}{3}} \times \frac{500}{0.61 \times 10^{-10}}$$

$$= 6.7 \times 10^{12} \text{ rad/s.}$$

7.21. *A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5 m/s.*

(a) How far will the cylinder go up the plane?

(b) How long will it take to return to the bottom?

Ans. Here, $\theta = 30°$, $v = 5$ m/s

Let the cylinder go up the plane upto a height h.

From $$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

$$\frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 = mgh$$

$$\frac{3}{4}mv^2 = mgh$$

$$h = \frac{3v^2}{4g} = \frac{3 \times 5^2}{4 \times 9.8} = 1.913 \text{ m}$$

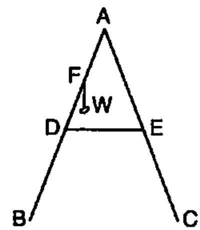
If s is the distance up the inclined plane, then as

$$\sin\theta = \frac{h}{s}, \quad s = \frac{h}{\sin\theta} = \frac{1.913}{\sin 30^\circ} \; 3.856 \text{ m}$$

Time taken to return to the bottom

$$t = \sqrt{\frac{2s\left(1+\frac{k^2}{r^2}\right)}{g\sin\theta}} = \sqrt{\frac{2\times 3.826\left(1+\frac{1}{2}\right)}{9.8\sin 30^\circ}} = 1.53\text{s.}$$

7.22. *As shown in Fig. the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied half way up. A weight 40 kg is suspended from a point F, 1.2 m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take $g = 9.8$ m s^2)*



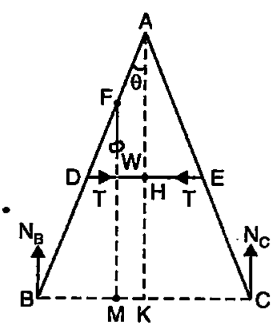
***(Hint:** Consider the equilibrium of each side of the ladder separately.)*

Ans. The forces acting on the ladder are shown in fig.

Here, W = 40 kg = 40 × 9.8 N = 392 N, $AB = AC$ = 1.6 m,

$$BD = \frac{1}{2} \times 1.6 \text{ m} = 0.8 \text{ m}, \; BF = 1.2 \text{ m and } DE = 0.5 \text{ m.}$$

In the Fig. ΔADE and ΔABC are similar triangles, hence



$$BC = DE \times \frac{AB}{AD}$$

$$= \frac{0.5\times 1.6}{0.8} = 1.0 \text{ m}$$

Now, considering equilibrium at point B,

$\Sigma\tau = 0$

$\therefore \quad W \times (MB) = N_C \times (CB) \quad ...(i)$

But $\quad MB = \dfrac{KB\times BF}{BA} = \dfrac{0.5\times 1.2}{1.6} = 0.375$ m

Substituting this value in (i), we get

$$\therefore \quad N_C = \frac{W\times(MB)}{(CB)} = \frac{392\times 0.375}{1} = 147 \text{ N}$$

Again considering equilibrium at point C in similar manner, we have

$$W \times (MC) = N_B \times (BC)$$

$$\therefore \quad N_B = \frac{W \times (MC)}{(BC)} = \frac{W \times (BC - BM)}{(BC)}$$

$$= \frac{392 \times (1 - 0.375)}{1} = 245 \text{ N}$$

Now, it can be easily shown that tension in the string

$$T = N_B - N_C = 245 - 147 = 98 \text{ N}.$$

7.23. *A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minutes. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90 cm to 20 cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m^2.*

(a) What is his new angular speed? (Neglect friction)

(b) Is kinetic energy conserved in the process? If not, from where does the change come about?

Ans. Here,

$$I_1 = 7.6 + 2 \times 5\,(0.9)^2 = 15.7 \text{ kg m}^2$$

$$\omega_1 = 30 \text{ rpm}$$

$$I_2 = 7.6 + 2 \times 5\,(0.2)^2 = 8.0 \text{ kg m}^2$$

$$\omega_2 = ?$$

According to the principle of conservation of angular momentum,

$$I_2\omega_2 = I_1\omega_1$$

$$\omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{15.7 \times 30}{8.0} = 58.88 \text{ rpm}$$

No, kinetic energy is not conserved in the process. In fact, as moment of inertia decreases, K.E. of rotation increases. This change comes about as work is done by the man in bringing his arms closer to his body.

7.24. *A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.*

*(**Hint:** The moment of inertia of the door about the vertical axis at one end is $ML^2/3$.)*

Ans. Angular momentum imparted by the bullet, $L = mv \times r$

$$= (10 \times 10^{-3}) \times 500 \times \frac{1}{2} = 2.5$$

Also, $$I = \frac{ML^2}{3} = \frac{12 \times (1.0)^2}{3} = 4 \text{ kg m}^2$$

Since $$L = I\omega$$

$\therefore$ $$\omega = \frac{L}{I} = \frac{2.5}{4} = 0.625 \text{ rad/s.}$$

7.25. *Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speed ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident. (a) What is the angular speed of the two-disc system? (b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$.*

Ans. (*a*) Let I_1 and I_2 be the moments of inertia of two discs having angular speeds ω_1 and ω_2 respectively. When they are brought in contact, the moment of inertia of the two-disc system will be $I_1 + I_2$. Let the system now have an angular speed w. From the law of conservation of angular momentum, we know that

$$I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\,\omega$$

$\therefore$ The angular speed of the two-disc system,

$$\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

(*b*) The sum of kinetic energies of the two discs before coming in contact,

$$k_1 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2$$

The final kinetic energy of the two-disc system,

$$k_2 = \frac{1}{2}(I_1 + I_2)\,\omega^2$$

$$= \frac{1}{2}(I_1 + I_2) \times \left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2$$

$$= \frac{1}{2}\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2$$

Now, $$k_1 - k_2 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 - \frac{1}{2}\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2$$

$$= \frac{1}{2(I_1+I_2)} \times \Big[(I_1\omega_1^2 + I_2\omega_2^2)(I_1+I_2) - (I_1^2\omega_1^2 + I_2^2\omega_2^2 + 2I_1I_2\,\omega_1\omega_2)\Big]$$

$$= \frac{1}{2(I_1+I_2)} \times [I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_1I_2\omega_1^2 + I_1I_2\,\omega_2^2 - I_1^2\omega_1^2 - I_2^2\omega_2^2 - 2I_1I_2\omega_1\omega_2]$$

$$= \frac{1}{2(I_1+I_2)}[I_1I_2(\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2)]$$

$$= \frac{I_1I_2}{2(I_1+I_2)}(\omega_1 - \omega_2)^2$$

Now, $(\omega_1 - \omega_2)^2$ will be positive whether ω_1 is greater or smaller than ω_2.

Also, $I_1I_2/2(I_1 + I_2)$ is also positive because I_1 and I_2 are positive.

Thus, $k_1 - k_2$ is a positive quantity.

$\therefore$ $k_1 = k_2$ + a positive quantity or $k_1 > k_2$

$\therefore$ The kinetic energy of the combined system (k_2) is less than the sum of the kinetic energies of the two dics.

The loss of energy on combining the two discs is due to the energy being used up because of the frictional forces between the surfaces of the two discs. These forces, in fact, bring about a common angular speed of the two discs on combining.

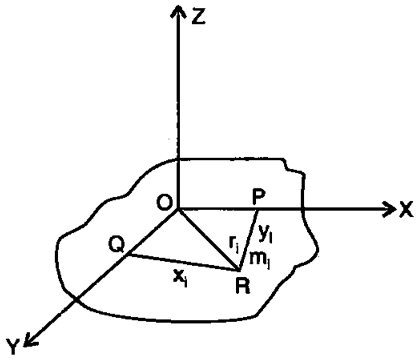
7.26. *(a) Prove the theorem of perpendicular axes.*

Hint: Square of the distance of a point (x, y) in the x – y plane from an axis through the origin perpendicular to the plane is $x^2 + y^2$]

(b) Prove the theorem of parallel axes.

Hint: If the centre of mass of chosen the origin $\sum m_i r_i = 0$]

Ans. (*a*) **The theorem of perpendicular axes:** According to this theorem, the moment of inertia of a plane lamina (*i.e.*, a two dimensional body of any shape/size) about any axis *OZ* perpendicular to the plane of the lamina is equal to sum of the moments of inertia of the lamina about any two mutually perpendicular axes *OX* and *OY* in the plane of lamina, meeting at a point where the given axis *OZ* passes through the lamina.



Suppose at the point 'R' m_i particle is situated moment of inertia about Z axis of lamina

$$I_z = \sum m_i r_i^2$$

$$= \sum m_i (x_i^2 + y_i^2) = \sum m_i x_i^2 + \sum m_i y_i^2$$

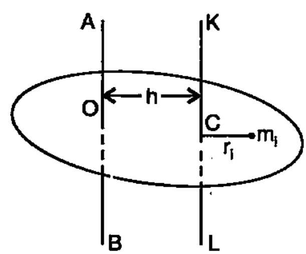
or $I_z = I_x + I_y$

where I_x = moment of inertia of body about x-axis

and I_y = moment of inertia of body about y-axis.

(Proved)

(*b*) **Theorem of parallel axes:** According to this theorem, moment of inertia of a rigid body about any axis *AB* is equal to moment of inertia of the body about another axis *KL* passing through centre of mass *C* of the body in a direction parallel to *AB*, plus the product of total mass *M* of the body and square of the perpendicular distance between the two parallel axes.



If *h* is perpendicular distance between the axes *AB* and *KL*, then

Suppose rigid body is made up of *n* particles $m_1, m_2, \ldots. m_i, \ldots..$ m_n at perpendicular distances $r_1, r_2, \ldots.. r_i \ldots. r_n$. respectively from the axis *KL* passing through centre of mass *C* of the body.

If r_i is the perpendicular distance of the particle of mass m_i from *KL*, then

$$I_{KL} = \sum_i m_i r_i^2 \quad \ldots(i)$$

The perpendicular distance of i^{th} particle from the axis

$$AB = (r_i + n)$$

or
$$I_{AB} = \sum_i m_i (r_i + h)^2$$

$$= \sum_i m_i (r_i^2 + h^2 + 2r_i h)$$

$$= \sum_i m_i r_i^2 + \sum_i m_i h^2 + 2h \sum_i m_i r_i \quad ...(ii)$$

As the body is balanced about the centre of mass, the algebraic sum of the moments of the weights of all particles about an axis passing through C must be zero.

$$\sum_i (m_i g) r_i = 0 \quad \text{or} \quad g \sum_i m_i r_i$$

or
$$\sum_i m_i r_i = 0 \quad ...(iii)$$

From equation (*ii*), we have

$$I_{AB} = \sum_i m_i r_i^2 + \left(\sum m_i\right) h^2 + 0$$

or
$$I_{AB} = I_{KL} + Mh^2$$

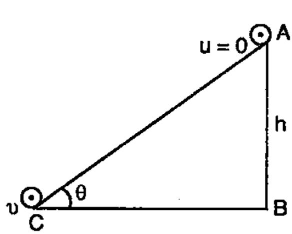
where
$$I_{KL} = \sum_i m_i r_i^2 \quad \text{and} \quad M = \sum m_i$$

7.27. *Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by,*

$$v^2 = \frac{2gh}{1 + k^2/R^2}$$

using dynamical consideration (i.e., by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

Ans. Let a rolling body ($I = Mk^2$) rolls down an inclined plane with an initial velocity $u = 0$; When it reaches the bottom of inclined plane, let its linear velocity be v. Then from conservation of mechanical energy, we have



Loss in $P.E.$ = Gain in translational K.E.

+ Gain in rotational K.E.

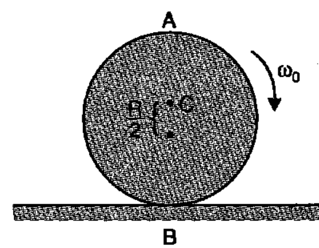
$$\therefore \qquad Mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}(mk^2)\left(\frac{v^2}{R^2}\right)$$

$$\therefore \qquad Mgh = \frac{1}{2}mv^2\left(1 + \frac{k^2}{R^2}\right)$$

$$\Rightarrow \qquad v^2 = \frac{2gh}{\left(1 + \frac{k^2}{R^2}\right)}.$$

7.28. *A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in Fig.? Will the disc roll in the direction indicated?*



Ans. Since $v = r\omega$,

For point, A, $v_A = R\omega_0$ in the direction of arrow.

For point, B, $v_B = R\omega_0$ in the opposite direction of arrow.

For point, C, $v_C = \frac{R}{2}\omega_0$ in the direction of arrow.

The disc will not roll in the given direction because friction is necessary for the same.

7.29. *Explain why friction is necessary to make the disc roll (refer to Q. 28) in the direction indicated.*

(a) Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.

(b) What is the force of friction after perfect rolling begins?

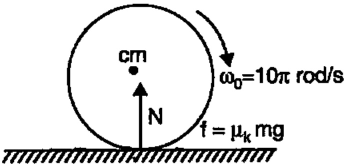
Ans. To roll a disc, we require a torque, which can be provided only by a tangential force. As force of friction is the only tangential force in this case, it is necessary.

(*a*) As frictional force at *B* opposes the velocity of point *B*, which is to the left, the frictional force must be to the right. The sense of frictional torque will be perpendicular to the plane of the disc and outwards.

(*b*) As frictional force at *B* decreases the velocity of the point of contact *B* with the surface, the perfect rolling begins only when velocity of point B becomes zero. Also, force of friction would become zero at this stage.

7.30. *A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10π rad/s. Which of two will start to roll earlier? The coefficient of kinetic friction is $\mu_k = 0.2$.*

Ans. When a disc or ring starts rotatory motion on a horizontal surface, initial translational velocity of centre of mass is zero. The frictional force causes the centre of mass to accelerate linearly but frictional torque causes angular retardation. As force of normal reaction $N = mg$, hence frictional force $f = \mu_k N = \mu_k mg$.



For linear motion $\quad f = \mu_k \cdot mg = ma$ ...(*i*)

and for rotational motion, $\tau = f \cdot R = \mu_k mg \cdot R = -I\alpha$ ...(*ii*)

Let perfect rolling motion starts at time t, when $v = R\omega$

From (*i*) $\quad a = \mu_k \cdot g$

$\therefore \quad v = u + at = 0 + \mu_k \cdot g \cdot t$ ...(*iii*)

From (*ii*) $\quad \alpha = -\dfrac{\mu_k \cdot mgR}{I} = -\dfrac{\mu_k \cdot mgR}{mK^2} = -\dfrac{\mu_k \cdot gR}{K^2}$

$\therefore \quad \omega = \omega_0 + \alpha t = \omega_0 - \dfrac{\mu_k \cdot gR}{K^2} t$ ...(*iv*)

Since $\quad v = R\omega$, hence $\mu_k \cdot g \cdot t = R\left[\omega_0 - \mu_k \cdot \dfrac{gR}{K^2} t\right]$

$$\Rightarrow \quad t^2 = \frac{R\omega_0}{\mu_k \cdot g\left(1 + \dfrac{R^2}{K^2}\right)}$$

For disc, $\quad K^2 = \dfrac{R^2}{2}$, hence $t = \dfrac{\omega_0 R}{\mu_k \cdot g\left(1 + \dfrac{R^2}{R^2/2}\right)} = \dfrac{\omega_0 R}{3\mu_k \cdot g}$

For ring, $\quad K^2 = R^2$, hence $t = \dfrac{\omega_0 R}{\mu_k \cdot g\left(1 + \dfrac{R^2}{R^2}\right)} = \dfrac{\omega_0 R}{2\mu_k \cdot g}$

Thus, it is clear that disc will start to roll earlier. The actual time at which disc starts rolling will be

$$t = \frac{\omega_0 R}{2\mu_k \cdot g} = \frac{(10\pi) \times (0.1)}{3 \times (0.2) \times 9.8} = 0.538.$$

7.31. *A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The coefficient of static friction* μ_s *= 0.25.*

(a) How much is the force of friction acting on the cylinder?

(b) What is the work done against friction during rolling?

(c) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly?

Ans. (*a*)

$$f = \frac{1}{2} mg \sin \theta$$

$$= \frac{1}{3} \times 10 \times 9.8 \times \sin 30° \text{ N} = \mathbf{16.3\ N.}$$

(*b*) **No work is done** against friction during rolling.

(*c*)

$$\mu = \frac{1}{3} \tan \theta \quad \text{or} \quad \tan \theta = 3\mu$$

$$\text{tam } \theta = 3 \times 0.25 = 0.75$$

$$\theta = \tan^{-1} (0.75) = 36.87° = \mathbf{37°.}$$

7.32. *Read each statement below carefully, and state, with reasons, if it is true or false:*

(a) During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.

(b) The instantaneous speed of the point of contact during rolling is zero.

(c) The instantaneous acceleration of the point of contact during rolloing is zero.

(d) For perfect rolling motion, work done against friction is zero.

(e) A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion.

Ans. (*a*) **True.** When a body rolls without slipping, the force of friction acts in the same direction as the direction of motion of the centre of mass of rolling body.

(*b*) **True.** This is because rolling body can be imagined to be rotating about an axis passing through the point of contact of the body with the ground. Hence its instantaneous speed is zero.

(*c*) **False.** This is because when the body is rotating, its instantaneous acceleration is not zero.

(*d*) **True.** For perfect rolling motion as there is no relative motion at the point of contact, hence work done against friction is zero.

(*e*) **True.** This is because rolling occurs only on account of friction which is a tangential force capable of providing torque. When the inclined plane is perfectly smooth, it will simply slip under the effect of its own weight.

7.33. *Separation of Motion of a system of particles into motion of the centre of mass and motion about the centre of mass:*

(a) *Show* $\vec{p_i} = \vec{p_i}' + m_i \vec{V}$

where p_i *is the momentum of the* i^{th} *particle (of mass* m_i*) and* $\vec{p_i}' + m_i \overrightarrow{v'_i}$. *Note* $\overrightarrow{v'_i}$ *is the veloity of* i^{th} *particle relative to the centre of mass.*

Also, prove using the definition of the centre of mass $\sum \vec{p_i} = 0$

(b) *Show* $K = K' + \frac{1}{2} MV^2$

where K is the total kinteic energy of the system of particles, K′ is the total kinetic energy of the system when the particle velocities are taken with respect to the centre of mass and $MV^2/2$ *is the kinetic energy of the translation of the system as a whole (i.e., of the centre of mass motion of the system). The result has been used in Sec. 7.14.*

(c) *Show* $\vec{L} = m_i \overrightarrow{L'} + M\vec{R} \times \vec{V}$

where $\vec{L} = \sum \vec{r'_i} \times \vec{p'_i}$ *is the angular momentum of the system about the centre of mass with velocities taken relative to the centre of mass. Remember* $\vec{r'_i} = \vec{r_i} - \vec{R}$; *rest of the notation is the standard notation used in the chapter. Note* $\vec{L'}$ and $\overrightarrow{MR} \times \vec{V}$ *can be said to be angular momenta respectively, about and of the centre of mass of the system of particles.*

(d) *Show*

$$\frac{d\vec{L'}}{dt} = \sum \vec{r'_i} \times \frac{d\vec{p'}}{dt}$$

Further, show that:

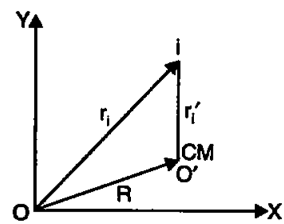
$$\frac{d\vec{L'}}{dt} = \overrightarrow{\tau'_{ext}}$$

where $\overrightarrow{\tau'_{ext}}$ *is the sum of all external torques acting on the system about the centre of mass.*

[Hint: use the definition of centre of mass and Newton's Third Law. Assume the internal forces between any two particles act along the line joining the particles.]

Ans. Here $\vec{r_i} = \vec{r'_i} + \vec{R}$ + R and $\vec{V_i} = \overrightarrow{V'_i} + \vec{V}$

where $\vec{r'_i}$ and $\vec{v'_i}$ denote the radius vector and velocity of the i^{th} particle referred to centre of mass O' as the new origin and $\vec{V}$ is the velocity of centre of mass relative to O.



(*a*) Momentum of i^{th} particle

$$\vec{P} = m_i \vec{V_i'}$$

$$= m_i (\vec{V_i'} + \vec{V}) \text{ (since } \vec{V_i} = \vec{V_i'} + \vec{V})$$

or
$$\vec{P} = m_i \vec{V_i'} + m_i \vec{V}$$

$$\vec{P} = \vec{P_i} + m_i \vec{V}$$

(*b*) Kinetic energy of the system of particles.

$$K = \frac{1}{2}\sum m_i V_i^2 = \frac{1}{2}\sum m_i \vec{V_i} \cdot \vec{V_i}$$

$$= \frac{1}{2}\sum m_i \left(\overrightarrow{V_i'} + \vec{V}\right)\cdot\left(\overrightarrow{V_i'} + \vec{V}\right)$$

$$= \frac{1}{2}\sum m_i \left(V_i'^2 + V^2 + 2\overrightarrow{V_i'} \cdot \vec{V}\right)$$

$$= \frac{1}{2}\sum m_i V_i'^2 + \frac{1}{2}\sum m_i V_i'^2 + \sum m_i \vec{V_i} \cdot \vec{V}$$

$$= \frac{1}{2} MV^2 + K'$$

where $M = \sum m_i$

= total mass of the system

$$K' = \frac{1}{2}\sum_i m_i V_i'^2$$

= kinetic energy of motion about the centre of mass

or $\frac{1}{2}Mv^2$ = kinetic energy of motion of centre of mass.

(Proved)

since
$$\sum_i m_i \overrightarrow{V'_i} \cdot \vec{V} = \sum m_i \frac{d\vec{r_i}}{dt} \cdot \vec{V}$$
$$= \frac{d}{dt}\left(\sum m_i \overrightarrow{r'_i}\right) \cdot \vec{V} = \frac{d}{dt}\left(M\vec{R} . \vec{V}\right)$$
$$= 0$$

(*c*) Total angular momentum of the system of particles.

$$\vec{L} = \vec{r_i} \times \vec{p}$$
$$= (\vec{r_i} + \vec{R}) \times \sum_i m_i \vec{V_i}$$
$$= (\vec{r'}_i + \vec{R}) \times \sum_i m_i (\vec{V_i}' + \vec{V})$$
$$= \sum_i \left(\vec{R} \times m_i \vec{V}\right) + \sum_i \vec{r'}_i \times m_i \vec{V'}_i + \left(\sum_i m_i \vec{r'}_i\right) \times \vec{V} + \vec{R} \times \sum_i m_i \vec{V}_i$$
$$= \sum_i \left(\vec{R} \times m_i \vec{V}\right) + \sum_i \vec{r_i}' \times m_i \vec{V_i} + \left(\sum_i m_i \vec{r_i}'\right) \times \vec{V} + \vec{R} \times \frac{d}{dt}\left(\sum_i m_i \vec{r_i}'\right)$$

The last two terms vanish for both contain the factor $\sum m_i \vec{r'}_i$ which is equal to

$$\sum_i m_i \vec{r_i} = \sum_i m_i \left(\vec{r'}_i - \vec{R}\right) = M\vec{R} - M\vec{R} = \vec{0}$$

from the definition of centre of mass. Also

$$\sum_i \left(\vec{R} \times m_i \vec{V}\right) = \vec{R} \times M\vec{V}$$

so that $$\vec{L} = \vec{R} \times M\vec{V} + \sum_i \vec{r}'_i \times \vec{P}_i$$

or $$\vec{L} = \vec{R} \times M\vec{V} + \vec{L}'$$

where $$\vec{L}' = \sum \vec{r}'_i \times \vec{P}_i$$

(*d*) From previous solution

$$\vec{L}' = \sum \vec{r}'_i \times \vec{P}_i$$

$$\frac{d\vec{L}'}{dt} = \sum \vec{r}_i{}' \times \frac{d\vec{P}_i}{dt} + \sum \frac{d\vec{r}'_i}{dt} \times \vec{P}_i$$

$$= \sum \vec{r}'_i \times \frac{d\vec{P}_i}{dt}$$

$$= \sum \vec{r}_i \times \vec{F}_i^{ext} = \vec{\tau}'_{ext}$$

Since $$\sum \frac{d\vec{r}'_i}{dt} \times \vec{P}_i = \sum \frac{d\vec{r}'_i}{dt} \times m\vec{v}_i = 0$$

Total torque $$\vec{\tau} = \sum \vec{r}_i \times \vec{F}_i^{ext}$$

$$= \sum \left(\vec{r}_i + \vec{R}\right) \times \vec{F}_i^{ext}$$

$$= \sum \vec{r}'_i \times \vec{F}_i^{ext} + \vec{R} \times \sum_i \vec{F}_i^{ext}$$

$$= \vec{\tau}'_{ext} + \vec{\tau}_0^{(ext)}$$

where $\vec{\tau}_{ext}$ is the total torque about the centre of mass as origin and $\vec{\tau}_0^{ext}$, that about the origin *O*.

$$\vec{\tau}'_{ext} = \sum \vec{r}'_i \times \vec{F}_i^{ext}$$

$$= \sum \vec{r}_i \times \frac{d\vec{P}'_i}{dt} = \frac{d}{dt} \sum_i \left(\vec{r}_i \times \vec{P}_i\right) = \frac{d\vec{L}'}{dt}$$

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