14 Oscillations

Lesson at a Glance

• Periodic Motion

Motions, processes or phenomena, which repeat themselves at regular intervals, are called periodic.

• Oscillatory Motion

The motion of a body is said to be oscillatory motion if it moves to and fro about a fixed point after regular intervals of time. The fixed point about which the body oscillates is called mean position or equilibrium position.

• Simple Harmonic Motion

Simple harmonic motion is a special type of periodic oscillatory motion in which

(*i*) The particle oscillates on a straight line

(*ii*) The acceleration of the particle is always directed towards a fixed point on the line.

(*iii*) The magnitude of acceleration is proportional to the displacement of the particle from the fixed point, *i.e.*, $a \propto -x$

Acceleration $= -\omega^2 x$

• Characteristics of SHM

The displacement x in SHM at time t is given by

$$x = A \sin(\omega t + \phi)$$

where the three constants A, ω and ϕ characterize the SHM, *i.e.*, they distinguish one SHM from another. A SHM can also be described by a cosine function as follows:

$$x = A \cos(\omega t + \delta)$$

• The time taken by an oscillating particle to complete one full oscillation to and fro about its mean (equilibrium) position is called the "time period" of SHM.

It is given by $$T = 2\pi\sqrt{\frac{m}{k}}$$

where, k is force constant (or spring factor) of spring.

• Frequency

The number of oscillations in one second is called frequency. It is expressed in sec^{-1} or Hertz. Frequency and time period are independent of amplitude.

$$\text{Frequency } (\nu) = \frac{1}{\text{Time period}}\left(\frac{1}{T}\right).$$

• Phase

The quantity $(\omega t + \phi)$ is called the phase of SHM at time t; it describes the state of motion at that instant. The quantity ϕ is the phase at time $t = 0$ and is called the phase constant or initial phase or epoch of the SHM. The phase constant is the time–independent term in the cosine or sine function.

• Springs in Series

If two springs, having spring constant k_1 and k_2, are joined in series, the spring constant of the combination is given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

• Springs in Parallel

If two springs, having spring constants k_1 and k_2, are joined in parallel, the spring constant of the combination is given by

$$k = k_1 + k_2$$

- When one spring is attached to two masses m_1 and m_2, then

$$M = \frac{m_1 m_2}{(m_1 + m_2)} \quad \therefore \quad T = 2\pi\sqrt{\frac{M}{K}}$$

• Simple Pendulum

A simple pendulum is the most common example of bodies executing S.H.M. An ideal simple pendulum consists of a heavy point mass body suspended by a weightless inextensible and perfectly flexible string from a rigid support about which it is free to oscillate.

- The time period of simple pendulum of length 'l' is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

The time period of a simple pendulum depends on

(*i*) length of the pendulum and

(*ii*) the acceleration due to gravity (g).

• A second's pendulum is a pendulum whose time period is 2s. At a place where $g = 9.8\ ms^{-2}$, the length of a second's pendulum is found to be 99.3 cm (≈ 1 m).

• Undamped and Damped Simple Harmonic Oscillations

Undamped Simple Harmonic oscillations: When a simple harmonic system oscillates with a constant amplitude which does not change with time, its oscillations are called undamped simple harmonic oscillations.

Damped Simple Harmonic oscillations: When a simple harmonic system oscillates with a decreasing amplitude with time, its oscillations are called damped simple harmonic oscillations.

The angular frequency of the damped oscillator is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

where b is damping constant.

Time period of damped oscillations is given by

$$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

• Resonance is the phenomenon of setting a body into oscillations with large amplitude under the influence of some external periodic force whose frequency is exactly equal to the natural frequency of the given body. Such oscillations are called the "resonant oscillations".

TEXTBOOK QUESTIONS SOLVED

14.1. *Which of the following examples represent periodic motion?*

(a) A swimmer completing one (return) trip from one bank of a river to the other and back.

(b) A freely suspended bar magnet displaced from its N-S direction and released.

(c) A hydrogen molecule rotating about its centre of mass.

(d) An arrow released from a bow.

Ans. (*a*) It is not a periodic motion. Though the motion of a swimmer is to and fro but will not have a definite period.

(*b*) Since a freely suspended magnet if once displaced from N-S direction and released, it oscillates about this position, it is a periodic motion.

(*c*) The rotating motion of a hydrogen molecule about its centre of mass is periodic.

(*d*) Motion of an arrow released from a bow is non-periodic.

14.2. *Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?*

(a) the rotations of earth about its axis.

(b) motion of an oscillating mercury column in a U-tube.

(c) motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.

(d) general vibrations of a polyatomic molecule about its equilibrium position.

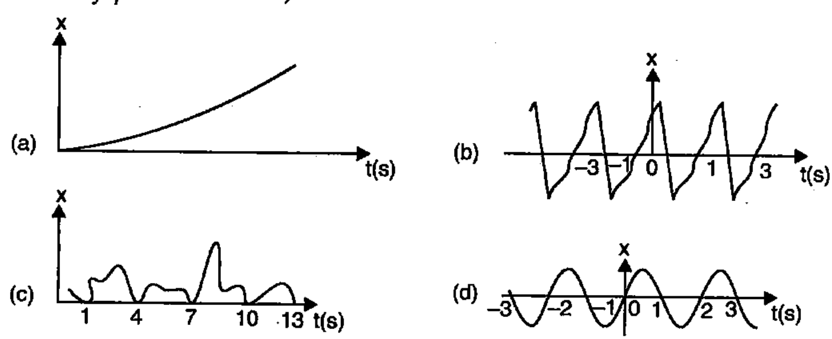
Ans. (*a*) Since the rotation of earth is not to and fro mot.on about a fixed point, thus it is periodic but not S.H.M.

(*b*) It is S.H.M.

(*c*) It is S.H.M.

(*d*) General vibrations of a polyatomic molecule about its equilibrium position is periodic but non SHM. Infact, it is a result of superposition of SHMs executed by individual vibrations of atoms of the molecule.

14.3. *Fig. depicts four x-t plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion)?*



Ans. Figure (*b*) and (*d*) represent periodic motions and the time period of each of these is 2 seconds. (*a*) and (*c*) are non-periodic motions.

14.4. *Which of the following function of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant).*

(*a*) $\sin \omega t - \cos \omega t$ (*b*) $\sin^2 \omega t$ (*c*) $3 \cos \left(\frac{\pi}{4} - 2\omega t\right)$

(d) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$ (e) $\exp(-\omega^2 t^2)$

(f) $1 + \omega t + \omega^2 t^2$.

Ans. The function will represent a periodic motion, if it is identically repeated after a fixed interval of time and will represent S.H.M if it can be written uniquely in the form of a $\cos\left(\frac{2\pi t}{T}+\phi\right)$ or a $\sin\left(\frac{2\pi t}{T}+\phi\right)$, where T is the time period.

(a)
$$\sin\omega t - \cos\omega t = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\omega t - \frac{1}{\sqrt{2}}\cos\omega t\right]$$
$$= \sqrt{2}\left[\sin\omega t\cos\frac{\pi}{4} - \cos\omega t\sin\frac{\pi}{4}\right]$$
$$= \sqrt{2}\sin\left(\omega t - \frac{\pi}{4}\right)$$

It is a S.H.M. and its period is $2\pi/\omega$

(b) $\sin^3 \omega t = \frac{1}{3}[3\sin\omega t - \sin 3\omega t]$

Here each term $\sin \omega t$ and $\sin 3\omega t$ individually represents S.H.M. But (ii) which is the outcome of the superposition of two SHMs will only be periodic but not **SHMs.** Its time period is $2\pi/\omega$.

(c)
$$3\cos\left(\frac{\pi}{4} - 2\omega t\right) = 3\cos\left(2\omega t - \frac{\pi}{4}\right).$$

[$\because \cos(-\theta) = \cos\theta$]

Clearly it represents SHM and its time period is $2\pi/2\omega$.

(d) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$. It represents the periodic but not S.H.M. Its time period is $2\pi/\omega$

(e) $e^{-\omega^2 t^2}$. It is an exponential function which never repeats itself. Therefore it represents non-periodic motion.

(f) $1 + \omega t + \omega^2 t^2$ also represents non periodic motion.

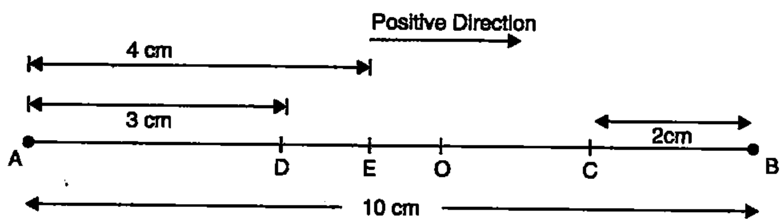
14.5. *A particle is in linear simple harmonic motion between two points, A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is*

(a) at the end A,

(b) at the end B,

(c) at the mid-point of AB going towards A,
(d) at 2 cm away from B going towards A,
(e) at 3 cm away from A going towards B, and
(f) at 4 cm away from B going towards A.

Ans. In the fig. (given below), the points A and B, 10 cm apart, are the extreme positions of the particle in SHM, and the point O is the mean position. The direction from A to B is positive, as indicated.



(*a*) At the end A, *i.e.*, extreme position, velocity is zero, acceleration and force are directed towards O and are positive.

(*b*) At the end B, *i.e.*, second extreme position, velocity is zero whereas the acceleration and force are directed towards the point O and are negative.

(*c*) At the mid point O, while going towards A, velocity is negative and maximum. The acceleration and force both are zero.

(*d*) At 2 cm away from B, that is, at C and going towards A: v is negative; acceleration and F, being directed towards O, are also negative.

(*e*) At 3 cm away from A, that is, at D and going towards B: v is positive; acceleration and F, being directed towards O, are also positive.

(*f*) At a distance of 4 cm away from A and going towards A, velocity is directed along BA, therefore, it is positive. Since acceleration and force are directed towards OB, both of them are positive.

14.6. *Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?*

(*a*) $a = 0.7x$ (*b*) $a = -200x^2$
(*c*) $a = -10x$ (*d*) $a = 100x^3$

Ans. Only (c) *i.e.*, $a = -10x$ represents SHM. This is because acceleration is proportional and opposite to displacement (x).

14.7. *The motion of a particle executing simple harmonic motion is described by the displacement function.*

$$x(t) = A \cos(\omega t + \phi).$$

If the initial (t = 0) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle? The angular frequency of the particle is π s⁻¹. If instead of the cosine function, we choose the sine function to describe the SHM: x = B sin (ωt + α), what are the amplitude and initial phase of the particle with the above initial conditions?

Ans. The given displacement function is

$$x(t) = A \cos(\omega t + \phi) \quad ...(i)$$

At $t = 0$, $x(0) = 1$ cm. Also, $\omega = \pi\ s^{-1}$

$\therefore$ $1 = A \cos(\pi \times 0 + \phi)$

$\Rightarrow$ $A \cos \phi = 1$ ...(ii)

Also, differentiating eqn. (i) w.r.t. 't'.

$$v = \frac{d}{dt}x(t) = -A\,\omega \sin(\omega t + \phi) \quad ...(iii)$$

Now at $t = 0$, $v = \omega$

$\therefore$ from eqn. (iii), $\omega = -A\,\omega \sin(\pi \times 0 + \phi)$

or $A \sin \phi = -1$ ...(iv)

Squaring and adding eqns. (ii) and (iv).

$$A^2 \cos^2 \phi + A^2 \sin^2 \phi = 1^2 + 1^2 \quad \text{or} \quad A = \sqrt{2}\text{ cm}$$

Dividing eqns. (ii) and (iv),

$$\frac{A \sin \phi}{A \cos \phi} = \frac{-1}{1} \quad \therefore \quad \tan \phi = -1 \quad \Rightarrow \quad \phi = \frac{3\pi}{4}$$

If instead we use the sine function, *i.e.*,

$x = B \sin(\omega t + \alpha)$, then

$$v = \frac{d}{dt}B\omega \cos(\omega t + \alpha)$$

$\therefore$ At $t = 0$, using $x = 1$ and $v = \omega$, we get

$1 = B \sin(\omega \times 0 + \alpha)$

or $B \sin \alpha = 1$ ...(v)

and $\omega = B\omega \cos(\omega \times 0 + \alpha)$ or $B \cos \alpha = 1$ ...(vi)

Dividing (v) by (vi),

$$\tan \alpha = 1 \quad \text{or} \quad \alpha = \frac{\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

Squaring (*v*) and (*vi*), we get

$$B^2 \sin^2 \alpha + B^2 \cos^2 \alpha = 1^2 + 1^2 \Rightarrow B = \sqrt{2} \text{ cm.}$$

14.8. *A spring balance has a scale that reads from 0 to 50 kg. The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body?*

Ans. $M = 50\text{kg}, \; y = 20\text{cm} = 0.2 \text{ m}, T = 0.60 \text{ s}$

$$F = ky \text{ or } Mg = ky \text{ or } k = \frac{Mg}{0.2} = \frac{50 \times 9.8}{0.2} \text{Nm}^{-1}$$

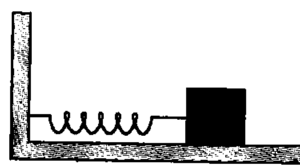
or $K = 2450 \text{ Nm}^{-1}$

Now, $T = 2\pi\sqrt{\frac{m}{k}}$ or $T^2 = 4\pi^2 \frac{m}{k}$ or $m = \frac{T^2 k}{4\pi^2}$

or $m = \frac{0.6 \times 0.6 \times 2460 \times 49}{4 \times 484} \text{kg} = 22.3 \text{ kg}$

$\Rightarrow$ $mg = 22.3 \times 9.8 \text{ N} = 218.5 \text{ N} = 22.3 \text{ kgf.}$

14.9. *A spring having with a spring constant 1200 Nm⁻¹ is mounted on a horizontal table as shown in Fig. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.*



Determine (i) the frequency of oscillations, (ii) maximum acceleration of the mass, and (iii) the maximum speed of the mass.

Ans. Here, $K = 1200 \text{ Nm}^{-1}$; $m = 3.0$ kg, $a = 2.0 \text{ cm} = 0.02 \text{ m}$

(*i*) Frequency, $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2 \times 3.14}\sqrt{\frac{1200}{3}} = 3.2 \text{ s}^{-1}$

(*ii*) Acceleration, $A = \omega^2 \; y = \frac{k}{m} y$

Acceleration will be maximum when y is maximum *i.e.*, $y = a$

$\therefore$ max. acceleration, $A_{max} = \frac{ka}{m} = \frac{1200 \times 0.02}{3} = 8 \text{ ms}^{-2}$

(*iii*) Max. speed of the mass will be when it is passing through mean position

$$V_{max} = a\omega = a\sqrt{\frac{k}{m}} = 0.02 \times \sqrt{\frac{1200}{3}} = 0.4 \text{ ms}^{-1}$$

14.10. *In Exercise 9, let us take the position of mass when the spring is unstreched as x = 0, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch (t = 0), the mass is*

(a) at the mean position,

(b) at the maximum stretched position, and

(c) at the maximum compressed position.

In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

Ans. $a = 2$ cm, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3}} s^{-1} = 20 s^{-1}$

(*a*) Since time is measured from mean position,

$$x = a \sin \omega t = 2 \sin 20t$$

(*b*) At the maximum stretched position, the body is at the extreme right position. The initial phase is $\frac{\pi}{2}$.

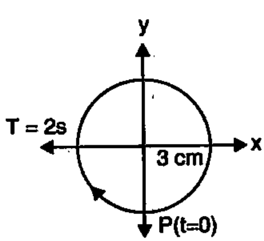
$$\therefore \quad x = a \sin (\omega t + \frac{\pi}{2}) = a \cos \omega t = 2 \cos 20\, t$$

(*c*) At the maximum compressed position, the body is at the extreme left position. The initial phase is $\frac{3\pi}{2}$.

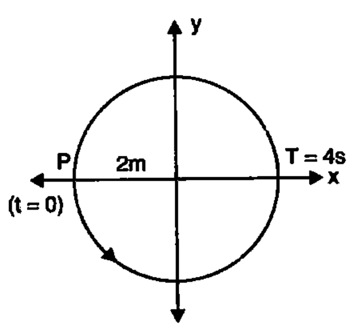
$$\therefore \quad x = a \sin (\omega t + \frac{3\pi}{2}) = -a \cos \omega t = -2 \cos 20t$$

Note: *The functions neither differ in amplitude nor in frequency. They differ only in initial phase.*

14.11. *Following figures correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e., clockwise or anticlockwise) are indicated on each figure.*



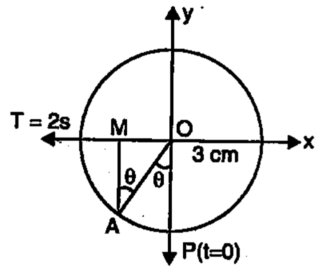
(a)



(b)

Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P in each case.

Ans. (1) Let A be any point on the circle of reference of the fig. (a) From A, draw BN perpen-dicular on x-axis.



If $\angle POA = \theta$, then

$\angle OAM = \theta = \omega t$

$\therefore$ In triangle OAM,

$$\frac{OM}{OA} = \sin\theta$$

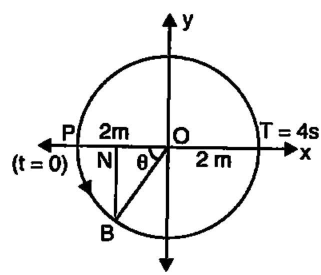
$$\therefore \quad \frac{-x}{3} = \sin\omega t = \sin\frac{2\pi}{T}t$$

[x is –ve in fourth quadrant]

$$\therefore \quad x = -3\sin\frac{2\pi}{2}t \quad \text{or} \quad x = -3\sin\pi t$$

which is the equation of SHM.

(2) Let B be any point on the circle of reference of fig. (b). From B, draw BN perpendicular on x-axis.



Then $\angle BON = \theta = \omega t$

$\therefore$ In ΔONB,

$$\cos\theta = \frac{ON}{OB}$$

or $\quad ON = OB\cos\theta$

$\therefore \quad -x = 2\cos\omega t$

$$\Rightarrow \quad x = -2\cos\frac{2\pi}{T}t = -2\cos\frac{2\pi}{4}t$$

$$\therefore \quad x = -2\cos\frac{\pi}{4}t \quad \text{which is equation of SHM}$$

14.12. *Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t = 0) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anti-clockwise in every case:*

(x is in cm and t is in s)

(a) $x = -2 \sin (3t + \pi/3)$ *(b)* $x = \cos (\pi/6 - t)$

(c) $x = 3 \sin (2\pi t + \pi/4)$ *(d)* $x = 2 \cos \pi t$.

Ans. (*a*) $x = 2 \cos \left(3t + \frac{\pi}{3} + \frac{\pi}{2}\right)$

Radius of the reference circle, r = amplitude of SHM = 2 cm,

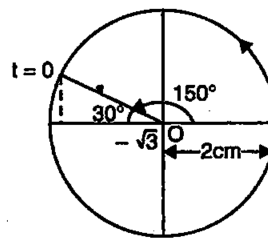
At $t = 0$, $x = -2 \sin \frac{\pi}{3} = \frac{-2\sqrt{3}}{2}$

$= -\sqrt{3}$ cm

Also $\omega t = 3t \quad \therefore \omega = 3$ rad/s

$\cos \phi_0 = -\frac{\sqrt{3}}{2}, \quad \phi_0 = 150°$

The reference circle is, thus, as plotted given in figure.



(*b*) $x = \cos\left(t - \frac{\pi}{6}\right)$

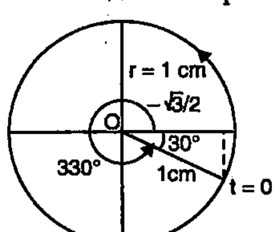
Radius of circle, r = amplitude of SHM = 1 cm.

At $t = 0, \quad x = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ cm

Also $\omega t = 1t \Rightarrow \omega = 1$ rad/s

$\cos \phi_0 = \frac{\sqrt{3}}{2}, \quad \phi_0 = -\frac{\pi}{6}$

The reference circle is, thus as plotted below.



(*c*) $x = 3 \cos \left(2\pi t + \frac{\pi}{4} + \frac{\pi}{2}\right)$

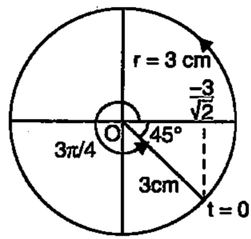
Here, radius of reference circle, r = 3 cm and at t = 0,

$x = 3 \sin \frac{\pi}{4} = \frac{\sqrt{3}}{2}$ cm

$$\omega t = 2\pi t \Rightarrow \omega = 2\pi \text{ rad/s}$$

$$\cos \phi_0 = \frac{\sqrt{\frac{3}{2}}}{3} = \frac{1}{\sqrt{2}}$$

Therefore, the reference circle is being shown below.



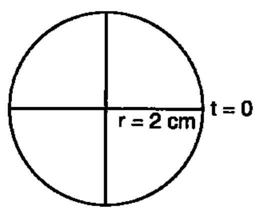
(*d*) $$x = 2 \cos \pi t$$

Radius of reference circle, $r = 2$ cm and at $t = 0, x = 2$ cm

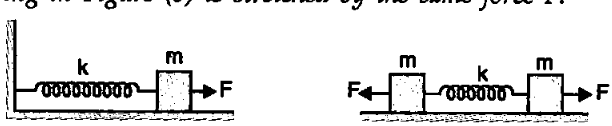
$\therefore$ $$\omega t = \pi t, \text{ or } \omega = \pi \text{ rad/s}$$

$$\cos \phi_0 = 1, \quad \phi_0 = 0$$

The reference circle is plotted below.



14.13. *Figure (a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force F applied at the free end stretches the spring. Figure (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Figure (b) is stretched by the same force F.*



(a) (b)

(a) What is the maximum extension of the spring in the two cases?

(b) If the mass in Fig. (a) and the two masses in Fig. (b) are released free, what is the period of oscillation in each case?

Ans. (*a*) Let y be the maximum extension produced in the spring in Fig. (*a*)

Then $F = ky$ (in magnitude) $\therefore \quad y = \frac{F}{k}$

If fig. (*b*), the force on one mass acts as the force of reaction due to the force on the other mass. Therefore, each mass behaves as if it is fixed with respect to the other.

Therefore, $F = ky \Rightarrow y = \frac{F}{k}$

(*b*) In fig. (*a*), $F = -ky$

$$\Rightarrow \quad ma = -ky \Rightarrow a = -\frac{k}{m}y$$

$$\therefore \quad \omega^2 = \frac{k}{m} \quad i.e., \quad \omega = \sqrt{\frac{k}{m}}$$

Therefore, period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

In fig. (*b*), we may consider that the centre of the system is O and there are two

springs each of length $\frac{l}{2}$ attached to the two masses, each *m*, so that k' is the spring factor of each of the springs.

Then, $K' = 2k$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{2k}}$$

14.14. *The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rev/min, what its maximum speed?*

Ans. Stroke of piston = 2 times the amplitude

Let A = amplitude, stroke = 1 m

$$\therefore \quad \Rightarrow A = \frac{1}{2} \text{m.}$$

Angular frequency, ω = 200 rad/min.

$V_{max} = ?$

We know that the maximum speed of the block when the amplitude is A,

$$V_{max} = \omega A = 200 \times \frac{1}{2} = 100 \text{ m/min.}$$

$$= \frac{100}{60} = \frac{5}{3}\text{ms}^{-1} = 1.67 \text{ ms}^{-1}.$$

14.15. *The acceleration due to gravity on the surface of moon is 1.7 ms⁻². What is the time period of a simple pendulum on the surface of moon if its time-period on the surface of Earth is 3.5 s? (g on the surface of Earth is 9.8 ms⁻².)*

Ans. Here, $g_m = 1.7 \text{ ms}^{-2}$; $g_e = 9.8 \text{ ms}^{-2}$; $T_m = ?$; $T_e = 3.5 \text{ s}^{-1}$

Since, $T_e = 2\pi\sqrt{\frac{1}{g_e}}$ and $T_m = 2\pi\sqrt{\frac{1}{g_m}}$

$\therefore$ $\frac{T_m}{T_e} = \sqrt{\frac{g_e}{g_m}} \Rightarrow T_m = T_e = \sqrt{\frac{g_e}{g_m}}$

$$= 3.5\sqrt{\frac{9.8}{1.7}} = 8.4 \text{ s.}$$

14.16. *Answer the following questions:*

(a) Time period of a particle in SHM depends on the force constant k and mass m of the particle:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

A simple pendulum executes SHM approximately.
Why then is the time period of a pendulum independent of the mass of the pendulum?

(b) The motion of a simple pendulum is approximately simple harmonic for small angle oscillations. For larger angles of oscillation, a more involved analysis shows that T is greater than $2\pi\sqrt{\frac{l}{g}}$. *Think of a qualitative argument to appreciate this result.*

(c) A man with a wrist watch on his hand falls from the top of a tower. Does the watch give correct time during the free fall?

(d) What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity?

Ans. (*a*) In case of a spring, k does not depend upon m. However, in case of a simple pendulum, k is directly proportional to m and hence the ratio $\frac{m}{k}$ is a constant quantity.

(*b*) The restoring force for the bob of the pendulum is given by

$$F = -mg \sin\theta$$

If θ is small, then $\sin\theta = \theta = \frac{y}{l}$ $\therefore$ $F = -\frac{mg}{l}y$

i.e., the motion is simple harmonic and time period is

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Cleraly, the above formula is obtained only if we apply the approximation $\sin\theta \approx \theta$. For large angles, this approximation is not valid and T is greater than $2\pi\sqrt{\frac{l}{g}}$.

(*c*) The wrist watch uses an electronic system or spring system to give the time, which does not change with acceleration due to gravity. Therefore, watch gives the correct time.

(*d*) During free fall of the cabin, the acceleration due to gravity is zero. Therefore, the frequency of oscillations is also zero *i.e.*, the pendulum will not vibrate at all.

14.17. *A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?*

Ans. In this case, the bob of the pendulum is under the action of two accelerations.

(*i*) Acceleration due to gravity 'g' acting vertically downwards.

(*ii*) Centripetal acceleration $a_c = \frac{v^2}{R}$ acting along the horizontal direction.

$\therefore$ Effective acceleration, $g' = \sqrt{g^2 + a_c^2}$ or $g' = \sqrt{g^2 + \frac{v^4}{R^2}}$

Now time period, $T' = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + \frac{v^4}{R^2}}}}$.

14.18. *A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_l. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period*

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}$$

where ρ is the density of cork. (Ignore damping due to viscosity of the liquid).

Ans. Say, initially in equilibrium, y height of cylinder is inside the liquid. Then,

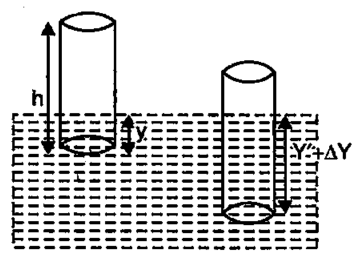
Weight of the cylinder = upthrust due to liquid displaced

$$\therefore \quad Ah\rho g = Ay\rho_l g$$

When the cork cylinder is depressed slightly by Δy and released, a restoring force, equal to additional upthrust, acts on it. The restoring force is

$$F = A(y + \Delta y)\,\rho_l g - Ay\rho_l g = A\rho_l g\Delta y$$

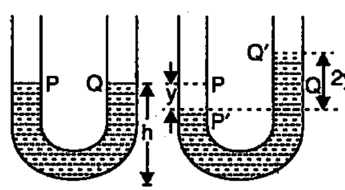
$\therefore$ Acceleration, $a = \frac{F}{m} = \frac{A\rho_l g\Delta y}{Ah\rho} = \frac{\rho_l g}{h\rho}.\,\Delta y$ and the acceleration is directed in a direction opposite to Δy. Obviously, as $a \propto -\Delta y$, the motion of cork cylinder is SHM, whose time period is given by



$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

$$= 2\pi\sqrt{\frac{\Delta y}{a}} = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}.$$

4.19. *One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.*



Ans. The suction pump creates the pressure difference, thus mercury rises in one limb of the U-tube. When it is removed, a net force acts on the liquid column due to the difference in levels of mercury in the two limbs and hence the liquid column executes S.H.M. which can be expressed as:

Consider the mercury contained in a vertical U-tube upto the level P and Q in its two limbs.

Let P = density of the mercury.

L = Total length of the mercury column in both the limbs.

A = internal cross-sectional area of U-tube.

m = mass of mercury in U-tube = LAP.

Assume, the mercury be depressed in left limb to P' by a small distance y, then it rises by the same amount in the right limb to position Q'.

∴ Difference in levels in the two limbs = $P'Q' = 2y$.

∴ Volume of mercury contained in the column of length

$$2y = A \times 2y$$

∴ $$m = A \times 2y \times \rho.$$

If W = weight of liquid contained in the column of length $2y$.

Then $$W = mg = A \times 2y \times \rho \times g$$

This weight produces the restoring force (F) which tends to bring back the mercury to its equilibrium position.

∴ $$F = -2\,Ay\rho g = -(2A\rho g)y$$

If a = acceleration produced in the liquid column, Then

$$a = \frac{F}{m}$$

$$= -\frac{(2A\rho g)y}{LA\rho} = -\frac{2A\rho g}{LA}$$

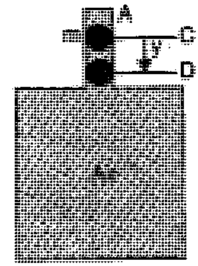
$$= -\frac{2\rho g}{2h\rho}y \qquad \text{...(i)} \quad (\because L = 2h)$$

where h = height of mercury in each limb. Now from eqn. (*i*), it is clear that $a \propto y$ and $-ve$ sign shows that it acts opposite to y, so the motion of mercury in u-tube is simple harmonic in nature having time period (T) given by

$$T = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{2h\rho}{2\rho g}} = 2\pi\sqrt{\frac{h\rho}{\rho g}}$$

$$T = 2\pi\sqrt{\frac{h}{g}}$$

4.20. *An air chamber of volume V has a neck area of cross section a into which a ball of mass m just fits and can move up and down without any friction (Fig.). Show that when the ball is pressed down a little*



and released, it executes SHM. Obtain an expression for the time period of oscillations assuming pressure-volume variations of air to be isothermal.

Ans. Consider an air chamber of volume V with a long neck of uniform area of cross-section A, and a frictionless ball of mass m fitted smoothly in the neck at position C, Fig. The pressure of air below the ball inside the chamber is equal to the atmospheric pressure. Increase the pressure on the ball by a little amount p, so that the ball is depressed to position D, where $CD = y$.

There will be decrease in volume and hence increase in pressure of air inside the chamber. The decrease in volume of the air inside the chamber, $\Delta V = Ay$

$$\text{Volumetric strain} = \frac{\text{change in volume}}{\text{original volume}}$$

$$= \frac{\Delta V}{V} = \frac{Ay}{V}$$

$\therefore$ Bulk Modulus of elasticity E, will be

$$E = \frac{\text{stress (or increase in pressure)}}{\text{volumetric strain}}$$

$$= \frac{-p}{Ay/V} = \frac{-pV}{Ay}$$

Here, negative sign shows that the increase in pressure will decrease the volume of air in the chamber.

Now, $$p = \frac{-E\,Ay}{V}$$

Due to this excess pressure, the restoring force acting on the ball is

$$F = p \times A = \frac{-E\,Ay}{V} \cdot A = \frac{-E\,A^2}{V} y \quad ...(i)$$

Since $F \propto y$ and negative sign shows that the force is directed towards equilibrium position. If the applied increased pressure is removed from the ball, the ball will start executing linear SHM in the neck of chamber with C as mean position.

In S.H.M., the restoring force,

$$F = -ky \quad ...(ii)$$

Comparing (*i*) and (*ii*), we have

Spring factor, $k = EA^2/V$

Here, inertia factor = mass of ball = m.

$$\text{Period, } T = 2\pi\sqrt{\frac{\text{inertia factor}}{\text{spring factor}}}$$

$$= 2\pi\sqrt{\frac{m}{EA^2/V}} = \frac{2\pi}{A}\sqrt{\frac{mV}{E}}$$

$$\therefore \quad \text{Frequency, } \nu = \frac{1}{T} = \frac{A}{2\pi}\sqrt{\frac{E}{mV}}$$

Note. If the ball oscillates in the neck of chamber under isothermal conditions, then $E = P$ = pressure of air inside the chamber, when ball is at equilibrium position. If the ball oscillates in the neck of chamber under adiabatic conditions, then $E = \gamma P$, where $\gamma = C_p/C_v$.

4.21. *You are riding an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg. ($g = 10$ m/s^2.)*

Ans. (*a*) Here, mass, M = 300 kg, displacement, x = 15 cm = 0.15 m, g = 10 m/s^2. There are four spring systems. If k is the spring constant of each spring, then total spring constant of all the four springs in parallel is

$$K_p = 4k \quad \therefore \quad M_g = k_p x = 4kx$$

$$\Rightarrow \qquad K = \frac{Mg}{4x} = \frac{3000\times 10}{4\times 0.15} = 5 \times 10^4 \text{ N.}$$

(*b*) For each spring system supporting 750 kg of weight,

$$t = 2\pi\sqrt{\frac{m}{k}} = 2 \times 3.14 \times \sqrt{\frac{750}{5\times 10^4}} = 0.77 \text{ sec.}$$

$\therefore$ Using $x = x_0\, e^{-\frac{bt}{2m}}$, we get

$$\frac{50}{100}x_0 = x_0\, e^{-\frac{b\times 0.77}{2\times 750}} \quad \text{or} \quad e^{\frac{0.77b}{1500}} = 2$$

Taking logarithm of both sides,

$$\frac{0.77b}{1500} = \ln 2 = 2.303 \log 2$$

$$\therefore \quad b = \frac{1500}{0.77} \times 2.303 \times 0.3010 = 1350.4 \text{ kg s}^{-1}$$

4.22. *Show that for a particle in linear SHM the average kinetic energy over a period of oscillation equals the average potential energy over the same period.*

Ans. Let the particle executing SHM starts oscillating from its mean position. Then displacement equation is

$$x = A \sin \omega t$$

$$\therefore \quad \text{Particle velocity, } v = A\omega \cos \omega t$$

$$\therefore \quad \text{Instantaneous K.E., } K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2 \cos^2 \omega t$$

∴ Average value of K.E. over one complete cycle

$$K_{av} = \frac{1}{T}\int_0^T \frac{1}{2}mA^2\omega^2 \cos^2 \omega t \, dt = \frac{mA^2\omega^2}{2T}\int_0^T \cos^2 \omega t \, dt$$

$$= \frac{mA^2\omega^2}{2T}\int_0^T \frac{(1+\cos 2\omega t)}{2} dt$$

$$= \frac{mA^2\omega^2}{4T}\left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{mA^2\omega^2}{4T}\left[(T-0) + \left(\frac{\sin 2\omega t - \sin 0}{2\omega}\right)\right]$$

$$= \frac{1}{4}mA^2\omega^2 \qquad \ldots(i)$$

Again instantaneous P.E.,

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2A^2 \sin^2 \omega t$$

∴ Average value of P.E. over one complete cycle

$$U_{av} = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2A^2 \sin^2 \omega t = \frac{m\omega^2A^2}{2T}\int_0^T \sin^2 \omega t \, dt$$

$$= \frac{m\omega^2A^2}{2T}\int_0^T \frac{(1-\cos 2\omega t)}{2} dt$$

$$= \frac{m\omega^2 A^2}{4T}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{m\omega^2 A^2}{4T}\left[(T-0) - \frac{(\sin 2\omega t - \sin 0)}{2\omega}\right]$$

$$= \frac{1}{4} m\omega^2 A^2 \qquad ...(ii)$$

Simple comparison of (*i*) and (*ii*), shows that

$$K_{av} = U_{av} = \frac{1}{4} m\omega^2 A^2$$

4.23. *A circular disc, of mass 10 kg, is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations of found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire. (Torsional spring constant a is defined by the relation J = - αθ, where J is the restoring couple and θ the angle of twist).*

Ans. $T = 2\pi\sqrt{\frac{I}{\alpha}}$ or $T^2 = \frac{4\pi^2 I}{\alpha}$

or $\alpha = \frac{4\pi^2 I}{T^2}$ or $\alpha = \frac{4\pi^2}{T^2}\left(\frac{1}{2}MR^2\right)$

or $\alpha = \frac{2\pi^2 MR^2}{T^2}$

or $\alpha = \frac{2(3.14)^2 \times 10 \times (0.15)^2}{(1.5)^2}$ Nm rad^{-1}

$= \mathbf{1.97}$ **Nm rad^{-1}.**

4.24. *A body describes simple harmonic motion with an amplitude of 5 cm and a period of 0.2s. Find the acceleration and velocity of the body when the displacement is (a) 5 cm (b) 3 cm (c) 0 cm.*

Ans. Here, r = 5 cm = 0.05 m; T = 0.2 s; $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi$ rad/s

When displacement is y, then

acceleration, $A = -\omega^2 y$

velocity, $V = \omega\sqrt{r^2 - y^2}$

Case (*a*) When $y = 5\text{ cm} = 0.05\text{ m}$

$$A = -(10\pi)^2 \times 0.05 = -5\pi^2\text{ m/s}^2$$

$$V = 10\pi\sqrt{(0.05)^2 - (0.05)^2} = 0.$$

Case (*b*) When $y = 3\text{ cm} = 0.03\text{ m}$

$$A = -(10\pi)^2 \times 0.03 = -3\pi^2\text{ m/s}^2$$

$$V = 10\pi\sqrt{(0.05)^2 - (0.03)^2}$$

$$= 10\pi \times 0.04 = 0.4\pi\text{ m/s}$$

Case (c) When $y = 0, \quad A = -(10\pi)^2 \times 0 = 0$

$$V = 10\pi\sqrt{(0.05)^2 - 0^2}$$

$$= 10\pi \times 0.05 = 0.5\pi\text{ m/s.}$$

4.25. *A mass attached to a spring is free to oscillate, with angular velocity w, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time $t = 0$. Determine the amplitude of the resulting oscillations in terms of the parameters w, x_0 and v_0.*

Ans.

$$x = \alpha\cos(\omega t + \theta)$$

$$v = \frac{dx}{dt} = -a\omega\sin(\omega t + \theta)$$

When $t = 0, \quad x = x_0$ and $\frac{dx}{dt} = -v_0$

$\therefore$ $x_0 = a\cos\theta$ ...(*i*)

and $-v_0 = -a\,\omega\sin\theta$ or $a\sin\theta = \frac{v_0}{\omega}$ ...(*ii*)

Squaring and adding (*i*) and (*ii*), we get

$$a^2(\cos^2\theta + \sin^2\theta) = x_0^2 + \frac{v_0^2}{\omega^2}$$

or

$$a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}.$$

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