2 Inverse Trigonometric Functions

2.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Sol. We know that $\frac{5\pi}{6}\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\frac{13\pi}{6}\notin[0,\pi]$

$$\therefore\ \tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

$$=\tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right]+\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]+\cos^{-1}\left(\cos\frac{\pi}{6}\right)$$

$$=\tan^{-1}\left(-\tan\frac{\pi}{6}\right)+\cos^{-1}\left(\cos\frac{\pi}{6}\right)$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\cos^{-1}\left(\cos\frac{\pi}{6}\right)\qquad[\because \tan^{-1}(-x)=-\tan^{-1}x]$$

$$=-\frac{\pi}{6}+\frac{\pi}{6}=0$$

Hence, $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=0$

Q2. Evaluate: $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

Sol. $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

$$=\cos\left[\pi-\cos^{-1}\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right]\qquad[\because \cos^{-1}(-x)=\pi-\cos^{-1}x]$$

$$=\cos\left[\pi-\frac{\pi}{6}+\frac{\pi}{6}\right]=\cos\pi=-1$$

Hence, $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]=-1.$

Q3. Prove that: $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$.

Sol. L.H.S. $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)$

$$=\cot\left[\tan^{-1}(1)-2\tan^{-1}\frac{1}{3}\right] \qquad \left[\because \cot^{-1}x=\tan^{-1}\frac{1}{x}\right]$$

$$=\cot\left[\tan^{-1}(1)-\tan^{-1}\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right] \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$=\cot\left[\tan^{-1}(1)-\tan^{-1}\frac{\frac{2}{3}}{\frac{8}{9}}\right]$$

$$=\cot\left[\tan^{-1}(1)-\tan^{-1}\frac{3}{4}\right]$$

$$=\cot\left[\tan^{-1}\left(\frac{1-\frac{3}{4}}{1+1\times\frac{3}{4}}\right)\right]=\cot\left[\tan^{-1}\left(\frac{\frac{1}{4}}{\frac{7}{4}}\right)\right]$$

$$=\cot\left[\tan^{-1}\frac{1}{7}\right] \qquad \left[\because \tan^{-1}\frac{1}{x}=\cot^{-1}x\right]$$

$=\cot[\cot^{-1}(7)]=7$ R.H.S.

Hence proved.

Q4. Find the value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

Sol. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

$$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(\sqrt{3}\right)+\tan^{-1}(-1) \qquad \left[\begin{array}{l}\because \tan^{-1}(-x) = -\tan^{-1} x \\ \tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right) \\ \sin\left(\frac{-\pi}{2}\right) = -1\end{array}\right]$$

$$= -\frac{\pi}{6}+\frac{\pi}{3}-\frac{\pi}{4} = \frac{-\pi}{12} \qquad \left[\because \tan^{-1}(-1) = \frac{-\pi}{4}\right]$$

Hence, $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right] = \frac{-\pi}{12}$

Q5. Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

Sol. We know that $\frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \quad \tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{3}\right)\right] = \tan^{-1}\left(-\tan\frac{\pi}{3}\right)$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{3}\right) \quad [\because \tan^{-1}(-x) = -\tan^{-1} x]$$

$$= -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Hence, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \frac{-\pi}{3}$.

Q6. Show that: $2\tan^{-1}(-3) = \frac{-\pi}{2}+\tan^{-1}\left(\frac{-4}{3}\right)$

Sol. L.H.S. $\quad 2\tan^{-1}(-3) = -2\tan^{-1}(3)$

$$= -\cos^{-1}\left[\frac{1-(3)^2}{1+(3)^2}\right] \qquad \left[\because 2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= -\cos^{-1}\left(\frac{1-9}{1+9}\right) = -\cos^{-1}\left(\frac{-8}{10}\right)$$

$$= -\cos^{-1}\left(\frac{-4}{5}\right) = -\left[\pi-\cos^{-1}\left(\frac{4}{5}\right)\right] = -\pi+\cos^{-1}\frac{4}{5}$$

$$= -\pi+\tan^{-1}\left(\frac{3}{4}\right) \qquad \left[\because \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}\right]$$

$$= -\pi + \frac{\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right) \qquad \left[\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x\right]$$

$$= \frac{-\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right)$$

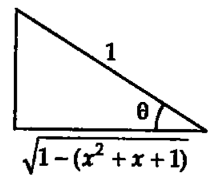
$$= \frac{-\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right) \qquad \left[\because \tan^{-1} x = \cot^{-1}\frac{1}{x}\right]$$

$$= \frac{-\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right) \text{ R.H.S.}$$

Hence proved.

Q7. Find the real solutions of the equation

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$



Sol. $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \dfrac{\pi}{2}$

Let $\quad \theta = \sin^{-1}\sqrt{x^2+x+1}$

$\therefore \quad \sin\theta = \sqrt{x^2+x+1}$

$$\Rightarrow \quad \tan\theta = \frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}} \quad \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}\right)$$

$$\Rightarrow \sin^{-1}\sqrt{x^2+x+1} = \tan^{-1}\left(\sqrt{\frac{x^2+x+1}{-x^2-x}}\right)$$

So, $\tan^{-1}\sqrt{x(x+1)} + \tan^{-1}\left(\sqrt{\dfrac{x^2+x+1}{-x^2-x}}\right) = \dfrac{\pi}{2}$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{x(x+1)} + \sqrt{\dfrac{x^2+x+1}{-x(x+1)}}}{1 - \sqrt{x(x+1)} \times \sqrt{\dfrac{x^2+x+1}{-x(x+1)}}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\dfrac{x(x+1) + \sqrt{-(x^2+x+1)}}{\sqrt{x(x+1)}}}{1 - \sqrt{-(x^2+x+1)}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \frac{x^2+x-\sqrt{-(x^2+x+1)}}{\left[1-\sqrt{-(x^2+x+1)}\right]\sqrt{x^2+x}} = \tan\frac{\pi}{2} = \frac{1}{0}$$

$$\Rightarrow \left[1-\sqrt{-(x^2+x+1)}\right]\sqrt{x^2+x} = 0$$

$$\Rightarrow \left[1-\sqrt{-(x^2+x+1)}\right] = 0 \quad \text{or} \quad \sqrt{x^2+x} = 0$$

Here, $\sqrt{-(x^2+x+1)} \notin R$

$\therefore \quad \sqrt{x^2+x} = 0$

$\Rightarrow \quad x^2+x=0 \quad \Rightarrow \quad x(x+1)=0$

$\Rightarrow \quad x=0 \quad \text{or} \quad x+1=0 \quad \Rightarrow \quad x=0 \quad \text{or} \quad x=-1$

Hence the real solutions are $x=0$ and $x=-1$.

Alternate Method:

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\sqrt{x^2+x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1}$$

$$\Rightarrow \quad \tan^{-1}\sqrt{x^2+x} = \cos^{-1}\sqrt{x^2+x+1} \quad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \cos^{-1}\left[\frac{1}{\sqrt{1+x^2+x}}\right] = \cos^{-1}\sqrt{x^2+x+1}$$

$$\left[\because \tan^{-1}x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{x^2+x+1}} = \sqrt{x^2+x+1}$$

$\Rightarrow \quad x^2+x+1 = 1 \quad \Rightarrow x^2+x=0$

$\Rightarrow \quad x(x+1) = 0 \quad \Rightarrow x=0 \quad \text{or} \quad x+1=0$

$\therefore \quad x=0, x=-1$

Q8. Find the value of the expression

$$\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$$

Sol. $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)\right]+\cos\left[\cos^{-1}\frac{1}{\sqrt{1+(2\sqrt{2})^2}}\right]$$

$$\left[\because \tan^{-1}x=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)\right]+\cos\left[\cos^{-1}\left(\frac{1}{3}\right)\right]$$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{3}{4}\right)\right]+\frac{1}{3} \quad \Rightarrow \quad \sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right]+\frac{1}{3}$$

$$\Rightarrow \frac{3}{5}+\frac{1}{3} \quad \Rightarrow \frac{14}{15} \qquad \left[\because \tan^{-1}x=\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right]$$

Hence, $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)=\frac{14}{15}$.

Q9. If $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$, then show that $\theta=\frac{\pi}{4}$

Sol. $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right)=\tan^{-1}(2\operatorname{cosec}\theta)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \frac{2\cos\theta}{1-\cos^2\theta}=2\operatorname{cosec}\theta \Rightarrow \frac{2\cos\theta}{\sin^2\theta}=\frac{2}{\sin\theta}$$

$\Rightarrow \cos\theta\sin\theta=\sin^2\theta$

$\Rightarrow \cos\theta\sin\theta-\sin^2\theta=0 \Rightarrow \sin\theta(\cos\theta-\sin\theta)=0$

$\Rightarrow \sin\theta=0$ or $\cos\theta-\sin\theta=0$

$\Rightarrow \sin\theta=0$ or $1-\tan\theta=0$

$\Rightarrow \theta=0$ or $\tan\theta=1$

$\Rightarrow \theta=0°$ or $\theta=\frac{\pi}{4}$ **Hence proved.**

Q10. Show that: $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

Sol. L.H.S. $\cos\left(2\tan^{-1}\frac{1}{7}\right)$

$$= \cos\left[\cos^{-1}\frac{1-\frac{1}{49}}{1+\frac{1}{49}}\right] \qquad \left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}\right]$$

$$= \cos\left[\cos^{-1}\frac{48}{50}\right] = \cos\left[\cos^{-1}\frac{24}{25}\right] = \frac{24}{25}$$

R.H.S. $\sin\left[4\tan^{-1}\frac{1}{3}\right]$

$$= \sin\left[2\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}\right)\right] \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \sin\left[2\tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)\right] = \sin\left[2\tan^{-1}\frac{3}{4}\right]$$

$$= \sin\left[\sin^{-1}\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right] \qquad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= \sin\left[\sin^{-1}\frac{24}{25}\right] \Rightarrow \frac{24}{25}$$

L.H.S. = R.H.S. **Hence proved.**

Q11. Solve the following equation: $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

Sol. Given that $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$$\Rightarrow \cos\left[\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \sin\left[\sin^{-1}\frac{4}{5}\right]$$

$$\left[\because \tan^{-1}x = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right), \quad \cot^{-1}x = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^2}} = \frac{4}{5}$$

Squaring both sides we get,

$$\frac{1}{1+x^2} = \frac{16}{25} \quad \Rightarrow \quad 1+x^2 = \frac{25}{16}$$

$$\Rightarrow \quad x^2 = \frac{25}{16} - 1 = \frac{9}{16} \quad \Rightarrow x = \pm\frac{3}{4}$$

Hence, $x = \frac{-3}{4}, \frac{3}{4}$.

LONG ANSWER TYPE QUESTIONS

Q12. Prove that: $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

Sol. L.H.S. $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Put $x^2 = \cos\theta \quad \therefore \theta = \cos^{-1}x^2$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta/2}+\sqrt{2\sin^2\theta/2}}{\sqrt{2\cos^2\theta/2}-\sqrt{2\sin^2\theta/2}}\right] \quad \left[\because \begin{matrix} 1+\cos\theta = 2\cos^2\theta/2 \\ 1-\cos\theta = 2\sin^2\theta/2 \end{matrix}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{1+\tan\theta/2}{1-\tan\theta/2}\right] \quad \text{[Dividing the Nr. and Den. by } \cos\theta/2]$$

$$\Rightarrow \tan\left[\tan\left(\frac{\pi}{4}\quad\frac{\theta}{2}\right)\right] \quad \left[\because \frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$\Rightarrow \frac{\pi}{4}+\frac{\theta}{2} \quad \Rightarrow \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \text{ R.H.S.} \quad [\text{Putting } \theta = \cos^{-1}x^2]$$

Hence proved.

Q13. Find the simplified form of $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$, where $x \in \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$

Sol. Given that $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$

Put $\frac{3}{5} = \cos y$

$\therefore \quad \sqrt{1-\cos^2 y} = \sin y \Rightarrow \sqrt{1-\frac{9}{25}} = \sin y \Rightarrow \frac{4}{5} = \sin y$

$\therefore \cos^{-1}\left[\frac{3}{5}\cos x + \frac{4}{5}\sin x\right] = \cos^{-1}[\cos y \cos x + \sin y \sin x]$

$= \cos^{-1}[\cos(y-x)] = y - x$

$= \tan^{-1}\frac{4}{3} - x \qquad \left[\tan y = \frac{\sin y}{\cos y} = \frac{4}{3}\right]$

Q14. Prove that: $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$

Sol. L.H.S. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$

Using $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\left[\frac{8}{17}.\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}.\sqrt{1-\left(\frac{8}{17}\right)^2}\right]$

$= \sin^{-1}\left[\frac{8}{17}.\sqrt{1-\frac{9}{25}} + \frac{3}{5}.\sqrt{1-\frac{64}{289}}\right]$

$= \sin^{-1}\left[\frac{8}{17}.\sqrt{\frac{16}{25}} + \frac{3}{5}.\sqrt{\frac{225}{289}}\right]$

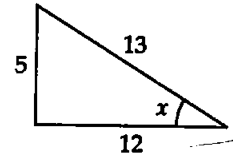
$= \sin^{-1}\left[\frac{8}{17}.\frac{4}{5} + \frac{3}{5}.\frac{15}{17}\right] = \sin^{-1}\left[\frac{32}{85} + \frac{45}{85}\right]$

$= \sin^{-1}\frac{77}{85}$ R.H.S. **Hence proved.**

Q15. Show that: $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$

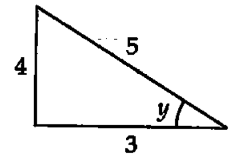
Sol. Let $\sin^{-1}\frac{5}{13}=x \Rightarrow \sin x=\frac{5}{13}$

$\Rightarrow \tan x=\frac{5}{12}$



Let $\cos^{-1}\frac{3}{5}=y \Rightarrow \cos y=\frac{3}{5}$

$\Rightarrow \tan y=\frac{4}{3}$



Now $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$

$$\Rightarrow \tan(x+y)=\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}=\frac{\frac{15+48}{36}}{\frac{36-20}{36}}=\frac{63}{16}$$

$$\Rightarrow x+y=\tan^{-1}\frac{63}{16}$$

$\therefore \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$ **Hence proved.**

Q16. Prove that: $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\sin^{-1}\frac{1}{\sqrt{5}}$

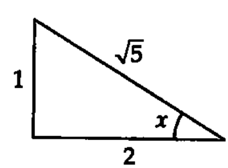
Sol. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\tan^{-1}\left[\dfrac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right]$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right]=\tan^{-1}\left[\frac{17}{34}\right]$$

Let $\tan^{-1}\left[\frac{17}{34}\right]=x$

$\therefore \tan x=\frac{17}{34}=\frac{1}{2}$



$$\sin x = \frac{1}{\sqrt{5}}$$

$$\therefore \qquad \tan^{-1}\frac{1}{2} = \sin^{-1}\frac{1}{\sqrt{5}} \text{ R.H.S.}$$

$$\text{Hence, } \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$$

Q17. Find the value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

Sol. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$\Rightarrow 2.\left(2\tan^{-1}\frac{1}{5}\right) - \tan^{-1}\frac{1}{239}$$

$$\Rightarrow 2\left[\tan^{-1}\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right] - \tan^{-1}\frac{1}{239} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow 2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{239} \quad \Rightarrow \tan^{-1}\left(\frac{2\times\frac{5}{12}}{1-\frac{25}{144}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right] \qquad \left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{120\times 239-119}{119\times 239+120}\right] \Rightarrow \tan^{-1}\left[\frac{28680-119}{28441+120}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{28561}{28561}\right] = \tan^{-1}(1) = \frac{\pi}{4}$$

Q18. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

Sol. To prove that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

L.H.S. Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = \tan^{-1}\theta$ $\qquad [\therefore \tan(\tan^{-1}\theta) = \theta]$

$$\Rightarrow \quad \sin^{-1}\frac{3}{4} = 2\tan^{-1}\theta \Rightarrow \sin^{-1}\frac{3}{4} = \sin^{-1}\left(\frac{2\theta}{1+\theta^2}\right)$$

$$\left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$\Rightarrow \quad \frac{2\theta}{1+\theta^2} = \frac{3}{4} \Rightarrow 3 + 3\theta^2 = 8\theta$$

$$\Rightarrow \quad 3\theta^2 - 8\theta + 3 = 0$$

$$\Rightarrow \quad \theta = \frac{-(-8) \pm \sqrt{(-8)^2 - 4\times 3\times 3}}{2\times 3}$$

$$= \frac{8 \pm \sqrt{64-36}}{6} = \frac{8\pm\sqrt{28}}{6} = \frac{8\pm 2\sqrt{7}}{6} = \frac{2\left(4\pm\sqrt{7}\right)}{6}$$

$$\Rightarrow \quad \theta = \frac{4\pm\sqrt{7}}{3}$$

$$\therefore \quad \theta = \frac{4+\sqrt{7}}{3} \text{ or } \frac{4-\sqrt{7}}{3}$$

$\theta = \dfrac{4+\sqrt{7}}{3}$ is ignored.

Because $\dfrac{-\pi}{2} \le \sin^{-1}\dfrac{3}{4} \le \dfrac{\pi}{2}$

$$\Rightarrow \frac{-\pi}{4} \le \frac{1}{2}\sin^{-1}\frac{3}{4} \le \frac{\pi}{4}$$

$$\Rightarrow \tan\left(\frac{-\pi}{4}\right) \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow -1 \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le 1$$

Hence, $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right) = \dfrac{4-\sqrt{7}}{3}$

Q19. If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then evaluate the following expression

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + \ldots \right.$$

$$\left. \ldots + \tan^{-1}\left(\frac{d}{1+a_{n-1}.a_n}\right)\right]$$

Sol. If $a_1, a_2, a_3, \ldots, a_n$ are the terms of an arithmetic progression

$\therefore d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 \ldots.$

$$\therefore \tan\left[\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{a_4-a_3}{1+a_3a_4}\right)+\ldots\right.$$
$$\left.\ldots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}.a_n}\right)\right]$$

$$\Rightarrow \tan[(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)+(\tan^{-1}a_4-\tan^{-1}a_3)$$
$$+\ldots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})]$$
$$\left[\because \tan^{-1}\frac{x-y}{1+xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$\Rightarrow \tan[\tan^{-1}a_2-\tan^{-1}a_1+\tan^{-1}a_3-\tan^{-1}a_2+\tan^{-1}a_4-\tan^{-1}a_3$$
$$+\ldots+\tan^{-1}a_n-\tan^{-1}a_{n-1}]$$

$$\Rightarrow \tan[\tan^{-1}a_n-\tan^{-1}a_1]$$

$$\Rightarrow \tan\left[\tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)\right] \Rightarrow \frac{a_n-a_1}{1+a_1a_n} \qquad [\because \tan(\tan^{-1}x)=x]$$

OBJECTIVE TYPE QUESTIONS

Choose the correct answers from the given four options in each of the Exercises from 20 to 37 (M.C.Q.).

Q20. Which of the following is the principal value branch of $\cos^{-1}x$?

(*a*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (*b*) $(0,\pi)$

(*c*) $[0,\pi]$ (*d*) $(0,\pi)-\left\{\frac{\pi}{2}\right\}$

Sol. Principal value branch of $\cos^{-1}x$ is $[0,\pi]$. Hence the correct answer is (*c*).

Q21. Which of the following is the principal value branch of $\text{cosec}^{-1}x$?

(*a*) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ (*b*) $[0,\pi]-\left\{\frac{\pi}{2}\right\}$

(*c*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (*d*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Sol. Principal value branch of $\text{cosec}^{-1}x$ is

$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$ as $\text{cosec}^{-1}(0)=\infty$ (not defined).

Hence, the correct answer is (*d*).

Q22. If $3\tan^{-1}x + \cot^{-1}x = \pi$, then x equals

(*a*) 0 (*b*) 1 (*c*) −1 (*d*) $\frac{1}{2}$

Sol. Given that $3\tan^{-1}x + \cot^{-1}x = \pi$

$\Rightarrow 2\tan^{-1}x + \tan^{-1}x + \cot^{-1}x = \pi$

$\Rightarrow \quad 2\tan^{-1}x + \frac{\pi}{2} = \pi \qquad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow \quad 2\tan^{-1}x = \pi - \frac{\pi}{2} \quad \Rightarrow 2\tan^{-1}x = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{4} \quad \Rightarrow \tan^{-1}x = \tan^{-1}(1)$

$\therefore \quad x = 1$

Hence, the correct answer is (*b*).

Q23. The value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ is

(*a*) $\frac{3\pi}{5}$ (*b*) $\frac{-7\pi}{5}$ (*c*) $\frac{\pi}{10}$ (*d*) $\frac{-\pi}{10}$

Sol. $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right] = \sin^{-1}\left[\cos\left(6\pi + \frac{3\pi}{5}\right)\right]$

$= \sin^{-1}\left[\cos\frac{3\pi}{5}\right] \qquad [\because \cos(2n\pi + x) = \cos x]$

$= \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right]$

$= \sin^{-1}\left[-\sin\left(\frac{\pi}{10}\right)\right] \quad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$

$= \sin^{-1}\left[\sin\left(\frac{-\pi}{10}\right)\right] = \frac{-\pi}{10}$

Hence, the correct answer is (*d*).

Q24. The domain of the function $\cos^{-1}(2x-1)$ is

(*a*) [0, 1] (*b*) [−1, 1] (*c*) (−1, 1) (*d*) [0, π]

Sol. The given function is $\cos^{-1}(2x-1)$

Let $\quad f(x) = \cos^{-1}(2x-1)$

$-1 \le 2x - 1 \le 1 \Rightarrow -1 + 1 \le 2x \le 1 + 1$

$0 \le 2x \le 2 \quad \Rightarrow 0 \le x \le 1$

$\therefore$ domain of the given function is [0, 1].

Hence, the correct answer is (*a*)

Q25. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(*a*) [1, 2] (*b*) [– 1, 1] (*c*) [0, 1] (*d*) None of these

Sol. Let $f(x) = \sin^{-1}\sqrt{x-1}$

$\because \sqrt{x-1} \geq 0$ and $-1 \leq \sqrt{x-1} \leq 1$

$\Rightarrow 0 \leq x-1 \leq 1 \Rightarrow 1 \leq x \leq 2 \Rightarrow x \in [1, 2]$

Hence, the correct answer is (*a*).

Q26. If $\cos\left[\sin^{-1}\frac{2}{5} + \cos^{-1}x\right] = 0$, then x is equal to

(*a*) $\frac{1}{5}$ (*b*) $\frac{2}{5}$ (*c*) 0 (*d*) 1

Sol. Given that $\cos\left[\sin^{-1}\frac{2}{5} + \cos^{-1}x\right] = 0$

$\Rightarrow \sin^{-1}\frac{2}{5} + \cos^{-1}x = \cos^{-1}(0)$

$\Rightarrow \sin^{-1}\frac{2}{5} + \cos^{-1}x = \frac{\pi}{2} \Rightarrow \sin^{-1}\frac{2}{5} = \frac{\pi}{2} - \cos^{-1}x$

$\Rightarrow \sin^{-1}\frac{2}{5} = \sin^{-1}x \quad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow x = \frac{2}{5}$

Hence, the correct answer is (*b*).

Q27. The value of sin [2 tan^{-1} (0.75)] is equal to

(*a*) 0.75 (*b*) 1.5 (*c*) 0.96 (*d*) sin 1.5

Sol. Given that sin [2 tan^{-1} (0.75)]

$$= \sin\left[2\tan^{-1}\frac{3}{4}\right]$$

$$= \sin\left[\sin^{-1}\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right] \quad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= \sin\left[\sin^{-1}\frac{\frac{3}{2}}{\frac{25}{16}}\right] = \sin\left[\sin^{-1}\frac{24}{25}\right]$$

$$= \sin[\sin^{-1}(0.96)]$$

$= 0.96$

Hence, the correct answer is (*c*).

Q28. The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is equal to

(*a*) $\frac{\pi}{2}$ (*b*) $\frac{3\pi}{2}$ (*c*) $\frac{5\pi}{2}$ (*d*) $\frac{7\pi}{2}$

Sol. $\cos^{-1}\left(\cos\frac{\pi}{2}\right) \neq \frac{3\pi}{2} \quad \because \frac{3\pi}{2} \notin [0, \pi]$

$\Rightarrow \cos^{-1}\left[\cos\left(\pi+\frac{\pi}{2}\right)\right] \Rightarrow \cos^{-1}\left[-\cos\frac{\pi}{2}\right] \Rightarrow \cos^{-1}[0] = \frac{\pi}{2}$

Hence, the correct answer is (*a*).

Q29. The value of expression $2\sec^{-2} 2 + \sin^{-1}\left(\frac{1}{2}\right)$ is

(*a*) $\frac{\pi}{6}$ (*b*) $\frac{5\pi}{6}$ (*c*) $\frac{7\pi}{6}$ (*d*) 1

Sol. $2\sec^{-1}2 + \sin^{-1}\frac{1}{2} = 2\sec^{-1}\left(\sec\frac{\pi}{3}\right) + \sin^{-1}\left(\sin\frac{\pi}{6}\right)$

$= 2.\frac{\pi}{3} + \frac{\pi}{6} = \frac{2\pi}{3} + \frac{\pi}{6} = \frac{5\pi}{6}$

Hence, the correct answer is (*b*).

Q30. If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, then $\cot^{-1}x + \cot^{-1}y$ equals

(*a*) $\frac{\pi}{5}$ (*b*) $\frac{2\pi}{5}$ (*c*) $\frac{3\pi}{5}$ (*d*) π

Sol. Given that $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$

$\Rightarrow \frac{\pi}{2} - \cot^{-1}x + \frac{\pi}{2} - \cot^{-1}y = \frac{4\pi}{5} \quad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow \pi - (\cot^{-1}x + \cot^{-1}y) = \frac{4\pi}{5}$

$\Rightarrow \cot^{-1}x + \cot^{-1}y = \pi - \frac{4\pi}{5}$

$\Rightarrow \cot^{-1}x + \cot^{-1}y = \frac{\pi}{5}$

Hence, the correct answer is (*a*).

Q31. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $a, x \in]0, 1$, then the value of x is

(*a*) 0 (*b*) $\frac{a}{2}$ (*c*) a (*d*) $\frac{2a}{1-a^2}$

Sol. $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right)+\cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$\Rightarrow \quad 2\tan^{-1}a + 2\tan^{-1}a = 2\tan^{-1}x$

$$\left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$\Rightarrow \quad 4\tan^{-1}a = 2\tan^{-1}x \quad \Rightarrow 2\tan^{-1}a = \tan^{-1}x$

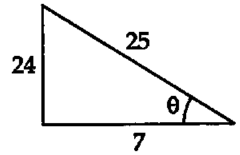
$\Rightarrow \quad \tan^{-1}\dfrac{2a}{1-a^2} = \tan^{-1}x \quad \Rightarrow x = \dfrac{2a}{1-a^2}$

Hence, the correct answer is (*d*).

Q32. The value of $\cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right]$ is

(*a*) $\dfrac{25}{24}$ (*b*) $\dfrac{25}{7}$ (*c*) $\dfrac{24}{25}$ (*d*) $\dfrac{7}{24}$

Sol. We have, $\cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right]$



Let $\quad \cos^{-1}\dfrac{7}{25} = \theta$

$\therefore \quad \cos\theta = \dfrac{7}{25} \Rightarrow \cot\theta = \dfrac{7}{24}$

$\therefore \cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right] = \cot\left[\cot^{-1}\left(\dfrac{7}{24}\right)\right] = \dfrac{7}{24}$

Hence, the correct answer is (*d*).

Q33. The value of expression $\tan\left[\dfrac{1}{2}\cos^{-1}\dfrac{2}{\sqrt{5}}\right]$ is

(*a*) $2+\sqrt{5}$ (*b*) $\sqrt{5}-2$ (*c*) $\dfrac{\sqrt{5}+2}{2}$ (*d*) $5+\sqrt{2}$

Sol. We have, $\tan\left[\dfrac{1}{2}\cos^{-1}\dfrac{2}{\sqrt{5}}\right]$

Let $\quad \theta = \dfrac{1}{2}\cos^{-1}\dfrac{2}{\sqrt{5}}$

$\Rightarrow \quad 2\theta = \cos^{-1}\dfrac{2}{\sqrt{5}} \Rightarrow \cos 2\theta = \dfrac{2}{\sqrt{5}}$

$\Rightarrow \quad \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{2}{\sqrt{5}} \qquad \left[\because \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

$\Rightarrow \quad 2+2\tan^2\theta = \sqrt{5}-\sqrt{5}\tan^2\theta$

$\Rightarrow \sqrt{5}\tan^2\theta + 2\tan^2\theta = \sqrt{5}-2 \Rightarrow (\sqrt{5}+2)\tan^2\theta = \sqrt{5}-2$

$$\Rightarrow \quad \tan^2\theta = \frac{\sqrt{5}-2}{\sqrt{5}+2}$$

$$\Rightarrow \quad \tan^2\theta = \frac{(\sqrt{5}-2)(\sqrt{5}-2)}{(\sqrt{5}+2)(\sqrt{5}-2)} \Rightarrow \tan^2\theta = \frac{(\sqrt{5}-2)^2}{5-4}$$

$$\Rightarrow \quad \tan\theta = \pm(\sqrt{5}-2)$$

$$\Rightarrow \quad \tan\theta = \sqrt{5}-2, \quad [-(\sqrt{5}-2) \text{ is not required}]$$

Hence, the correct answer is (*b*).

Q34. If $|x| \le 1$, then $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

(*a*) $4\tan^{-1}x$ (*b*) 0 (*c*) $\frac{\pi}{2}$ (*d*) π

Sol. Here, we have $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$= 2\tan^{-1}x + 2\tan^{-1}x \quad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= 4\tan^{-1}x$$

Hence, the correct answer is (*a*).

Q35. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ equals

(*a*) 0 (*b*) 1 (*c*) 6 (*d*) 12

Sol. We have $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

$\Rightarrow \quad \cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = \pi + \pi + \pi$

$\Rightarrow \quad \cos^{-1}\alpha = \pi, \cos^{-1}\beta = \pi$ and $\cos^{-1}\gamma = \pi$

$\Rightarrow \quad \alpha = \cos\pi, \beta = \cos\pi$ and $\gamma = \cos\pi$

$\therefore \quad \alpha = -1, \beta = -1$ and $\gamma = -1$

Which gives $\alpha = \beta = \gamma = -1$

So $\quad \alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$

$\Rightarrow \quad (-1)(-1-1) + (-1)(-1-1) + (-1)(-1-1)$

$\Rightarrow \quad (-1)(-2) + (-1)(-2) + (-1)(-2) \Rightarrow 2+2+2 \Rightarrow 6$

Hence, the correct answer is (*c*).

Q36. The number of real solutions of the equation

$$\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x) \text{ in } \left[\frac{\pi}{2}, \pi\right] \text{ is}$$

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) infinite

Sol. We have $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow \sqrt{2\cos^2 x} = \sqrt{2}x$ $[\because \cos^{-1}(\cos x) = x]$

$\Rightarrow \sqrt{2}\cos x = \sqrt{2}x \Rightarrow \cos x = x$

Which does not satisfy for any value of x.

Hence, the correct answer is (*d*).

Q37. If $\cos^{-1} x > \sin^{-1} x$, then

(*a*) $\frac{1}{\sqrt{2}} < x \leq 1$ (*b*) $0 \leq x < \frac{1}{\sqrt{2}}$

(*c*) $-1 \leq x < \frac{1}{\sqrt{2}}$ (*d*) $x > 0$

Sol. Here, given that $\cos^{-1} x > \sin^{-1} x$

$\Rightarrow \sin[\cos^{-1} x] > x$

$\Rightarrow \sin\left[\sin^{-1}\sqrt{1-x^2}\right] > x \Rightarrow \sqrt{1-x^2} > x$

$\Rightarrow x < \sqrt{1-x^2} \Rightarrow x^2 < 1 - x^2 \Rightarrow 2x^2 < 1$

$\Rightarrow x^2 < \frac{1}{2} \Rightarrow x < \pm\frac{1}{\sqrt{2}}$

We know that $-1 \leq x \leq 1$

So $-1 \leq x < \frac{1}{\sqrt{2}}$.

Hence, the correct answer is (c).

Fill in the Blanks in each of the Exercises 38 to 48.

Q38. The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is ....

Sol. Let $\cos^{-1}\left(-\frac{1}{2}\right) = x \Rightarrow \cos x = -\frac{1}{2}$

$\Rightarrow \cos x = \cos\left(-\frac{\pi}{3}\right) \Rightarrow \cos x = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\therefore x = \frac{2\pi}{3} \in [0, \pi]$

Hence, Principal value of $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

Q39. The value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is ....

Sol. $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) \neq \frac{3\pi}{5}$ as $\frac{3\pi}{5} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

So $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\sin\left(\pi - \frac{2\pi}{5}\right)$

$$= \sin^{-1}\sin\left(\frac{2\pi}{5}\right) = \frac{2\pi}{5} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Hence, the value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}$

Q40. If $\cos\left(\tan^{-1}x + \cot^{-1}\sqrt{3}\right) = 0$, then value of x is ....

Sol. Given that

$$\cos\left[\tan^{-1}x + \cot^{-1}\sqrt{3}\right] = 0$$

$$\Rightarrow \quad \tan^{-1}x + \cot^{-1}\sqrt{3} = \cos^{-1}(0)$$

$$\Rightarrow \quad \tan^{-1}x + \cot^{-1}\sqrt{3} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{2} - \cot^{-1}\sqrt{3} \qquad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \tan^{-1}x = \tan^{-1}\sqrt{3} \Rightarrow x = \sqrt{3}$$

Hence, the value of x is $\sqrt{3}$.

Q41. The set of values of $\sec^{-1}\left(\frac{1}{2}\right)$ is ....

Sol. Let $\quad \sec^{-1}\left(\frac{1}{2}\right) = x \Rightarrow \sec x = \frac{1}{2}$

Since, the domain of $\sec^{-1}x$ is $R - \{-1, 1\}$ and $\frac{1}{2} \notin R - \{-1, 1\}$.

Hence, $\sec^{-1}\left(\frac{1}{2}\right)$ has no set of values.

Q42. The principal value of $\tan^{-1}\sqrt{3}$ is ....

Sol. $\tan^{-1}\sqrt{3} = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Hence the principal value of $\tan^{-1}\sqrt{3}$ is $\frac{\pi}{3}$.

Q43. The value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is ....

Sol. $$\cos^{-1}\left(\cos\frac{14\pi}{3}\right) = \cos^{-1}\left[\cos\left(5\pi - \frac{\pi}{3}\right)\right]$$

$$= \cos^{-1}\left[\cos\left(\frac{-\pi}{3}\right)\right] = \cos^{-1}\left[\cos\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= \cos^{-1}\left[\cos\frac{2\pi}{3}\right] = \frac{2\pi}{3} \in [0, \pi]$$

Hence, the value of $\cos^{-1}\left[\cos\frac{14\pi}{3}\right] = \frac{2\pi}{3}$.

Q44. The value of $\cos(\sin^{-1}x + \cos^{-1}x)$, $|x| \le 1$ is ....

Sol. $\cos[\sin^{-1}x + \cos^{-1}x] = \cos\frac{\pi}{2} = 0$ $\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

Hence, the value of $\cos(\sin^{-1}x + \cos^{-1}x) = 0$.

Q45. The value of expression $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right)$, when $x = \frac{\sqrt{3}}{2}$ is ....

Sol. $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$ $\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

Hence, the value of the given expression is 1.

Q46. If $y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ for all x, then ..... $< y <$ ......

Sol.
$$y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \quad y = 2\tan^{-1}x + 2\tan^{-1}x$$
$$\Rightarrow \quad y = 4\tan^{-1}x \quad \left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x\right]$$

Now $\frac{-\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$

$\Rightarrow \quad -4\times\frac{\pi}{2} < 4\tan^{-1}x < 4\times\frac{\pi}{2} \quad \Rightarrow -2\pi < y < 2\pi$

Hence, the value of y is $(-2\pi, 2\pi)$.

Q47. The result $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when value of xy is ....

Sol. The given result is true when $xy > -1$.

Q48. The value of $\cot^{-1}(-x)$ for all $x \in R$ in terms of $\cot^{-1}x$ is ....

Sol. $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$ $[\because$ as $\cot^{-1}(-x) = \pi - \cot^{-1}x]$

State True or False for the Statement in Each of the Exercises 49 to 55.

Q49. All trigonometric functions have inverse over their respective domains.

Sol. False.

We know that all inverse trigonometric functions are restricted over their domains.

Q50. The value of expression $(\cos^{-1}x)^2$ is equal to $\sec^2 x$.

Sol. False.

We know that $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right) \neq \sec x$

So $(\cos^{-1} x)^2 \neq \sec^2 x$

Q51. The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

Sol. True.

We know that all trigonometric functions are restricted over their domains to obtain their inverse functions.

Q52. The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

Sol. True.

Q53. The graph of inverse trigonometric function can be obtained from the graph of their corresponding trigonometric function by interchanging x and y *axes*.

Sol. True.

We know that the domain and range are interchanged in the graph of inverse trigonometric functions to that of their corresponding trigonometric functions.

Q54. The minimum value of n for which $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$, $n \in N$ is valid is 5.

Sol. False.

Given that $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$

$\Rightarrow \quad \frac{n}{\pi} > \tan\frac{\pi}{4} \quad \Rightarrow \quad \frac{n}{\pi} > 1$

$\Rightarrow \quad n > \pi \quad \Rightarrow n > 3.14$

Hence, the value of n is 4.

Q55. The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ is $\frac{\pi}{3}$.

Sol. True.

$$\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right] = \sin^{-1}\left[\cos\left(\sin^{-1}\sin\frac{\pi}{6}\right)\right]$$

$$\sin^{-1}\left[\cos\frac{\pi}{6}\right] = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

❑❑❑