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Determinants

4.3 EXERCISE

■ SHORT ANSWER TYPE QUESTIONS

Using the properties of determinants in Exercise 1 to 6, evaluate:

$$\text{Q1. } \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

$$\text{Sol. Let } \Delta = \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2$$

$$= \begin{vmatrix} x^2 - 2x + 2 & x - 1 \\ 0 & x + 1 \end{vmatrix}$$

$$= (x + 1)(x^2 - 2x + 2) - 0$$

$$= x^3 - 2x^2 + 2x + x^2 - 2x + 2 = x^3 - x^2 + 2$$

$$\text{Q2. } \begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$$

$$\text{Sol. Let } \Delta = \begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} a + x + y + z & y & z \\ a + x + y + z & a + y & z \\ a + x + y + z & y & a + z \end{vmatrix} = (a + x + y + z) \begin{vmatrix} 1 & y & z \\ 1 & a + y & z \\ 1 & y & a + z \end{vmatrix}$$

(Taking $a + x + y + z$ common)

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= (a + x + y + z) \begin{vmatrix} 0 & -a & 0 \\ 0 & a & -a \\ 1 & y & a + z \end{vmatrix}$$

$$\text{Expanding along } C_1 = (a + x + y + z) |1(a^2 - 0)| = a^2(a + x + y + z)$$

$$\text{Q3. } \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

$$\text{Sol. Let } \Delta = \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

Taking x^2 , y^2 and z^2 common from C_1 , C_2 and C_3 respectively

$$= x^2y^2z^2 \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$$

Expanding along R_1

$$\begin{aligned} &= x^2y^2z^2 \left[0 \begin{vmatrix} 0 & y \\ z & 0 \end{vmatrix} - x \begin{vmatrix} y & y \\ z & 0 \end{vmatrix} + x \begin{vmatrix} y & 0 \\ z & z \end{vmatrix} \right] \\ &= x^2y^2z^2 [-x(0 - yz) + x(yz - 0)] \\ &= x^2y^2z^2(xyz + xyz) = x^2y^2z^2(2xyz) = 2x^3y^3z^3 \end{aligned}$$

$$\text{Q4. } \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

$$\text{Sol. Let } \Delta = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

Taking $(x+y+z)$ common from C_1

$$= (x+y+z) \begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= (x+y+z) \begin{vmatrix} 0 & -x-2y & -x+y \\ 0 & 2y+z & -y-2z \\ 1 & y-z & 3z \end{vmatrix}$$

Expanding along C_1

$$\begin{aligned}
 &= (x+y+z) \left[1 \begin{vmatrix} -x-2y & -x+y \\ 2y+z & -y-2z \end{vmatrix} \right] \\
 &= (x+y+z) [(-x-2y)(-y-2z) - (2y+z)(-x+y)] \\
 &= (x+y+z) (xy + 2zx + 2y^2 + 4yz + 2xy - 2y^2 + zx - zy) \\
 &= (x+y+z) (3xy + 3zx + 3yz) = 3(x+y+z) (xy + yz + zx)
 \end{aligned}$$

$$\text{Q5. } \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

$$\text{Sol. Let } \Delta = \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}$$

Taking $(3x+4)$ common from C_1

$$= (3x+4) \begin{vmatrix} 1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= (3x+4) \begin{vmatrix} 0 & -4 & 0 \\ 0 & 4 & -4 \\ 1 & x & x+4 \end{vmatrix}$$

Expanding along C_1

$$= (3x+4) \left[1 \begin{vmatrix} -4 & 0 \\ 4 & -4 \end{vmatrix} \right] = (3x+4) (16 - 0) = 16(3x+4)$$

$$\text{Q6. } \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$\text{Sol. Let } \Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$$

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ b+c+a & -(b+c+a) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

Taking $(b+c+a)$ from C_1 and C_2

$$= (a+b+c)^3 \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Expanding along R_1

$$= (a+b+c)^3 \left[1 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \right] = (a+b+c)^3.$$

Using the properties of determinants in Exercises 7 to 9, prove that:

$$\text{Q7. } \begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$$

$$\text{Sol. } \text{L.H.S.} = \begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix}$$

$R_1 \rightarrow xR_1, R_2 \rightarrow yR_2, R_3 \rightarrow zR_3$ and dividing the determinant by xyz .

$$= \frac{1}{xyz} \begin{vmatrix} xy^2z^2 & xyz & xy+zx \\ yz^2x^2 & yzx & yz+xy \\ zx^2y^2 & zxy & zx+zy \end{vmatrix}$$

Taking xyz common from C_1 and C_2

$$= \frac{xyz \cdot xyz}{xyz} \begin{vmatrix} yz & 1 & xy + zx \\ zx & 1 & yz + xy \\ xy & 1 & zx + zy \end{vmatrix}$$

$C_3 \rightarrow C_3 + C_1$

$$= xyz \begin{vmatrix} yz & 1 & xy + yz + zx \\ zx & 1 & xy + yz + zx \\ xy & 1 & xy + yz + zx \end{vmatrix}$$

Taking $(xy + yz + zx)$ common from C_3

$$= (xyz)(xy + yz + zx) \begin{vmatrix} yz & 1 & 1 \\ zx & 1 & 1 \\ xy & 1 & 1 \end{vmatrix}$$

$$= (xyz)(xy + yz + zx) \begin{vmatrix} yz & 1 & 1 \\ zx & 1 & 1 \\ xy & 1 & 1 \end{vmatrix} = 0$$

[$\because C_2$ and C_3 are identical]

L.H.S. = R.H.S. Hence proved.

Q8. $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

Sol. L.H.S. = $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$

$C_1 \rightarrow C_1 - (C_2 + C_3)$

$$= \begin{vmatrix} 0 & z & y \\ -2x & z+x & x \\ -2x & x & x+y \end{vmatrix}$$

Taking -2 common from C_1

$$= -2 \begin{vmatrix} 0 & z & y \\ x & z+x & x \\ x & x & x+y \end{vmatrix}$$

$R_2 \rightarrow R_2 - R_3$

$$= -2 \begin{vmatrix} 0 & z & y \\ 0 & z & -y \\ x & x & x+y \end{vmatrix}$$

Expanding along C_1

$$= -2 \left[x \begin{vmatrix} -zy & -zy \end{vmatrix} \right] = -2(-2xyz) = 4xyz \text{ R.H.S.}$$

L.H.S. = R.H.S.

Hence, proved.

$$\text{Q9. } \begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

$$\text{Sol. L.H.S.} = \begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= \begin{vmatrix} a^2 - 1 & a - 1 & 0 \\ 2a - 2 & a - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} (a+1)(a-1) & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Taking $(a-1)$ common from C_1 and C_2

$$= (a-1)(a-1) \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Expanding along C_3

$$= (a-1)^2 \left[1 \begin{vmatrix} a+1 & 1 \\ 2 & 1 \end{vmatrix} \right]$$

$$= (a-1)^2 (a+1-2) = (a-1)^2 (a-1) = (a-1)^3 \text{ R.H.S.}$$

L.H.S. = R.H.S.

Hence, proved.

Q10. If $A + B + C = 0$, then prove that

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

$$\text{Sol. L.H.S.} = \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$$

Expanding along C_1

$$= 1 \begin{vmatrix} 1 & \cos A \\ \cos A & 1 \end{vmatrix} - \cos C \begin{vmatrix} \cos C & \cos B \\ \cos A & 1 \end{vmatrix} + \cos B \begin{vmatrix} \cos C & \cos B \\ 1 & \cos A \end{vmatrix}$$

$$\begin{aligned}
&= 1(1 - \cos^2 A) - \cos C(\cos C - \cos A \cos B) \\
&\quad + \cos B(\cos A \cos C - \cos B) \\
&= \sin^2 A - \cos^2 C + \cos A \cos B \cos C \\
&\quad + \cos A \cos B \cos C - \cos^2 B \\
&= \sin^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C \\
&= -\cos(A+B) \cdot \cos(A-B) - \cos^2 C + 2 \cos A \cos B \cos C \\
&\quad [\because \sin^2 A - \cos^2 B = -\cos(A+B) \cdot \cos(A-B)] \\
&= -\cos(-C) \cdot \cos(A-B) + \cos C(2 \cos A \cos B - \cos C) \\
&\quad [\because A+B+C=0] \\
&= -\cos C(\cos A \cos B + \sin A \sin B) \\
&\quad + \cos C(2 \cos A \cos B - \cos C) \\
&= -\cos C(\cos A \cos B + \sin A \sin B - 2 \cos A \cos B + \cos C) \\
&= -\cos C(-\cos A \cos B + \sin A \sin B + \cos C) \\
&= \cos C(\cos A \cos B - \sin A \sin B - \cos C) \\
&= \cos C[\cos(A+B) - \cos C] \\
&= \cos C[\cos(-C) - \cos C] \quad [\because A+B=-C] \\
&= \cos C[\cos C - \cos C] = \cos C \cdot 0 = 0 \text{ R.H.S.}
\end{aligned}$$

L.H.S. = R.H.S.

Hence, proved.

Q11. If the coordinates of the vertices of an equilateral triangle with sides of length ' a ' are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Sol. Area of triangle whose vertices are (x_1, y_1) , (x_2, y_2) and (x_3, y_3)

$$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\text{Let } \Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow \Delta^2 = \frac{1}{4} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

But area of equilateral triangle whose side is ' a ' = $\frac{\sqrt{3}}{4} a^2$

$$\therefore \left(\frac{\sqrt{3}}{4} a^2 \right)^2 = \frac{1}{4} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

$$\Rightarrow \frac{3}{16} a^4 = \frac{1}{4} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{16} a^4 \times 4 = \frac{3}{4} a^4$$

Hence, proved.

Q12. Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$.

Sol. Let $A = \begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$

$$|A| = \begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$$

$$C_1 \rightarrow C_1 - C_2$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -7 & 3 & \cos 2\theta \\ 14 & -7 & -2 \end{vmatrix} = 0$$

Taking 7 common from C_1

$$\Rightarrow 7 \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0$$

Expanding along C_1

$$\Rightarrow 1 \begin{vmatrix} 1 & \sin 3\theta \\ -7 & -2 \end{vmatrix} + 2 \begin{vmatrix} 1 & \sin 3\theta \\ 3 & \cos 2\theta \end{vmatrix} = 0$$

$$\Rightarrow -2 + 7 \sin 3\theta + 2 (\cos 2\theta - 3 \sin 3\theta) = 0$$

$$\Rightarrow -2 + 7 \sin 3\theta + 2 \cos 2\theta - 6 \sin 3\theta = 0$$

$$\Rightarrow -2 + 2 \cos 2\theta + \sin 3\theta = 0$$

$$\Rightarrow -2 + 2(1 - 2 \sin^2 \theta) + 3 \sin \theta - 4 \sin^3 \theta = 0$$

$$\Rightarrow -2 + 2 - 4 \sin^2 \theta + 3 \sin \theta - 4 \sin^3 \theta = 0$$

$$\begin{aligned}
\Rightarrow & -4 \sin^3 \theta - 4 \sin^2 \theta + 3 \sin \theta = 0 \\
\Rightarrow & -\sin \theta (4 \sin^2 \theta + 4 \sin \theta - 3) = 0 \\
& \sin \theta = 0 \quad \text{or} \quad 4 \sin^2 \theta + 4 \sin \theta - 3 = 0 \\
\therefore & \theta = n\pi \quad \text{or} \quad 4 \sin^2 \theta + 6 \sin \theta - 2 \sin \theta - 3 = 0 \\
& \text{when } n \in \mathbb{I} \\
\Rightarrow & 2 \sin \theta (2 \sin \theta + 3) - 1 (2 \sin \theta + 3) = 0 \\
\Rightarrow & (2 \sin \theta + 3) (2 \sin \theta - 1) = 0 \\
\Rightarrow & 2 \sin \theta + 3 = 0 \quad \text{or} \quad 2 \sin \theta - 1 = 0 \\
& \sin \theta = \frac{-3}{2} \quad \text{or} \quad \sin \theta = \frac{1}{2}
\end{aligned}$$

$\sin \theta = \frac{-3}{2}$ is not possible as $-1 \leq x \leq 1$

$$\therefore \sin \theta = \frac{1}{2} \Rightarrow \sin \theta = \sin \frac{\pi}{6} \Rightarrow \theta = n\pi + (-1)^n \cdot \frac{\pi}{6}$$

Hence, $\theta = n\pi$ or $n\pi + (-1)^n \frac{\pi}{6}$

Q13. If $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$, then find values of x .

Sol. Let $A = \begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$

$$|A| = \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$\Rightarrow \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

Taking $(12+x)$ common from R_1 ,

$$\Rightarrow (12+x) \begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$$C_1 \rightarrow C_1 - C_2, \quad C_2 \rightarrow C_2 - C_3$$

$$(12+x) \begin{vmatrix} 0 & 0 & 1 \\ 2x & -2x & 4+x \\ 0 & 2x & 4-x \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow (12+x) \left[1 \cdot \begin{vmatrix} 2x & -2x \\ 0 & 2x \end{vmatrix} \right] = 0$$

$$(12+x)(4x^2-0) = 0 \Rightarrow 12+x=0 \text{ or } 4x^2=0$$

$$\Rightarrow x = -12 \text{ or } x = 0$$

Q14. If $a_1, a_2, a_3, \dots, a_r$ are in G.P., then prove that the determinant

$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix} \text{ is independent of } r.$$

Sol. If $a_1, a_2, a_3, \dots, a_r$ be the terms of G.P., then

$$a_n = AR^{n-1}$$

(where A is the first term and R is the common ratio of the G.P.)

$$\therefore a_{r+1} = AR^{r+1-1} = AR^r; a_{r+5} = AR^{r+5-1} = AR^{r+4}$$

$$a_{r+9} = AR^{r+9-1} = AR^{r+8}; a_{r+7} = AR^{r+7-1} = AR^{r+6}$$

$$a_{r+11} = AR^{r+11-1} = AR^{r+10}; a_{r+15} = AR^{r+15-1} = AR^{r+14}$$

$$a_{r+17} = AR^{r+17-1} = AR^{r+16}; a_{r+21} = AR^{r+21-1} = AR^{r+20}$$

\therefore The determinant becomes

$$\begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$$

Taking AR^r, AR^{r+6} and AR^{r+10} common from R_1, R_2 and R_3 respectively.

$$AR^r \cdot AR^{r+6} \cdot AR^{r+10} \begin{vmatrix} 1 & R^4 & R^8 \\ 1 & R^4 & R^8 \\ 1 & R^6 & R^{10} \end{vmatrix}$$

$$= AR^r \cdot AR^{r+6} \cdot AR^{r+10} |0|$$

[$\because R_1$ and R_2 are identical rows]

$$= 0$$

Hence, the given determinant is independent of r .

Q15. Show that the points $(a+5, a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line for any value of a .

Sol. If the given points lie on a straight line, then the area of the triangle formed by joining the points pairwise is zero.

$$\text{So, } \begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$\Rightarrow \begin{vmatrix} 7 & -7 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix}$$

Expanding along C_3

$$1 \cdot \begin{vmatrix} 7 & -7 \\ -2 & 3 \end{vmatrix} = 21 - 14 = 7 \text{ units}$$

As $7 \neq 0$. Hence,, the three points do not lie on a straight line for any value of a .

Q16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0.$$

$$\text{Sol. } \begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

$$C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos B & \cos B - \cos C & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \\ -\cos^2 B - \cos B & -\cos^2 C - \cos C & \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos B & \cos B - \cos C & 1 + \cos C \\ \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C & \cos^2 C + \cos C \\ +\cos A - \cos B & +\cos B - \cos C & \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos B & \cos B - \cos C & 1 + \cos C \\ (\cos A + \cos B) \times (\cos A - \cos B) & (\cos B + \cos C) \times (\cos B - \cos C) & \cos^2 C + \cos C \\ +(\cos A - \cos B) & +(\cos B + \cos C) & \end{vmatrix} = 0$$

Taking $(\cos A - \cos B)$ and $(\cos B - \cos C)$ common from C_1 and C_2 respectively.

$$\Rightarrow (\cos A - \cos B)(\cos B - \cos C) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 + \cos C \\ \cos A + \cos B + \cos^2 C + \cos B + 1 & \cos C + 1 & \cos C \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow (\cos A - \cos B)(\cos B - \cos C) \left[1 \begin{vmatrix} 1 & 1 \\ \cos A + \cos B + \cos B + 1 & \cos C + 1 \end{vmatrix} \right] = 0$$

$$\Rightarrow (\cos A - \cos B)(\cos B - \cos C) \left[(\cos B + \cos C + 1) - (\cos A + \cos B + 1) \right] = 0$$

$$\Rightarrow (\cos A - \cos B)(\cos B - \cos C) [\cos B + \cos C + 1 - \cos A - \cos B - 1] = 0$$

$$\Rightarrow (\cos A - \cos B)(\cos B - \cos C)(\cos C - \cos A) = 0$$

$$\cos A - \cos B = 0 \text{ or } \cos B - \cos C = 0$$

$$\text{or } \cos C - \cos A = 0$$

$$\Rightarrow \cos A = \cos B \text{ or } \cos B = \cos C \text{ or } \cos C = \cos A$$

$$\Rightarrow \angle A = \angle C \text{ or } \angle B = \angle C \Rightarrow \angle A = \angle B$$

Hence, ΔABC is an isosceles triangle.

Q17. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that $A^{-1} = \frac{A^2 - 3I}{2}$.

Sol. Here, $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A| = 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 0 - 1(0 - 1) + 1(1 - 0)$$

$$= 1 + 1 = 2 \neq 0 \text{ (non-singular matrix.)}$$

Now, co-factors,

$$a_{11} = + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad a_{12} = - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad a_{13} = + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$a_{21} = - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad a_{22} = + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad a_{23} = - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$a_{31} = + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1, \quad a_{32} = - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1, \quad a_{33} = + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\text{Adj}(A) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj}(A) = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} \text{Now, } A^2 &= A \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \end{aligned}$$

$$\text{Hence, } A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\text{Now, we have to prove that } A^{-1} = \frac{A^2 - 3I}{2}$$

$$\begin{aligned} \text{R.H.S.} &= \frac{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{2} \\ &= \frac{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}{2} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \\ &= A^{-1} = \text{L.H.S.} \end{aligned}$$

Hence, proved.

LONG ANSWER TYPE QUESTIONS

- Q18.** If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1} . Using A^{-1} , solve the system of linear equations $x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$.

Sol. Given that

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1 \begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} \\ &= 1(-1-2) - 2(-2-0) + 0 \\ &= -3 + 4 = 1 \neq 0 \text{ (non-singular matrix.)} \end{aligned}$$

Now co-factors,

$$a_{11} = + \begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} = -3, a_{12} = - \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} = 2, a_{13} = + \begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} = 2$$

$$a_{21} = - \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = -2, a_{22} = + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, a_{23} = - \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$a_{31} = + \begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} = -4, a_{32} = - \begin{vmatrix} 1 & 0 \\ -2 & -2 \end{vmatrix} = 2, a_{33} = + \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = 3$$

$$\text{Adj}(A) = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj}(A) = \frac{1}{1} \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Now, the system of linear equations is given by $x - 2y = 10$, $2x - y - z = 8$ and $-2y + z = 7$, which is in the form of $CX = D$.

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$\text{where } C = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$\therefore (A^T)^{-1} = (A^{-1})^T$$

$$\therefore C^T = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix} = A$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = C^{-1}D \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

Hence, $x = 0$, $y = -5$ and $z = -3$

Q19. Using matrix method, solve the system of equation

$$3x + 2y - 2z = 3, \quad x + 2y + 3z = 6, \quad 2x - y + z = 2.$$

Sol. Given that

$$3x + 2y - 2z = 3$$

$$x + 2y + 3z = 6$$

$$2x - y + z = 2$$

$$\text{Let } A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

$$|A| = 3 \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= 3(2 + 3) - 2(1 - 6) - 2(-1 - 4)$$

$$= 15 + 10 + 10 = 35 \neq 0 \text{ non-singular matrix}$$

Now, co-factors,

$$a_{11} = + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5, \quad a_{12} = - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5, \quad a_{13} = + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$a_{21} = - \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0, \quad a_{22} = + \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 7, \quad a_{23} = - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = 7$$

$$a_{31} = + \begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} = 10, \quad a_{32} = - \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -11, \quad a_{33} = + \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 4$$

$$\text{Adj}(A) = \begin{bmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{bmatrix}' = \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj}(A) = \frac{1}{35} \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 15+0+20 \\ 15+42-22 \\ -15+42+8 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $x=1$, $y=1$ and $z=1$.

Q20. If $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, then find BA and

use this to solve the system of equations $y+2z=7$, $x-y=3$ and $2x+3y+4z=17$.

Sol. We have, $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$$\begin{aligned} BA &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6I \end{aligned}$$

$$\therefore B^{-1} = \frac{1}{6} A = \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

The given equations can be re-write as,

$$x - y = 3, 2x + 3y + 4z = 17 \text{ and } y + 2z = 7$$

$$\therefore \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \end{aligned}$$

Hence, $x = 2$, $y = -1$ and $z = 4$

Q21. If $a + b + c \neq 0$ and $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 0$, then prove that $a = b = c$.

Sol. Given that: $a + b + c \neq 0$ and $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 0$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$\Rightarrow \begin{bmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{bmatrix} = 0$$

$$\Rightarrow (a+b+c) \begin{bmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{bmatrix} = 0 \quad \text{(Taking } a+b+c \text{ common from } C_1)$$

$$\Rightarrow a+b+c \neq 0 \quad \therefore \begin{bmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{bmatrix} = 0$$

$$R_1 \rightarrow R_1 - R_2 \text{ and } R_2 \rightarrow R_2 - R_3$$

$$\Rightarrow \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix} = 0$$

Expanding along C_1

$$\Rightarrow 1 \begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix} = 0$$

$$\Rightarrow (b-c)(a-b) - (c-a)^2 = 0$$

$$\Rightarrow ab - b^2 - ac + bc - c^2 - a^2 + 2ac = 0$$

$$\Rightarrow -a^2 - b^2 - c^2 + ab + bc + ac = 0$$

$$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ac = 0$$

$$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 0$$

(Multiplying both sides by 2)

$$\Rightarrow (a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (a^2 + c^2 - 2ac) = 0$$

$$\Rightarrow (a-b)^2 + (b-c)^2 + (a-c)^2 = 0$$

It is only possible when $(a-b)^2 = (b-c)^2 = (a-c)^2 = 0$

$\therefore a = b = c$ Hence, proved.

Q22. Prove that $\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ac - b^2 \end{vmatrix}$ is divisible by $a + b + c$

and find the quotient.

Sol. Let $\Delta = \begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ac - b^2 \end{vmatrix}$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$\Rightarrow \begin{vmatrix} ab + bc + ac - a^2 - b^2 - c^2 & ca - b^2 & ab - c^2 \\ ab + bc + ac - a^2 - b^2 - c^2 & ab - c^2 & bc - a^2 \\ ab + bc + ac - a^2 - b^2 - c^2 & bc - a^2 & ac - b^2 \end{vmatrix}$$

Taking $ab + bc + ac - a^2 - b^2 - c^2$ common from C_1

$$(ab + bc + ac - a^2 - b^2 - c^2) \begin{vmatrix} 1 & ca - b^2 & ab - c^2 \\ 1 & ab - c^2 & bc - a^2 \\ 1 & bc - a^2 & ac - b^2 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2 \text{ and } R_2 \rightarrow R_2 - R_3$$

$$\Rightarrow \begin{vmatrix} ab+bc+ac-a^2-b^2-c^2 & & \\ 0 & ca-b^2-ab+c^2 & ab-c^2-bc+a^2 \\ 0 & ab-c^2-bc+a^2 & bc-a^2-ac+b^2 \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} ab+bc+ac-a^2-b^2-c^2 & & \\ 0 & a(c-b)+(c+b)(c-b) & b(a-c)+(a+c)(a-c) \\ 0 & b(a-c)+(a+c)(a-c) & c(b-a)+(b+a)(b-a) \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} ab+bc+ac-a^2-b^2-c^2 & & \\ 0 & (c-b)(a+b+c) & (a-c)(a+b+c) \\ 0 & (a-c)(a+b+c) & (b-a)(a+b+c) \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow (ab+bc+ac-a^2-b^2-c^2)(a+b+c)(a+b+c)$$

$$\begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\Rightarrow (a+b+c)^2 (ab+bc+ac-a^2-b^2-c^2)$$

$$\begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & bc-a^2 & ac-b^2 \end{vmatrix}$$

Expanding along C_1

$$\Rightarrow (a+b+c)^2 (ab+bc+ac-a^2-b^2-c^2) \begin{vmatrix} c-b & a-c \\ a-c & b-a \end{vmatrix}$$

$$\Rightarrow (a+b+c)^2 (ab+bc+ac-a^2-b^2-c^2) [(c-b)(b-a)-(a-c)^2]$$

$$\Rightarrow (a+b+c)^2 (ab+bc+ac-a^2-b^2-c^2) (bc-ca-b^2+ab-a^2-c^2+2ac)$$

$$\Rightarrow (a+b+c)^2 (ab+bc+ac-a^2-b^2-c^2) (ab+bc+ca-a^2-b^2-c^2)$$

$$\Rightarrow (a+b+c)^2 (ab+bc+ac-a^2-b^2-c^2)^2$$

$$\Rightarrow (a+b+c)(a+b+c)(a^2+b^2+c^2-ab-bc-ac)^2$$

Hence, the given determinant is divisible by $a+b+c$ and the quotient is

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ac)^2$$

$$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ac)(a^2+b^2+c^2-ab-bc-ac)$$

$$\Rightarrow (a^3+b^3+c^3-3abc)(a^2+b^2+c^2-ab-bc-ac)$$

$$\Rightarrow -(a^3 + b^3 + c^3 - 3abc)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac)$$

$$\Rightarrow \frac{1}{2}(a^3 + b^3 + c^3 - 3abc)[(a-b)^2 + (b-c)^2 + (a-c)^2]$$

Q23. If $x + y + z = 0$, prove that
$$\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Sol. L.H.S.

$$\text{Let } \Delta = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$$

Expanding along R_1

$$\Rightarrow xa \begin{vmatrix} za & xb \\ xc & ya \end{vmatrix} - yb \begin{vmatrix} yc & xb \\ zb & ya \end{vmatrix} + zc \begin{vmatrix} yc & za \\ zb & xc \end{vmatrix}$$

$$\Rightarrow xa(yza^2 - x^2bc) - yb(y^2ac - xzb^2) + zc(xyc^2 - z^2ab)$$

$$\Rightarrow xyz a^3 - x^3 abc - y^3 abc + xyz b^3 + xyz c^3 - z^3 abc$$

$$\Rightarrow xyz(a^3 + b^3 + c^3) - abc(x^3 + y^3 + z^3)$$

$$\Rightarrow xyz(a^3 + b^3 + c^3) - abc(3xyz)$$

$$[(\because x + y + z = 0) (\therefore x^3 + y^3 + z^3 = 3xyz)]$$

$$\Rightarrow xyz(a^3 + b^3 + c^3 - 3abc)$$

$$\text{R.H.S. } xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$\Rightarrow xyz \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\Rightarrow xyz(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c & a & b \\ b & c & a \end{vmatrix} \quad (\text{Taking } a+b+c \text{ common from } R_1)$$

$$C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$$

$$\Rightarrow xyz(a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ c-a & a-b & b \\ b-c & c-a & a \end{vmatrix}$$

Expanding along R_1

$$\Rightarrow xyz(a+b+c) \begin{vmatrix} 1 & c-a & a-b \\ b-c & c-a & \end{vmatrix}$$

$$\Rightarrow xyz(a+b+c) [(c-a)^2 - (b-c)(a-b)]$$

$$\Rightarrow xyz(a+b+c) (c^2 + a^2 - 2ca - ab + b^2 + ac - bc)$$

$$\Rightarrow xyz(a+b+c) (a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\Rightarrow xyz(a^3 + b^3 + c^3 - 3abc)$$

$$[a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)]$$

L.H.S. = R.H.S.

Hence, proved.

OBJECTIVE TYPE QUESTIONS (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

Q24. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is

- (a) 3 (b) ± 3 (c) ± 6 (d) 6

Sol. Given that

$$\Rightarrow \begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$$

$$\Rightarrow 2x^2 - 40 = 18 + 14 \Rightarrow 2x^2 = 32 + 40$$

$$\Rightarrow 2x^2 = 72 \Rightarrow x^2 = 36$$

$$\therefore x = \pm 6$$

Hence, the correct option is (c).

Q25. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$ is

- (a) $a^3 + b^3 + c^3$ (b) $3bc$
 (c) $a^3 + b^3 + c^3 - 3abc$ (d) None of these

Sol. Here, we have $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$

$$C_2 \rightarrow C_2 + C_3$$

$$\Rightarrow \begin{vmatrix} a-b & a+b+c & a \\ b-a & a+b+c & b \\ c-a & a+b+c & c \end{vmatrix}$$

$$\Rightarrow (a+b+c) \begin{vmatrix} a-b & 1 & a \\ b-a & 1 & b \\ c-a & 1 & c \end{vmatrix} \quad \text{(Taking } a+b+c \text{ common from } C_2)$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$\Rightarrow (a+b+c) \begin{vmatrix} 2(a-b) & 0 & a-b \\ b-c & 0 & b-c \\ c-a & 1 & c \end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ common from R_1 and R_2 respectively

$$\Rightarrow (a+b+c)(a-b)(b-c) \begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ c-a & 1 & c \end{vmatrix}$$

Expanding along C_2

$$\Rightarrow (a+b+c)(a-b)(b-c) \left[-1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right]$$

$$\Rightarrow (a+b+c)(a-b)(b-c)(-1)$$

$$\Rightarrow (a+b+c)(a-b)(c-b)$$

Hence, the correct option is (d).

Q26. The area of a triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is 9 sq units. Then, the value of k will be

(a) 9 (b) 3 (c) -9 (d) 6

Sol. Area of triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) will be:

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow \Delta = \frac{1}{2} \begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$\Rightarrow = \frac{1}{2} \left[-3 \begin{vmatrix} 0 & 1 \\ k & 1 \end{vmatrix} - 0 \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & 0 \\ 0 & k \end{vmatrix} \right]$$

$$\Rightarrow = \frac{1}{2} [-3(-k) - 0 + 1(3k)]$$

$$\Rightarrow = \frac{1}{2} (3k + 3k) \Rightarrow \frac{1}{2} (6k) = 3k$$

$$3k = 9 \Rightarrow k = 3$$

Hence, the correct option is (b).

Q27. The determinant $\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$ equals

- (a) $abc(b-c)(c-a)(a-b)$ (b) $(b-c)(c-a)(a-b)$
 (c) $(a+b+c)(b-c)(c-a)(a-b)$ (d) None of these

Sol. Let $\Delta = \begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$

$$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix} \quad \text{(Taking } (b-a) \text{ common from } C_1 \text{ and } C_3)$$

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

$C_1 \rightarrow C_1 - C_3$

$$= (a-b)^2 \begin{vmatrix} b-c & b-c & c \\ a-b & a-b & b \\ c-a & c-a & a \end{vmatrix} \quad \text{(} C_1 \text{ and } C_2 \text{ are identical columns.)}$$

$$= (a-b)^2 \cdot 0$$

$$= 0$$

Hence, the correct option is (d).

Q28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ is

- (a) 0 (b) 2 (c) 1 (d) 3

Sol. Given that

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$\Rightarrow \begin{vmatrix} 2 \cos x + \sin x & \cos x & \cos x \\ 2 \cos x + \sin x & \sin x & \cos x \\ 2 \cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

Taking $2 \cos x + \sin x$ common from C_1

$$\Rightarrow (2 \cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$\Rightarrow (2 \cos x + \sin x) \begin{vmatrix} 0 & \cos x - \sin x & 0 \\ 0 & \sin x - \cos x & \cos x - \sin x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow (2 \cos x + \sin x) \begin{bmatrix} \cos x - \sin x & 0 \\ \sin x - \cos x & \cos x - \sin x \end{bmatrix}$$

$$\Rightarrow (2 \cos x + \sin x) (\cos x - \sin x)^2 = 0$$

$$2 \cos x + \sin x = 0$$

$$2 + \tan x = 0$$

$$\therefore \tan x = -2$$

$$\text{But } -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$$

So, x has no solution.

$$(\cos x - \sin x)^2 = 0$$

$$\cos x - \sin x = 0$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow \tan x = \tan \frac{\pi}{4}$$

$$\therefore x = \frac{\pi}{4} \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right]$$

So, it will have only one real root.

Hence, the correct option is (c).

Q29. If A , B and C are angles of a triangle, then the determinant

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is equal to}$$

(a) 0

(b) -1

(c) 1

(d) None of these

$$\text{Sol. Let } \Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

$$C_1 \rightarrow aC_1 + bC_2 + cC_3$$

$$\Rightarrow \begin{vmatrix} -a + b \cos C + c \cos B & \cos C & \cos B \\ a \cos C - b + c \cos A & -1 & \cos A \\ a \cos B + b \cos A - c & \cos A & -1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -a+a & \cos C & \cos B \\ -b+b & -1 & \cos A \\ -c+c & \cos A & -1 \end{vmatrix} \left[\begin{array}{l} \because \text{ From projection formula} \\ a = b \cos C + c \cos B \\ b = a \cos C + c \cos A \\ c = b \cos A + a \cos B \end{array} \right]$$

$$\Rightarrow \begin{bmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{bmatrix} = 0$$

Hence, the correct option is (a).

Q30. Let $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2 \sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$, then $\lim_{t \rightarrow 0} \frac{f(t)}{t^2}$ is equal to

- (a) 0 (b) -1 (c) 2 (d) 3

Sol. We have $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2 \sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$

Expanding along R_1

$$\begin{aligned} &= \cos t \begin{vmatrix} t & 2t \\ t & t \end{vmatrix} - t \begin{vmatrix} 2 \sin t & 2t \\ \sin t & t \end{vmatrix} + 1 \begin{vmatrix} 2 \sin t & t \\ \sin t & t \end{vmatrix} \\ &= \cos t (t^2 - 2t^2) - t(2t \sin t - 2t \sin t) + (2t \sin t - t \sin t) \\ &= -t^2 \cos t + t \sin t \end{aligned}$$

$$\therefore \frac{f(t)}{t^2} = \frac{-t^2 \cos t + t \sin t}{t^2}$$

$$\Rightarrow \frac{f(t)}{t^2} = -\cos t + \frac{\sin t}{t}$$

$$\Rightarrow \lim_{t \rightarrow 0} \frac{f(t)}{t^2} = \lim_{t \rightarrow 0} (-\cos t) + \lim_{t \rightarrow 0} \frac{\sin t}{t} = -1 + 1 = 0$$

Hence, the correct option is (a).

Q31. The maximum value of

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin \theta & 1 \\ 1 + \cos \theta & 1 & 1 \end{vmatrix} \text{ is } \quad (\theta \text{ is real number})$$

- (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{2}$ (d) $\frac{2\sqrt{3}}{4}$

Sol. Given that: $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin \theta & 1 \\ 1 + \cos \theta & 1 & 1 \end{vmatrix}$

$$C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -\sin \theta & \sin \theta & 1 \\ \cos \theta & 0 & 1 \end{vmatrix}$$

Expanding along R_1

$$= 1 \begin{vmatrix} -\sin \theta & \sin \theta \\ \cos \theta & 0 \end{vmatrix} = -\sin \theta \cos \theta$$

$$\Rightarrow = -\frac{1}{2} \cdot 2 \sin \theta \cos \theta = -\frac{1}{2} \sin 2\theta$$

but maximum value of $\sin 2\theta = 1 \Rightarrow \left| -\frac{1}{2} \cdot 1 \right| = \frac{1}{2}$

Hence, the correct option is (a).

Q32. If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

(a) $f(a) = 0$ (b) $f(b) = 0$ (c) $f(0) = 0$ (d) $f(1) = 0$

Sol. Given that: $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$

$$f(a) = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix}$$

Expanding along $R_1 = (a-b) \begin{vmatrix} 2a & 0 \\ a+b & a+c \end{vmatrix}$

$$= (a-b) [2a(a+c)] = (a-b) \cdot 2a \cdot (a+c) \neq 0$$

$$f(b) = \begin{vmatrix} 0 & b-a & 0 \\ b+a & 0 & b-c \\ 2b & b+c & 0 \end{vmatrix}$$

Expanding along R_1

$$= -(b-a) \begin{vmatrix} b+a & b-c \\ 2b & 0 \end{vmatrix}$$

$$= -(b-a) [(-2b)(b-c)] = 2b(b-a)(b-c) \neq 0$$

$$f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$\begin{aligned} \text{Expanding along } R_1 &= a \begin{vmatrix} a & -c \\ b & 0 \end{vmatrix} - b \begin{vmatrix} a & 0 \\ b & c \end{vmatrix} \\ &= a(bc) - b(ac) = abc - abc = 0 \end{aligned}$$

Hence, the correct option is (c).

Q33. If $A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, then A^{-1} exists if

- (a) $\lambda = 2$ (b) $\lambda \neq 2$ (c) $\lambda \neq -2$ (d) None of these

Sol. We have,

$$A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$$

$$\begin{aligned} \text{Expanding along } R_1 &= 2 \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} - \lambda \begin{vmatrix} 0 & 5 \\ 1 & 3 \end{vmatrix} - 3 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \\ &= 2(6-5) - \lambda(0-5) - 3(0-2) \\ &= 2 + 5\lambda + 6 = 8 + 5\lambda \end{aligned}$$

If A^{-1} exists then $|A| \neq 0$

$$\therefore 8 + 5\lambda \neq 0 \text{ so } \lambda \neq \frac{-8}{5}$$

Hence, the correct option is (d).

Q34. If A and B are invertible matrices, then which of the following is not correct?

- (a) $\text{adj } A = |A| \cdot A^{-1}$ (b) $\det(A)^{-1} = [\det(A)]^{-1}$
 (c) $(AB)^{-1} = B^{-1}A^{-1}$ (d) $(A+B)^{-1} = B^{-1} + A^{-1}$

Sol. If A and B are two invertible matrices then

(a) $\text{adj } A = |A| \cdot A^{-1}$ is correct

(b) $\det(A)^{-1} = [\det(A)]^{-1} = \frac{1}{\det(A)}$ is correct

(c) Also, $(AB)^{-1} = B^{-1}A^{-1}$ is correct

(d) $(A+B)^{-1} = \frac{1}{|A+B|} \cdot \text{adj}(A+B)$

$$\therefore (A+B)^{-1} \neq B^{-1} + A^{-1}$$

Hence, the correct option is (d).

Q35. If x, y, z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$,

then the value of $x^{-1} + y^{-1} + z^{-1}$ is

- (a) xyz (b) $x^{-1}y^{-1}z^{-1}$ (c) $-x - y - z$ (d) -1

Sol. Given that

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$$

Taking x , y and z common from R_1 , R_2 and R_3 respectively.

$$\Rightarrow xyz \begin{vmatrix} \frac{1}{x}+1 & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix} = 0$$

$R_1 \rightarrow R_1 + R_2 + R_3$

$$\Rightarrow xyz \begin{vmatrix} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 & \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 & \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 \\ \frac{1}{y} & \frac{1}{y} + 1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z} + 1 \end{vmatrix} = 0$$

Taking $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1$ common from R_1

$$\Rightarrow xyz \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 \right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y} & \frac{1}{y} + 1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z} + 1 \end{vmatrix} = 0$$

$C_1 \rightarrow C_1 - C_2$, $C_2 \rightarrow C_2 - C_3$

$$\Rightarrow xyz \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 \right) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{y} \\ 0 & -1 & \frac{1}{z} + 1 \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow xyz \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 \right) \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow xyz \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 \right) (1) = 0$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 = 0 \text{ and } xyz \neq 0 \quad (x \neq y \neq z \neq 0)$$

$$\therefore x^{-1} + y^{-1} + z^{-1} = -1$$

Hence, the correct option is (d).

Q36. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(a) $9x^2(x+y)$ (b) $9y^2(x+y)$ (c) $3y^2(x+y)$ (d) $7x^2(x+y)$

Sol. Let $\Delta = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$

$$= (3x+3y) \begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$

[Taking $(3x+3y)$ common from C_1]

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$\Rightarrow 3(x+y) \begin{vmatrix} 0 & y & y \\ 0 & -2y & y \\ 1 & x+2y & x \end{vmatrix}$$

Expanding along C_1

$$\Rightarrow 3(x+y) \left[1 \begin{vmatrix} y & y \\ -2y & y \end{vmatrix} \right]$$

$$\Rightarrow 3(x+y)(y^2+2y^2) \Rightarrow 3(x+y)(3y^2) \Rightarrow 9y^2(x+y)$$

Hence, the correct option is (b).

Q37. There are two values of 'a' which makes determinant,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86, \text{ then sum of these numbers is}$$

- (a) 4 (b) 5 (c) -4 (d) 9

Sol. Given that, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$

Expanding along C_1

$$\Rightarrow 1 \begin{vmatrix} a & -1 \\ 4 & 2a \end{vmatrix} - 2 \begin{vmatrix} -2 & 5 \\ 4 & 2a \end{vmatrix} + 0 \begin{vmatrix} -2 & 5 \\ a & -1 \end{vmatrix} = 86$$

$$\Rightarrow (2a^2 + 4) - 2(-4a - 20) = 86$$

$$\Rightarrow 2a^2 + 4 + 8a + 40 = 86$$

$$\Rightarrow 2a^2 + 8a + 4 + 40 - 86 = 0$$

$$\Rightarrow 2a^2 + 8a - 42 = 0$$

$$\Rightarrow a^2 + 4a - 21 = 0$$

$$\Rightarrow a^2 + 7a - 3a - 21 = 0$$

$$\Rightarrow a(a+7) - 3(a+7) = 0$$

$$\Rightarrow (a-3)(a+7) = 0$$

$$\therefore a = 3, -7$$

Required sum of the two numbers = $3 - 7 = -4$.

Hence, the correct option is (c).

Fill in the Blanks

Q38. If A is a matrix of order 3×3 , then $|3A| = \underline{\hspace{2cm}}$.

Sol. We know that for a matrix of order 3×3 ,

$$|KA| = K^3 |A|$$

$$\therefore |3A| = 3^3 |A| = 27|A|$$

Q39. If A is invertible matrix of order 3×3 , then $|A^{-1}| \underline{\hspace{2cm}}$.

Sol. We know that for an invertible matrix A of any order,

$$|A^{-1}| = \frac{1}{|A|}$$

Q40. If $x, y, z \in \mathbb{R}$, then the value of determinant

$$\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix} \text{ is equal to } \underline{\hspace{2cm}}.$$

Sol. We have, $\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$

$$C_1 \rightarrow C_1 - C_2$$

$$\Rightarrow \begin{vmatrix} (2^x + 2^{-x})^2 - (2^x - 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 - (3^x - 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 - (4^x - 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 4 \cdot 2^x \cdot 2^{-x} & (2^x - 2^{-x})^2 & 1 \\ 4 \cdot 3^x \cdot 3^{-x} & (3^x - 3^{-x})^2 & 1 \\ 4 \cdot 4^x \cdot 4^{-x} & (4^x - 4^{-x})^2 & 1 \end{vmatrix} \quad \begin{array}{l} \text{[applying} \\ (a+b)^2 - (a-b)^2 = 4ab \end{array}$$

$$\Rightarrow \begin{vmatrix} 4 & (2^x - 2^{-x})^2 & 1 \\ 4 & (3^x - 3^{-x})^2 & 1 \\ 4 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$\Rightarrow 4 \begin{vmatrix} 1 & (2^x - 2^{-x})^2 & 1 \\ 1 & (3^x - 3^{-x})^2 & 1 \\ 1 & (4^x - 4^{-x})^2 & 1 \end{vmatrix} \quad \text{(Taking 4 common from } C_1)$$

$$\Rightarrow 4 \cdot 0 = 0 \quad (\because C_1 \text{ and } C_3 \text{ are identical columns)}$$

Q41. If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos \theta & \sin \theta \\ \cos \theta & \sin \theta & 0 \\ \sin \theta & 0 & \cos \theta \end{vmatrix} = \underline{\hspace{2cm}}.$

Sol. Given that: $\cos 2\theta = 0$

$$\Rightarrow \cos 2\theta = \cos \frac{\pi}{2} \Rightarrow 2\theta = \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{4}$$

The determinant can be written as

$$\Rightarrow \begin{vmatrix} 0 & \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ \cos \frac{\pi}{4} & \sin \frac{\pi}{4} & 0 \\ \sin \frac{\pi}{4} & 0 & \cos \frac{\pi}{4} \end{vmatrix}^2 \Rightarrow \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}^2$$

$$\Rightarrow \left[\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \right]^2 \quad \left(\begin{array}{l} \text{Taking } \frac{1}{\sqrt{2}} \text{ common from} \\ C_1, C_2 \text{ and } C_3 \end{array} \right)$$

Expanding along C_1 ,

$$\Rightarrow \left[\frac{1}{2\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \right]^2 \Rightarrow \left[\frac{1}{2\sqrt{2}} \{-1(1) + 1(0-1)\} \right]^2$$

$$\Rightarrow \left[\frac{1}{2\sqrt{2}} \{-1-1\} \right]^2 \Rightarrow \frac{1}{8} \cdot (4) = \frac{1}{2}$$

Q42. If A is a matrix of order 3×3 , then $(A^2)^{-1} = \underline{\hspace{2cm}}$.

Sol. For any square matrix A , $(A^2)^{-1} = (A^{-1})^2$.

Q43. If A is a matrix of order 3×3 , then the number of minors in the determinants of A are $\underline{\hspace{2cm}}$.

Sol. The order of a matrix is 3×3

\therefore Total number of elements $= 3 \times 3 = 9$

Hence, the number of minors in the determinant is 9.

Q44. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to $\underline{\hspace{2cm}}$.

Sol. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to **the value of the determinant of the given matrix.**

$$\text{Let } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_1

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13}$$

(where M_{11} , M_{12} and M_{13} are the minors of the corresponding elements)

Q45. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are _____.

Sol. We have, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Expanding along R_1

$$\Rightarrow x \begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ 7 & x \end{vmatrix} + 7 \begin{vmatrix} 2 & x \\ 7 & 6 \end{vmatrix} = 0$$

$$\Rightarrow x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) = 0$$

$$\Rightarrow x^3 - 12x - 6x + 42 + 84 - 49x = 0$$

$$\Rightarrow x^3 - 67x + 126 = 0 \quad \dots(1)$$

The roots of the equation may be the factors of 126 i.e., $2 \times 7 \times 9$ 9 is given the root of the determinant put $x = 2$ in eq. (1)

$$(2)^3 - 67 \times 2 + 126 \Rightarrow 8 - 134 + 126 = 0$$

Hence, $x = 2$ is the other root.

Now, put $x = 7$ in eq. (1)

$$(7)^3 - 67(7) + 126 \Rightarrow 343 - 469 + 126 = 0$$

Hence, $x = 7$ is also the other root of the determinant.

Q46. $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} = \underline{\hspace{2cm}}$

Sol. Let $\Delta = \begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$

$$C_1 \rightarrow C_1 - C_3$$

$$= \begin{vmatrix} z-x & xyz & x-z \\ z-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$$

Taking $(z-x)$ common from C_1

$$= (z-x) \begin{vmatrix} 1 & xyz & x-z \\ 1 & 0 & y-z \\ 1 & z-y & 0 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= (z-x) \begin{vmatrix} 0 & xyz & x-y \\ 0 & y-z & y-z \\ 1 & z-y & 0 \end{vmatrix}$$

Taking $(y-z)$ common from R_2

$$= (z-x)(y-z) \begin{vmatrix} 0 & xyz & x-y \\ 0 & 1 & 1 \\ 1 & z-y & 0 \end{vmatrix}$$

Expanding along C_1

$$= (z-x)(y-z) \left[1 \begin{vmatrix} xyz & x-y \\ 1 & 1 \end{vmatrix} \right]$$

$$= (z-x)(y-z)(xyz-x+y) = (y-z)(z-x)(y-x+xyz)$$

$$\text{Q47. If } f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \dots$$

then $A =$ _____ .

Sol. Given that

$$\begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \dots$$

Taking $(1+x)^{17}$, $(1+x)^{23}$ and $(1+x)^{41}$ common from R_1 , R_2 and R_3 respectively

$$(1+x)^{17} \cdot (1+x)^{23} \cdot (1+x)^{41} \begin{vmatrix} 1 & (1+x)^2 & (1+x)^6 \\ 1 & (1+x)^6 & (1+x)^{11} \\ 1 & (1+x)^2 & (1+x)^6 \end{vmatrix}$$

$$\Rightarrow (1+x)^{17} \cdot (1+x)^{23} \cdot (1+x)^{41} \cdot 0 \quad (R_1 \text{ and } R_3 \text{ are identical})$$

$$\therefore 0 = A + Bx + Cx^2 + \dots$$

By comparing the like terms, we get $A = 0$.

State True or False for the statements of the following Exercises:

Q48. $(A^3)^{-1} = (A^{-1})^3$, where A is a square matrix and $|A| \neq 0$.

Sol. Since $(A^K)^{-1} = (A^{-1})^K$ where $K \in \mathbb{N}$

So, $(A^3)^{-1} = (A^{-1})^3$ is true

Q49. $(aA)^{-1} = \frac{1}{a} A^{-1}$, where a is any real number and A is a square matrix.

Sol. If A is a non-singular square matrix, then for any non-zero scalar ' a ', aA is invertible.

$$\therefore (aA) \cdot \left(\frac{1}{a} A^{-1} \right) = a \cdot \frac{1}{a} \cdot A \cdot A^{-1} = I$$

So, (aA) is inverse of $\left(\frac{1}{a} A^{-1} \right)$

$$\Rightarrow (aA)^{-1} = \frac{1}{a} A^{-1} \text{ is true.}$$

Q50. $|A^{-1}| \neq |A|^{-1}$, where A is a non-singular matrix.

Sol. False.

Since $|A^{-1}| = |A|^{-1}$ for a non-singular matrix.

Q51. If A and B are matrices of order 3 and $|A| = 5, |B| = 3$ then $|3AB| = 27 \times 5 \times 3 = 405$

Sol. True.

$$|3AB| = 3^3 |AB| = 27 |A| |B| = 27 \times 5 \times 3 \quad [\because |KA| = K^n |A|]$$

Q52. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.

Sol. True.

Since $|A| = 12$

If A is a square matrix of order n

then $|\text{Adj } A| = |A|^{n-1}$

$$\therefore |\text{Adj } A| = |A|^{3-1} = |A|^2 = (12)^2 = 144 \quad [n = 3]$$

Q53. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, where a, b, c are in A.P.

Sol. True.

$$\text{Let } \Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$$R_2 \rightarrow 2R_2 - (R_1 + R_3)$$

$$= \begin{vmatrix} x+1 & x+2 & x+a \\ 0 & 0 & 2b - (a+c) \\ x+3 & x+4 & x+c \end{vmatrix}$$

a, b, c are in A.P.

$$\begin{aligned} \therefore b - a = c - b &\Rightarrow 2b = a + c \\ &= \begin{vmatrix} x+1 & x+2 & x+a \\ 0 & 0 & 0 \\ x+3 & x+4 & x+c \end{vmatrix} = 0 \end{aligned}$$

Q54. $|\text{adj } A| = |A|^2$, where A is a square matrix of order two.

Sol. False.

Since $|\text{adj } A| = |A|^{n-1}$ where n is the order of the square matrix.

Q55. The determinant $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$ is equal to zero.

Sol. True.

$$\text{Let } \Delta = \begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$$

Splitting up C_3

$$= \begin{vmatrix} \sin A & \cos A & \sin A \\ \sin B & \cos A & \sin B \\ \sin C & \cos A & \sin C \end{vmatrix} + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix}$$

$$= 0 + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix} \quad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

$$= \cos A \cos B \begin{vmatrix} \sin A & 1 & 1 \\ \sin B & 1 & 1 \\ \sin C & 1 & 1 \end{vmatrix}$$

$$\begin{aligned} &[\text{Taking } \cos A \text{ and } \cos B \text{ common from } C_2 \text{ and } C_3 \text{ respectively}] \\ &= \cos A \cos B (0) \quad [\because C_2 \text{ and } C_3 \text{ are identical}] \\ &= 0 \end{aligned}$$

Q56. If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly K

determinants of order 3, each element of which contains only one term, then the value of K is 8.

Sol. True.

$$\text{Let } \Delta = \begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$$

Splitting up C_1

$$\Rightarrow \begin{vmatrix} x & p+u & l+f \\ y & q+v & m+g \\ z & r+w & n+h \end{vmatrix} + \begin{vmatrix} a & p+u & l+f \\ b & q+v & m+g \\ c & r+w & n+h \end{vmatrix}$$

Splitting up C_2 in both determinants

$$\Rightarrow \begin{vmatrix} x & p & l+f \\ y & q & m+g \\ z & r & n+h \end{vmatrix} + \begin{vmatrix} x & u & l+f \\ y & v & m+g \\ z & w & n+h \end{vmatrix} + \begin{vmatrix} a & p & l+f \\ b & q & m+g \\ c & r & n+h \end{vmatrix} \\ + \begin{vmatrix} a & u & l+f \\ b & v & m+g \\ c & w & n+h \end{vmatrix}$$

Similarly by splitting C_3 in each determinant, we will get 8 determinants.

Q57. Let $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$

then $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$

Sol. True.

Given that: $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$

L.H.S. $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix}$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 2p+2x+2a & a+x & a+p \\ 2q+2y+2b & b+y & b+q \\ 2r+2z+2c & c+z & c+r \end{vmatrix}$$

$$= 2 \begin{vmatrix} p+x+a & a+x & a+p \\ q+y+b & b+y & b+q \\ r+z+c & c+z & c+r \end{vmatrix}$$

[Taking 2 common from C_1]

$$C_1 \rightarrow C_1 - C_2 = 2 \begin{vmatrix} p & a+x & a+p \\ q & b+y & b+q \\ r & c+z & c+r \end{vmatrix}$$

$$C_3 \rightarrow C_3 - C_1^d = 2 \begin{vmatrix} p & a+x & a \\ q & b+y & b \\ r & c+z & c \end{vmatrix}$$

Splitting up C_2

$$= 2 \begin{vmatrix} p & a & a \\ q & b & b \\ r & c & c \end{vmatrix} + 2 \begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} = 2(0) + 2 \begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix}$$

$$= 2 \begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} \Rightarrow 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad (C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3)$$

$$= 2 \times 16 = 32$$

Q58. The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1 + \sin \theta) & 1 \\ 1 & 1 & 1 + \cos \theta \end{vmatrix}$ is $\frac{1}{2}$.

Sol. True.

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & (1 + \sin \theta) & 1 \\ 1 & 1 & 1 + \cos \theta \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -\sin \theta & \sin \theta & 1 \\ 0 & -\cos \theta & 1 + \cos \theta \end{vmatrix}$$

Expanding along C_3

$$= 1 \begin{vmatrix} -\sin \theta & \sin \theta \\ 0 & -\cos \theta \end{vmatrix} = \sin \theta \cos \theta - 0 = \sin \theta \cos \theta$$

$$= \frac{1}{2} \cdot 2 \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$$

$$= \frac{1}{2} \times 1 \quad [\text{Maximum value of } \sin 2\theta = 1]$$

$$= \frac{1}{2}$$

□□□