5 Continuity and Differentiability

5.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Examine the continuity of the function

$f(x) = x^3 + 2x^2 - 1$ at $x = 1$

Sol. We know that $y = f(x)$ will be continuous at $x = a$ if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)$$

Given: $f(x) = x^3 + 2x^2 - 1$

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1+h)^3 + 2(1+h)^2 - 1 = 1 + 2 - 1 = 2$$

$$\lim_{x \to 1} f(x) = (1)^3 + 2(1)^2 - 1$$
$$= 1 + 2 - 1 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{\to} (1+h)^3 + 2(1+h)^2 - 1$$
$$= 1 + 2 - 1 = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} f(x) = \lim_{x \to 1^+} f(x) = 2.$$

Hence, $f(x)$ is continuous at $x = 1$.

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

Q2. $f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ at $x = 2$

Sol.

$$\lim_{x \to 2^+} f(x) = 3x + 5$$
$$= \lim_{h \to 0} 3(2 + h) + 5 = 11$$

$$\lim_{x \to 2} f(x) = 3x + 5 = 3(2) + 5 = 11$$

$$\lim_{x \to 2^-} f(x) = x^2 = \lim_{h \to 0} (2 - h)^2$$
$$= \lim_{h \to 0} (2)^2 + h^2 - 4h = (2)^2 = 4$$

Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) \neq \lim_{x \to 2} f(x)$

Hence $f(x)$ is discontinuous at $x = 2$.

Q3. $f(x) = \begin{cases} \dfrac{1 - \cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. $\lim_{x\to 0^-} f(x) = \dfrac{1-\cos 2x}{x^2}$

$$= \lim_{h\to 0}\frac{1-\cos 2(0-h)}{(0-h)^2} = \lim_{h\to 0}\frac{1-\cos(-2h)}{h^2}$$

$$= \lim_{h\to 0}\frac{1-\cos 2h}{h^2}$$

$$= \lim_{h\to 0}\frac{2\sin^2 h}{h^2} \qquad \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \lim_{h\to 0}\frac{2\sin h}{h}\cdot\frac{\sin h}{h} = 2.1.1 = 2 \qquad \left[\lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$\lim_{x\to 0^+} f(x) = \dfrac{1-\cos 2x}{x^2}$

$$= \lim_{h\to 0}\frac{1-\cos 2(0+h)}{(0+h)^2} = \lim_{h\to 0}\frac{1-\cos 2h}{h^2}$$

$$= \lim_{h\to 0}\frac{2\sin^2 h}{h^2} = \frac{2\sin h}{h}\cdot\frac{\sin h}{h} = 2.1.1 = 2$$

$\lim_{x\to 0} f(x) = 5$

As $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0} f(x)$

$\therefore f(x)$ is discontinuous at $x = 0$.

Q4. $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$ at $x = 2$

Sol. $f(x) = \dfrac{2x^2-3x-2}{x-2}$

$$= \frac{2x^2-4x+x-2}{x-2} = \frac{2x(x-2)+1(x-2)}{x-2}$$

$$= \frac{(2x+1)(x-2)}{x-2} = 2x+1$$

$\lim_{x\to 2^-} f(x) = 2x+1$

$$= \lim_{h\to 0} 2(2-h)+1 = 4+1 = 5$$

$\lim_{x\to 2^+} f(x) = 2x+1$

$$= \lim_{h\to 0} 2(2+h)+1 = 4+1 = 5$$

$\lim_{x\to 2} f(x) = 5$

As $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 5$

Hence, $f(x)$ is continuous at $x = 2$.

Q5. $f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ at $x = 4$

Sol. $\lim_{x \to 4^-} f(x) = \dfrac{|x-4|}{2(x-4)}$ $\left[\begin{array}{l}\text{for } x < 4, |x-4| = -(x-4) \\ \text{for } x > 4, |x-4| = (x-4)\end{array}\right]$

$$= \lim_{h \to 0} \frac{-[4-h-4]}{2[4-h-4]} = \lim_{h \to 0} \frac{h}{-2h} = -\frac{1}{2}$$

$$\lim_{x \to 4^+} f(x) = \frac{|x-4|}{2(x-4)} = \lim_{h \to 0} \frac{[4+h-4]}{2[4+h-4]} = \frac{1}{2}$$

$$\lim_{x \to 4} f(x) = 0$$

$$\therefore \quad \lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x) \neq \lim_{x \to 4} f(x)$$

Hence, $f(x)$ is discontinuous at $x = 4$.

Q6. $f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

Sol. $\lim_{x \to 0^-} f(x) = |x| \cos \dfrac{1}{x}$

$$= \lim_{h \to 0} |0-h| \cos \frac{1}{(0-h)} = \lim_{h \to 0} h \cos \frac{1}{h}$$

$$= 0 \qquad \left[\because \cos \frac{1}{x} \text{ oscillate between } -1 \text{ and } 1\right]$$

$$\lim_{x \to 0^-} f(x) = |x| \cos \frac{1}{x}$$

$$= \lim_{h \to 0} |0+h| \cos \frac{1}{(0+h)} = \lim_{h \to 0} h . \cos \frac{1}{h} = 0$$

$$\lim_{x \to 0} f(x) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 0$$

Hence, $f(x)$ is continuous at $x = 0$.

Q7. $f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & \text{if } x \neq 0 \\ 0, & \text{if } x = a \end{cases}$ at $x = a$.

Sol. $\lim_{x \to a^-} f(x) = |x-a| \sin \frac{1}{x-a}$

$$= \lim_{h \to 0} |a-h-a| . \sin \frac{1}{a-h-a} = \lim_{h \to 0} h . \sin \frac{1}{-h}$$

$$= \lim_{h \to 0} -h . \sin \frac{1}{h} \qquad [\because \sin(-\theta) = -\sin\theta]$$

$= 0 \times$ [a number oscillating between -1 and 1]

$= 0$

$\lim_{x \to a^+} f(x) = |x-a| \sin \frac{1}{x-a}$

$$= \lim_{h \to 0} |a+h-a| . \sin \frac{1}{a+h-a} = \lim_{h \to 0} h . \sin \frac{1}{h}$$

$= 0 \times$ [a number oscillating between -1 and 1]

$\lim_{x \to a} f(x) = 0$

As $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = 0$

Hence, $f(x)$ is continuous at $x = a$.

Q8. $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

Sol. $\lim_{x \to 0^-} f(x) = \dfrac{e^{1/x}}{1+e^{1/x}}$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{0-h}}}{1+e^{\frac{1}{0-h}}} = \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{1/h}\left(1-e^{-1/h}\right)} = \lim_{h \to 0} \frac{1}{e^{1/h}-1} = \frac{1}{e^{1/0}-1}$$

$$= \frac{1}{e^{\infty}-1} = \frac{1}{0-1} = -1 \qquad [\because e^{\infty} = 0]$$

$\lim_{x \to 0^+} f(x) = \dfrac{e^{1/x}}{1+e^{1/x}}$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{0+h}}}{1+e^{\frac{1}{0+h}}} = \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{-1/h}(1 + e^{1/h})} = \lim_{h \to 0} \frac{1}{e^{-1/h} + 1}$$

$$= \frac{1}{e^{-\infty} + 1} = \frac{1}{0 + 1} = 1 \qquad [e^{-\infty} = 0]$$

$$\lim_{x \to 0} f(x) = 0$$

As $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0} f(x)$

Hence, $f(x)$ is discontinuous at $x = 0$.

Q9. $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$.

Sol. $\lim_{x \to 1^-} f(x) = \dfrac{x^2}{2} = \lim_{h \to 0} \dfrac{(1-h)^2}{2} = \dfrac{1}{2}$

$\lim_{x \to 1} f(x) = \dfrac{x^2}{2} = \dfrac{(1)^2}{2} = \dfrac{1}{2}$

$\lim_{x \to 1^+} f(x) = 2x^2 - 3x + \dfrac{3}{2} = 2(1)^2 - 3(1) + \dfrac{3}{2} = 2 - 3 + \dfrac{3}{2} = \dfrac{1}{2}$

As $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = \dfrac{1}{2}$

Hence, $f(x)$ is continuous at $x = 1$.

Q10. $f(x) = |x| + |x - 1|$ at $x = 1$.

Sol. $\lim_{x \to 1^-} f(x) = |x| + |x - 1| = \lim_{h \to 0} |1 - h| + |1 - h - 1|$

$= |1 - 0| + |1 - 0 - 1| = 1 + 0 = 1$

$\lim_{x \to 1^+} f(x) = |x| + |x - 1|$

$= \lim_{h \to 0} |1 + h| + |1 + h - 1| = 1 + 0 = 1$

$\lim_{x \to 1} f(x) = |x| + |x - 1| = |1| + |1 - 1| = 1 + 0 = 1$

As $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x)$

Hence, $f(x)$ is continuous at $x = 1$.

Find the value of k in each of the Exercises 11 to 14 so that the function f is continuous at the indicated point:

Q11. $f(x) = \begin{cases} 3x - 8, & \text{if } x \le 5 \\ 2k, & \text{if } x > 5 \end{cases}$ at $x = 5$

Sol. $\lim_{x \to 5^-} f(x) = 3x - 8$

$= \lim_{h \to 0} 3(5-h) - 8 = 15 - 8 = 7$

$\lim_{x \to 5^+} f(x) = 2k$

As the function is continuous at $x = 5$

$\therefore \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x)$

$\therefore \quad 7 = 2k \Rightarrow k = \frac{7}{2}$

Hence, the value of k is $\frac{7}{2}$.

Q12. $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$ at $x = 2$

Sol. $f(x) = \dfrac{2^{x+2} - 16}{4^x - 16} = \dfrac{2^2 . 2^x - 16}{(2^x)^2 - (4)^2} = \dfrac{4(2^x - 4)}{(2^x - 4)(2^x + 4)}$

$f(x) = \dfrac{4}{2^x + 4}$

$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} \dfrac{4}{2^{2-h} + 4} = \dfrac{4}{2^2 + 4} = \dfrac{4}{4+4} = \dfrac{4}{8} = \dfrac{1}{2}$

$\lim_{x \to 2} f(x) = k$

As the function is continuous at $x = 2$.

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x)$

$\therefore \quad k = \frac{1}{2}$

Hence, value of k is $\frac{1}{2}$.

Q13. $f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \leq x \leq 1 \end{cases}$ at $x = 0$

Sol. $\lim_{x \to 0^-} f(x) = \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$

$= \lim_{x \to 0^-} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x} \times \dfrac{\sqrt{1+kx} + \sqrt{1-kx}}{\sqrt{1+kx} + \sqrt{1-kx}}$

$$= \lim_{x\to 0^-} \frac{(1+kx)-(1-kx)}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x\to 0^-} \frac{1+kx-1+kx}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x\to 0^-} \frac{2kx}{x\left[\sqrt{1+kx}+\sqrt{1-kx}\right]}$$

$$= \lim_{x\to 0^-} \frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}}$$

$$= \lim_{h\to 0} \frac{2k}{\sqrt{1+k(0-h)}+\sqrt{1-k(0-h)}}$$

$$= \frac{2k}{\sqrt{1}+\sqrt{1}} = \frac{2k}{2} = k$$

$$\lim_{x\to 0} f(x) = \frac{2x+1}{x-1} = \frac{2(0)+1}{0-1} = \frac{1}{-1} = -1$$

As the function is continuous at $x = 0$.

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$$

$$k = -1$$

Hence, the value of k is -1.

Q14. $f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$

Sol. $\displaystyle \lim_{x\to 0^-} f(x) = \frac{1-\cos kx}{x \sin x}$

$$= \lim_{h\to 0} \frac{1-\cos k(0-h)}{(0-h)\sin(0-h)} = \lim_{h\to 0} \frac{1-\cos(-kh)}{-h.\sin(-h)}$$

$$= \lim_{h\to 0} \frac{1-\cos kh}{h \sin h} \quad \left[\because \begin{matrix} \sin(-\theta) = -\sin\theta \\ \cos(-\theta) = \cos\theta \end{matrix}\right]$$

$$= \lim_{h\to 0} \frac{2\sin^2 \frac{kh}{2}}{h \sin h}$$

$$= \lim_{\substack{h\to 0 \\ kh\to 0}} \frac{2\sin\frac{kh}{2}}{\frac{kh}{2}} \times \frac{kh}{2} \times \frac{\sin\frac{kh}{2}}{\frac{kh}{2}} \times \frac{kh}{2} . \frac{1}{h.\frac{\sin h}{h}.h}$$

$$= 2.1.\frac{kh}{2}.1.\frac{kh}{2}.\frac{1}{h^2}.1 \qquad \left[\lim_{h\to 0}\frac{\sin h}{h} = 1 \text{ and } \lim_{kh\to 0}\frac{\sin kh}{kh} = 1\right]$$

$$= \frac{k^2}{2}$$

$$\lim_{x\to 0} f(x) = \frac{1}{2}$$

As $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$

$$\therefore \quad \frac{k^2}{2} = \frac{1}{2}$$

$$\Rightarrow \quad k^2 = 1 \;\Rightarrow k = \pm 1$$

Hence, the value of k is ± 1.

Q15. Prove that the function f defined by

$$f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

remains discontinuous at $x = 0$, regardless the choice of k.

Sol.
$$\lim_{x\to 0^-} f(x) = \frac{x}{|x| + 2x^2} = \lim_{h\to 0}\frac{0-h}{|0-h| + 2(0-h)^2}$$

$$= \lim_{h\to 0}\frac{-h}{h + 2h^2} = \lim_{h\to 0}\frac{-h}{h(1+2h)}$$

$$= \lim_{h\to 0}\frac{-1}{1+2h} = \frac{-1}{1+2(0)} = -1$$

$$\lim_{x\to 0^+} f(x) = \frac{x}{|x| + 2x^2} = \lim_{h\to 0}\frac{0+h}{|0+h| + 2(0+h)^2}$$

$$= \lim_{h\to 0}\frac{h}{h + 2h^2} = \lim_{h\to 0}\frac{h}{h(1+2h)} = \frac{1}{1+0} = 1$$

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Hence, $f(x)$ is discontinuous at $x = 0$ regardless the choice of k.

Q16. Find the values of a and b such that the function f defined by

$$f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a + b, & \text{if } x = 4 \\ \dfrac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$$

is a continuous function at $x = 4$.

Sol. $\lim_{x \to 4^-} f(x) = \frac{x-4}{|x-4|} + a$

$= \lim_{h \to 0} \frac{4-h-4}{|4-h-4|} + a = \lim_{h \to 0} \frac{-h}{h} + a = -1 + a$

$\lim_{x \to 4} f(x) = a + b$

$\lim_{x \to 4^+} f(x) = \frac{x-4}{|x-4|} + b$

$= \lim_{h \to 0} \frac{4+h-4}{|4+h-4|} + b = \lim_{h \to 0} \frac{h}{h} + b = 1 + b$

As the function is continuous at $x = 4$.

$\therefore \quad \lim_{\to} f(x) = \lim_{x \to 4} f(x) = \lim_{x \to 4^+} f(x)$

$-1 + a = a + b = 1 + b$

$\therefore \quad -1 + a = a + b \Rightarrow b = -1$

$1 + b = a + b \Rightarrow a = 1$

Hence, the value of $a = 1$ and $b = -1$.

Q17. Given the function $f(x) = \frac{1}{x+2}$. Find the point of discontinuity of the composite function $y = f[f(x)]$.

Sol. $f(x) = \frac{1}{x+2}$

$f[f(x)] = \frac{1}{f(x)+2} = \frac{1}{\frac{1}{x+2}+2} = \frac{1}{\frac{1+2x+4}{x+2}} = \frac{x+2}{2x+5}$

$\therefore \quad f[f(x)] = \frac{x+2}{2x+5}$

This function will not be defined and continuous where $2x + 5 = 0 \Rightarrow x = \frac{-5}{2}$.

Hence, $x = \frac{-5}{2}$ is the point of discontinuity.

Q18. Find all the points of discontinuity of the function

$f(t) = \frac{1}{t^2+t-2}$, where $t = \frac{1}{x-1}$.

Sol. We have $f(t) = \frac{1}{t^2+t-2}$

$\Rightarrow \quad f(t) = \frac{1}{\frac{1}{(x-1)^2} + \frac{1}{(x-1)} - 2}$ $\left[\text{putting } t = \frac{1}{x-1}\right]$

$$= \frac{1}{\frac{1+x-1-2(x-1)^2}{(x-1)^2}} = \frac{(x-1)^2}{x-2x^2-2+4x}$$

$$= \frac{(x-1)^2}{-2x^2+5x-2} = \frac{(x-1)^2}{-(2x^2-5x+2)}$$

$$= \frac{(x-1)^2}{-[2x^2-4x-x+2]} = \frac{(x-1)^2}{-[2x(x-2)-1(x-2)]}$$

$$= \frac{(x-1)^2}{-(x-2)(2x-1)} = \frac{(x-1)^2}{(2-x)(2x-1)}$$

So, if $f(t)$ is discontinuous, then $2-x=0 \quad \therefore x=2$

and $2x-1=0 \quad \therefore x=\frac{1}{2}$

Hence, the required points of discontinuity are 2 and $\frac{1}{2}$.

Q19. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x=\pi$.

Sol. Given that $f(x) = |\sin x + \cos x|$ at $x=\pi$

Put $g(x) = \sin x + \cos x$ and $h(x) = |x|$

$\therefore \quad h[g(x)] = h(\sin x + \cos x) = |\sin x + \cos x|$

Now, $g(x) = \sin x + \cos x$ is a continuous function since $\sin x$ and $\cos x$ are two continuous functions at $x=\pi$.

We know that every modulus function is a continuous function everywhere.

Hence, $f(x) = |\sin x + \cos x|$ is continuous function at $x=\pi$.

Q20. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases} \quad \text{at } x=2.$$

Sol. We know that a function f is differentiable at a point 'a' in its domain if

$$Lf'(c) = Rf'(c)$$

where $Lf'(c) = \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$ and

$$Rf'(c) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Here, $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 2 \end{cases}$ at $x=2$.

at $x=2$

$$Lf'(c) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{(2-h)[2-h]-(2-1)2}{-h}$$

$$= \lim_{h\to 0} \frac{(2-h).1-2}{-h} \qquad [\because [2-h]=1]$$

$$= \lim_{h\to 0} \frac{2-h-2}{-h} = 1$$

$$Rf'(c) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h-1)(2+h)-(2-1).2}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)(2+h)-2}{h} = \lim_{h\to 0} \frac{2+h+2h+h^2-2}{h}$$

$$= \lim_{h\to 0} \frac{3h+h^2}{h} = \lim_{h\to 0} \frac{h(3+h)}{h} = 3$$

$$Lf'(2) \neq Rf'(2)$$

Hence, $f(x)$ is not differentiable at $x = 2$.

Q21. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0.$$

Sol. Given that:

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$

For differentiability we know that:

$$Lf'(c) = Rf'(c)$$

$$\therefore Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(0-h)^2 \sin\frac{1}{(0-h)} - 0}{-h} = \frac{h^2 .\sin\left(-\frac{1}{h}\right)}{-h}$$

$$= h.\sin\left(\frac{1}{h}\right) = 0 \times \left[-1 \leq \sin\left(\frac{1}{h}\right) \leq 1\right]$$

$$= 0$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^2 \sin\left(\frac{1}{0+h}\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h . \sin\left(\frac{1}{h}\right)$$

$$= 0 \times \left[-1 \leq \sin\left(\frac{1}{h}\right) \leq 1\right] = 0$$

So, $Lf'(0) = Rf'(0) = 0$

Hence, $f(x)$ is differentiable at $x = 0$.

Q22. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} 1 + x, \text{ if } x \leq 2 \\ 5 - x, \text{ if } x > 2 \end{cases} \quad \text{at } x = 2.$$

Sol. $f(x)$ is differentiable at $x = 2$ if

$Lf'(2) = Rf'(2)$

$$\therefore \quad Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{(1+2-h) - (1+2)}{-h} = \lim_{h \to 0} \frac{3-h-3}{-h} = \frac{-h}{-h} = 1$$

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{[5-(2+h)] - (1+2)}{h} = \lim_{h \to 0} \frac{3-h-3}{h}$$

$$= \frac{-h}{h} = -1$$

So, $Lf'(2) \neq Rf'(2)$

Hence, $f(x)$ is not differentiable at $x = 2$.

Q23. Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

Sol. We have $f(x) = |x - 5|$

$$\Rightarrow \quad f(x) = \begin{cases} -(x-5) \text{ if } x - 5 < 0 \text{ or } x < 5 \\ x - 5 \text{ if } x - 5 > 0 \text{ or } x > 5 \end{cases}$$

For continuity at $x = 5$

L.H.L. $\lim_{h \to 5^-} f(x) = -(x-5)$

$$= \lim_{h \to 0} -(5-h-5) = \lim_{h \to 0} h = 0$$

R.H.L. $\lim_{x \to 5^+} f(x) = x - 5$

$$= \lim_{h \to 0} (5+h-5) = \lim_{h \to 0} h = 0$$

L.H.L. = R.H.L.

So, $f(x)$ is continuous at $x = 5$.

Now, for differentiability

$$Lf'(5) = \lim_{h \to 0} \frac{f(5-h) - f(5)}{-h}$$

$$= \lim_{h \to 0} \frac{-(5-h-5)-(5-5)}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

$$Rf'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$$= \lim_{h \to 0} \frac{(5+h-5)-(5-5)}{h} = \lim_{h \to 0} \frac{h-0}{h} = 1$$

$\because \quad Lf'(5) \neq Rf'(5)$

Hence, $f(x)$ is not differentiable at $x = 5$.

Q24. A function $f : R \to R$ satisfies the equation $f(x + y) = f(x).f(y)$ $\forall\, x, y \in R, f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

Sol. Given that: $f : R \to R$ satisfies the equation $f(x + y) = f(x).f(y)$ $\forall\, x, y \in R, f(x) \neq 0$.

Let us take any point $x = 0$ at which the function $f(x)$ is differentiable.

$$\therefore \quad f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$2 = \lim_{h \to 0} \frac{f(0).f(h) - f(0)}{h} \quad [\because f(0) = f(h)] \qquad ...(i)$$

$$\Rightarrow \quad 2 = \lim_{h \to 0} \frac{f(0)[f(h)-1]}{h}$$

Now $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x).f(h) - f(x)}{h} \quad [\because f(x+y) = f(x).f(y)]$$

$$= \lim_{h \to 0} \frac{f(x)[f(h)-1]}{h} = 2f(x) \qquad \text{from eqn. } (i)$$

Hence, $f'(x) = 2f(x)$.

Differentiate each of the following w.r.t. x (Exercises 25 to 43):

Q25. $2^{\cos^2 x}$

Sol. Let $\quad y = 2^{\cos^2 x}$

Taking log on both sides, we get

$$\log y = \log 2^{\cos^2 x} \Rightarrow \log y = \cos^2 x \,.\, \log 2$$

Differentiating both sides w.r.t. x

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 2.\frac{d}{dx}\cos^2 x$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 2\left[2\cos x.\frac{d}{dx}\cos x\right]$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 2\ [2\cos x(-\sin x)]$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 2\,(-\sin 2x)$$

$$\frac{dy}{dx} = -y\,.\log 2\sin 2x$$

Hence, $\dfrac{dy}{dx} = -2^{\cos^2 x}(\log 2\sin 2x)$

Q26. $\dfrac{8^x}{x^8}$

Sol. Let $\quad y = \dfrac{8^x}{x^8}$

Taking log on both sides, we get, $\log y = \log \dfrac{8^x}{x^8}$

$\Rightarrow \quad \log y = \log 8^x - \log x^8 \ \Rightarrow\ \log y = x\log 8 - 8\log x$

Differentiating both sides w.r.t. x

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \log 8.1 - \frac{8}{x} \quad \Rightarrow \quad \frac{dy}{dx} = y\left[\log 8 - \frac{8}{x}\right]$$

Hence, $\dfrac{dy}{dx} = \dfrac{8^x}{x^8}\left[\log 8 - \dfrac{8}{x}\right]$

Q27. $\log\left(x + \sqrt{x^2 + a}\right)$

Sol. Let $\quad y = \log\left(x + \sqrt{x^2 + a}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log\left(x + \sqrt{x^2 + a}\right)$$

$$= \frac{1}{x + \sqrt{x^2 + a}}.\frac{d}{dx}\left(x + \sqrt{x^2 + a}\right)$$

$$= \frac{1}{x + \sqrt{x^2 + a}}.\left[1 + \frac{1}{2\sqrt{x^2 + a}} \times \frac{d}{dx}(x^2 + a)\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}}.\left[1+\frac{1}{2\sqrt{x^2+a}}.2x\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}}.\left[1+\frac{x}{\sqrt{x^2+a}}\right]$$

$$= \frac{1}{x+\sqrt{x^2+a}}.\left(\frac{\sqrt{x^2+a}+x}{\sqrt{x^2+a}}\right) = \frac{1}{\sqrt{x^2+a}}$$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+a}}$.

Q28. $\log [\log (\log x^5)]$

Sol. Let $y = \log [\log (\log x^5)]$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\log [\log (\log x^5)]$$

$$= \frac{1}{\log (\log x^5)} \times \frac{d}{dx}\log (\log x^5)$$

$$= \frac{1}{\log (\log x^5)} \times \frac{1}{\log(x^5)} \times \frac{d}{dx}\log x^5$$

$$= \frac{1}{\log (\log x^5)}.\frac{1}{\log(x^5)}.\frac{1}{x^5}.\frac{d}{dx}x^5$$

$$= \frac{1}{\log (\log x^5)}.\frac{1}{\log(x^5)}.\frac{1}{x^5}.5x^4$$

$$= \frac{5}{x\log (x^5).\log (\log x^5)}$$

Hence, $\frac{dy}{dx} = \frac{5}{x\log (x^5).\log (\log x^5)}$.

Q29. $\sin\sqrt{x} + \cos^2\sqrt{x}$

Sol. Let $y = \sin\sqrt{x} + \cos^2\sqrt{x}$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin\sqrt{x}\right) + \frac{d}{dx}\left(\cos^2\sqrt{x}\right)$$

$$= \cos\sqrt{x}.\frac{d}{dx}\left(\sqrt{x}\right) + 2\cos\sqrt{x}.\frac{d}{dx}\left(\cos\sqrt{x}\right)$$

$$= \cos\sqrt{x}.\frac{1}{2\sqrt{x}} + 2\cos\sqrt{x}\left(-\sin\sqrt{x}\right).\frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{2\sqrt{x}}.\cos\sqrt{x} - 2\cos\sqrt{x}.\sin\sqrt{x}.\frac{1}{2\sqrt{x}}$$

$$= \frac{\cos\sqrt{x}}{2\sqrt{x}} - \frac{\sin 2\sqrt{x}}{2\sqrt{x}}$$

Hence, $\frac{dy}{dx} = \frac{\cos\sqrt{x}}{2\sqrt{x}} - \frac{\sin 2\sqrt{x}}{2\sqrt{x}}$.

Q30. $\sin^n(ax^2 + bx + c)$

Sol. Let $y = \sin^n(ax^2 + bx + c)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^n(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\frac{d}{dx}\sin(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c).\frac{d}{dx}(ax^2 + bx + c)$$

$$= n.\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c).(2ax + b)$$

Hence, $\frac{dy}{dx} = n(2ax + b).\sin^{n-1}(ax^2 + bx + c).\cos(ax^2 + bx + c)$

Q31. $\cos(\tan\sqrt{x+1})$

Sol. Let $y = \cos\left(\tan\sqrt{x+1}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left(\tan\sqrt{x+1}\right)$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\frac{d}{dx}\left(\tan\sqrt{x+1}\right)$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}.\frac{d}{dx}\sqrt{x+1}$$

$$= -\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}.\frac{1}{2\sqrt{x+1}}.1$$

Hence, $\frac{dy}{dx} = -\frac{1}{2\sqrt{x+1}}\sin\left(\tan\sqrt{x+1}\right).\sec^2\sqrt{x+1}$

Q32. $\sin x^2 + \sin^2 x + \sin^2(x^2)$

Sol. Let $y = \sin x^2 + \sin^2 x + \sin^2(x^2)$

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = \frac{d}{dx}\sin(x^2) + \frac{d}{dx}\sin^2 x + \frac{d}{dx}\sin^2(x^2)$$

$$= \cos x^2 . \frac{d}{dx}(x^2) + 2\sin x . \frac{d}{dx}(\sin x) + 2\sin(x^2)\frac{d}{dx}\sin(x^2)$$

$$= \cos x^2 . 2x + 2\sin x . \cos x + 2\sin x^2 . \cos x^2 . \frac{d}{dx}(x^2)$$

$$= 2x . \cos x^2 + \sin 2x + 2\sin x^2 . \cos x^2 . 2x$$

Hence, $\frac{dy}{dx} = 2x . \cos x^2 + \sin 2x + 2x \sin 2x^2$

Q33. $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Sol. Let $y = \sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right) = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}} . \frac{d}{dx}\left(\frac{1}{\sqrt{x+1}}\right)$$

$$= \frac{1}{\sqrt{1-\frac{1}{x+1}}} . \frac{d}{dx}(x+1)^{-1/2}$$

$$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}} . \frac{-1}{2}(x+1)^{-3/2} . \frac{d}{dx}(x+1)$$

$$= \frac{\sqrt{x+1}}{\sqrt{x}} . \frac{-1}{2}(x+1)^{-3/2} . 1$$

$$= \frac{-1}{2} . \frac{\sqrt{x+1}}{\sqrt{x}} . \frac{1}{(x+1)^{3/2}} = -\frac{1}{2\sqrt{x}(x+1)}$$

Hence, $\frac{dy}{dx} = -\frac{1}{2\sqrt{x}(x+1)}$

Q34. $(\sin x)^{\cos x}$

Sol. Let $y = (\sin x)^{\cos x}$

Taking log on both sides,

$$\log y = \log(\sin x)^{\cos x}$$

$\Rightarrow \quad \log y = \cos x . \log(\sin x)$ $\quad [\because \log x^y = y \log x]$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}\cos x.\log(\sin x)$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \cos x.\frac{d}{dx}\log(\sin x) + \log(\sin x).\frac{d}{dx}\cos x$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \cos x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x) + \log(\sin x).(-\sin x)$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \cot x\,.\cos x - \sin x\,.\log(\sin x)$$

$$\frac{dy}{dx} = y[\cot x.\cos x - \sin x.\log(\sin x)]$$

Hence, $\frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x.\log(\sin x)\right]$

Q35. $\sin^m x\,.\cos^n x$

Sol. Let $\quad y = \sin^m x\,.\cos^n x$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^m x.\cos^n x)$$

$$= \sin^m x.\frac{d}{dx}(\cos^n x) + \cos^n x.\frac{d}{dx}\sin^m x$$

$$= \sin^m x.n.\cos^{n-1} x\frac{d}{dx}(\cos x) + \cos^n x.m.\sin^{m-1} x\frac{d}{dx}(\sin x)$$

$$= n.\sin^m x.\cos^{n-1} x.(-\sin x) + m.\cos^n x.\sin^{m-1} x.\cos x$$

$$= -n.\sin^{m+1} x.\cos^{n-1} x + m\cos^{n+1} x.\sin^{m-1} x$$

$$= \sin^m x.\cos^n x\left[-n\frac{\sin x}{\cos x} + m.\frac{\cos x}{\sin x}\right]$$

Hence, $\frac{dy}{dx} = \sin^m x.\cos^n x[-n\tan x + m.\cot x]$

Q36. $(x+1)^2(x+2)^3(x+3)^4$

Sol. Let $\quad y = (x+1)^2(x+2)^3(x+3)^4$

Taking log on both sides,

$$\log y = \log[(x+1)^2.(x+2)^3.(x+3)^4]$$

$$\Rightarrow \quad \log y = \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

$[\because \log xy = \log x + \log y]$

$$\Rightarrow \quad \log y = 2\log(x+1) + 3\log(x+2) + 4\log(x+3)$$
$$[\because \log x^y = y \log x]$$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}.\frac{dy}{dx} = 2.\frac{d}{dx}\log(x+1) + 3.\frac{d}{dx}\log(x+2) + 4.\frac{d}{dx}\log(x+3)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = 2.\frac{1}{x+1} + 3.\frac{1}{x+2} + 4.\frac{1}{x+3}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+1)^2(x+2)^3(x+3)^4\left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$= (x+1)^2(x+2)^3(x+3)^4 \left[\frac{2(x+2)(x+3) + 3(x+1)(x+3) + 4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$= (x+1)(x+2)^2(x+3)^3(2x^2 + 10x + 12 + 3x^2 + 12x + 9 + 4x^2 + 12x + 8)$$

$$= (x+1)(x+2)^2(x+3)^3(9x^2 + 34x + 29)$$

Hence, $\dfrac{dy}{dx} = (x+1)(x+2)^2(x+3)^3(9x^2 + 34x + 29)$

Q37. $\cos^{-1}\left(\dfrac{\sin x + \cos x}{\sqrt{2}}\right), \ -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

Sol. Let $y = \cos^{-1}\left(\dfrac{\sin x + \cos x}{\sqrt{2}}\right)$

$$= \cos^{-1}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]$$

$$= \cos^{-1}\left[\sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}.\cos x\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{4} - x\right)\right]$$

$$y = \frac{\pi}{4} - x \qquad \left[\because -\frac{\pi}{4} < x < \frac{\pi}{4}\right]$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = -1$$

Q38. $\tan^{-1}\left[\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right], \ -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

Sol. Let $$y = \tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right]$$

$$= \tan^{-1}\left[\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}\right] \quad \left[\begin{array}{l}\because 1-\cos x = 2\sin^2 x/2 \\ \quad 1+\cos x = 2\cos^2 x/2\end{array}\right]$$

$$= \tan^{-1}\left[\frac{\sin x/2}{\cos x/2}\right] = \tan^{-1}\left[\tan\frac{x}{2}\right]$$

$$\therefore \quad y = \frac{x}{2}$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2}.1 = \frac{1}{2}$$

Hence, $\dfrac{dy}{dx} = \dfrac{1}{2}$

Q39. $\tan^{-1}(\sec x + \tan x)$, $\dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

Sol. Let $y = \tan^{-1}(\sec x + \tan x)$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\sec x + \tan x)]$$

$$= \frac{1}{1+(\sec x + \tan x)^2}.\frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{1}{1+\sec^2 x + \tan^2 x + 2\sec x \tan x}.(\sec x \tan x + \sec^2 x)$$

$$= \frac{1}{(1+\tan^2 x) + \sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{\sec^2 x + \sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{2\sec^2 x + 2\sec x \tan x}.\sec x(\tan x + \sec x)$$

$$= \frac{1}{2\sec x(\sec x + \tan x)}.\sec x(\tan x + \sec x) = \frac{1}{2}$$

Hence, $\dfrac{dy}{dx} = \dfrac{1}{2}$

Alternate solution

Let $y = \tan^{-1}(\sec x + \tan x)$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right)$$

$$= \tan^{-1}\left[\frac{\cos^2 x/2 + \sin^2 x/2 + 2\sin x/2 \cos x/2}{\cos^2 x/2 - \sin^2 x/2}\right]$$

$$\left[\begin{array}{l}\because \sin 2x = 2\sin x \cos x \\ \cos 2x = \cos^2 x - \sin^2 x\end{array}\right]$$

$$= \tan^{-1}\left[\frac{(\cos x/2 + \sin x/2)^2}{(\cos x/2 + \sin x/2)(\cos x/2 - \sin x/2)}\right]$$

$$= \tan^{-1}\left[\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right]$$

$$= \tan^{-1}\left[\frac{1 + \tan x/2}{1 - \tan x/2}\right]$$ [Dividing the Nr. and Den. by $\cos x/2$]

$$= \tan^{-1}\left[\frac{\tan \pi/4 + \tan x/2}{1 - \tan \pi/4 . \tan x/2}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$$

$\therefore \quad y = \frac{\pi}{4} + \frac{x}{2}$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2}.1 = \frac{1}{2}$$

Hence, $\frac{dy}{dx} = \frac{1}{2}$.

Q40. $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1$.

Sol. Let $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x} - \frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x} + \frac{a\sin x}{b\cos x}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x)$$

$$\left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$$

$$\Rightarrow \quad y = \tan^{-1}\frac{a}{b} - x$$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) - \frac{d}{dx}(x) = 0 - 1 = -1$$

Hence, $\frac{dy}{dx} = -1.$

Q41. $\sec^{-1}\left(\frac{1}{4x^3 - 3x}\right)$, $0 < x < \frac{1}{\sqrt{2}}$.

Sol. Let $\quad y = \sec^{-1}\left(\frac{1}{4x^3 - 3x}\right)$

Put $x = \cos\theta \quad \therefore \theta = \cos^{-1}x$

$$y = \sec^{-1}\left(\frac{1}{4\cos^3\theta - 3\cos\theta}\right)$$

$$\Rightarrow \quad y = \sec^{-1}\left(\frac{1}{\cos 3\theta}\right) \quad [\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta]$$

$$\Rightarrow \quad y = \sec^{-1}(\sec 3\theta) \Rightarrow y = 3\theta$$

$$y = 3\cos^{-1}x$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = 3.\frac{d}{dx}\cos^{-1}x = 3\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-3}{\sqrt{1-x^2}}$$

Hence, $\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$.

Q42. $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$, $\frac{-1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$.

Sol. Let $\quad y = \tan^{-1}\left[\frac{3a^2x - x^3}{a^3 - 3ax^2}\right]$

Put $x = a\tan\theta \quad \therefore \theta = \tan^{-1}\frac{x}{a}$

$$y = \tan^{-1}\left[\frac{3a^2.a\tan\theta - a^3\tan^3\theta}{a^3 - 3a.a^2\tan^2\theta}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right]$$

$$\Rightarrow \quad y = \tan^{-1}[\tan 3\theta] \qquad \left[\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right]$$

$$\Rightarrow \quad y = 3\theta \Rightarrow y = 3\tan^{-1}\frac{x}{a}$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = 3.\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right)$$

$$= 3.\frac{1}{1+\frac{x^2}{a^2}}.\frac{d}{dx}.\left(\frac{x}{a}\right) = 3.\frac{a^2}{a^2+x^2}.\frac{1}{a} = \frac{3a}{a^2+x^2}$$

Hence, $\dfrac{dy}{dx} = \dfrac{3a}{a^2+x^2}$.

Q43. $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), -1 < x < 1, x \neq 0.$

Sol. Let $\quad y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Putting $x^2 = \cos 2\theta \quad \therefore \theta = \frac{1}{2}\cos^{-1}x^2$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$\Rightarrow \quad y = \tan\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{1 + \tan\theta}{1 - \tan\theta}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}.\tan\theta}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{4} + \theta \quad \Rightarrow y = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{1}{2}\frac{d}{dx}(\cos^{-1} x^2)$$

$$= 0 + \frac{1}{2} \times \frac{-1}{\sqrt{1-x^4}}.\frac{d}{dx}(x^2) = \frac{-1.2x}{2\sqrt{1-x^4}} = -\frac{x}{\sqrt{1-x^4}}$$

Hence, $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{1-x^4}}$.

Find $\dfrac{dy}{dx}$ of each of the functions expressed in parametric form in Exercises from 44 to 48:

Q44. $x = t + \dfrac{1}{t}, \; y = t - \dfrac{1}{t}$

Sol. Given that:

$$x = t + \frac{1}{t}, \; y = t - \frac{1}{t}$$

Differentiating both the given parametric functions w.r.t. t

$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \; \frac{dy}{dt} = 1 + \frac{1}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}} = \frac{t^2+1}{t^2-1}$$

Hence, $\frac{dy}{dx} = \frac{t^2+1}{t^2-1}$.

Q45. $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Sol. Given that:

$$x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), \quad y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$

Differentiating both the parametric functions w.r.t. θ.

$$\frac{dx}{d\theta} = e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \left(\theta + \frac{1}{\theta}\right).e^{\theta}$$

$$\frac{dx}{d\theta} = e^{\theta}\left(1 - \frac{1}{\theta^2} + \theta + \frac{1}{\theta}\right) \Rightarrow e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)$$

$$= \frac{e^{\theta}(\theta^3 + \theta^2 + \theta - 1)}{\theta^2}$$

$$y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$

$$\frac{dy}{d\theta} = e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right).(-e^{-\theta})$$

$$\frac{dy}{d\theta} = e^{-\theta}\left(1 + \frac{1}{\theta^2} - \theta + \frac{1}{\theta}\right) \Rightarrow e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)$$

$$= e^{-\theta}\frac{(-\theta^3 + \theta^2 + \theta + 1)}{\theta^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^2}\right)}{e^{\theta}\left(\frac{\theta^3 + \theta^2 + \theta - 1}{\theta^2}\right)}$$

$$= e^{-2\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1}\right)$$

Hence, $\dfrac{dy}{dx} = e^{-2\theta}\left(\dfrac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta-1}\right)$.

Q46. $x = 3\cos\theta - 2\cos^3\theta,\ y = 3\sin\theta - 2\sin^3\theta$.

Sol. Given that: $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$.

Differentiating both the parametric functions w.r.t. θ

$$\frac{dx}{d\theta} = -3\sin\theta - 6\cos^2\theta.\frac{d}{d\theta}(\cos\theta)$$
$$= -3\sin\theta - 6\cos^2\theta.(-\sin\theta)$$
$$= -3\sin\theta + 6\cos^2\theta.\sin\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta - 6\sin^2\theta.\frac{d}{d\theta}(\sin\theta)$$
$$= 3\cos\theta - 6\sin^2\theta.\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta.\sin\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos\theta(3-6\sin^2\theta)}{\sin\theta(-3+6\cos^2\theta)} = \frac{\cos\theta\left[3-6(1-\cos^2\theta)\right]}{\sin\theta\left[-3+6\cos^2\theta\right]}$$
$$= \cot\theta\left(\frac{3-6+6\cos^2\theta}{-3+6\cos^2\theta}\right) = \cot\theta\left(\frac{-3+6\cos^2\theta}{-3+6\cos^2\theta}\right)$$
$$= \cot\theta$$

Hence, $\dfrac{dy}{dx} = \cot\theta$.

Q47. $\sin x = \dfrac{2t}{1+t^2},\ \tan y = \dfrac{2t}{1-t^2}$

Sol. Given that $\sin x = \dfrac{2t}{1+t^2}$ and $\tan y = \dfrac{2t}{1-t^2}$

$\therefore$ Taking $\sin x = \dfrac{2t}{1+t^2}$

Differentiating both sides w.r.t t, we get

$$\cos x.\frac{dx}{dt} = \frac{(1+t^2).\frac{d}{dt}(2t) - 2t.\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\Rightarrow \quad \cos x.\frac{dx}{dt} = \frac{2(1+t^2)-2t.2t}{(1+t^2)^2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2+2t^2-4t^2}{(1+t^2)^2}\times\frac{1}{\cos x}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2-2t^2}{(1+t^2)^2} \times \frac{1}{\sqrt{1-\sin^2 x}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\sqrt{\frac{(1+t^2)^2-4t^2}{(1+t^2)^2}}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1+t^2}{\sqrt{1+t^4+2t^2-4t^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1\quad {}^2)^2} \times \frac{(1+t^2)}{\sqrt{1+t^4-2t^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)} \times \frac{1}{\sqrt{(1-t^2)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)} \times \frac{1}{(1-t^2)} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{2}{1+t^2}$$

Now taking, $\tan y = \frac{2}{1-t^2}$

Differentiating both sides w.r.t, t, we get

$$\frac{d}{dt}(\tan y) = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\,.\,\frac{d}{dt}(2t) - 2t\,.\,\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\,.\,2 - 2t\,.\,(-2t)}{(1-t^2)^2}$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2+2t^2}{(1-t^2)^2} \times \frac{1}{\sec^2 y}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{1+\tan^2 y}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{1+\left(\frac{2t}{1-t^2}\right)^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{1}{\frac{(1-t^2)^2+4t^2}{(1-t^2)^2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{1+t^2+2t^2+4t^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{1+t^4+2t^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{(1+t^2)^2} \Rightarrow \frac{dy}{dt} = \frac{2}{1+t^2}$$

$$\therefore \quad \frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}} = 1$$

Hence $\frac{dy}{dt} = 1$

Q48. $x = \dfrac{1+\log t}{t^2}$, $y = \dfrac{3+2\log t}{t}$.

Sol. Given that: $x = \dfrac{1+\log t}{t^2}$, $y = \dfrac{3+2\log t}{t}$.

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = \frac{t^2 . \frac{d}{dt}(1+\log t) - (1+\log t).\frac{d}{dt}(t^2)}{t^4}$$

$$= \frac{t^2 . \left(\frac{1}{t}\right) - (1+\log t).2t}{t^4} = \frac{t-(1+\log t).2t}{t^4}$$

$$= \frac{t[1-2-2\log t]}{t^4} = \frac{-(1+2\log t)}{t^3}$$

$$y = \frac{3+2\log t}{t}$$

$$\frac{dy}{dt} = \frac{t.\frac{d}{dt}(3+2\log t)-(3+2\log t).\frac{d}{dt}(t)}{t^2}$$

$$= \frac{t(2/t)-(3+2\log t).1}{t^2}$$

$$= \frac{2-3-2\log t}{t^2} = \frac{-(1+2\log t)}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-(1+2\log t)}{t^2}}{\frac{-(1+2\log t)}{t^3}} = \frac{t^3}{t^2} = t$$

Hence, $\frac{dy}{dx} = t.$

Q49. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = \frac{-y\log x}{x\log y}$.

Sol. Given that: $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

$\Rightarrow \cos 2t = \log x$ and $\sin 2t = \log y$.

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = e^{\cos 2t}.\frac{d}{dt}(\cos 2t) = e^{\cos 2t}(-\sin 2t).\frac{d}{dt}(2t)$$

$$= -e^{\cos 2t}.\sin 2t.2 = -2e^{\cos 2t}.\sin 2t$$

Now $\quad y = e^{\sin 2t}$

$$\frac{dy}{dt} = e^{\sin 2t}.\frac{d}{dt}(\sin 2t) = e^{\sin 2t}.\cos 2t.\frac{d}{dt}(2t)$$

$$= e^{\sin 2t}.\cos 2t.2 = 2e^{\sin 2t}.\cos 2t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t}.\cos 2t}{-2e^{\cos 2t}.\sin 2t} = \frac{e^{\sin 2t}.\cos 2t}{-e^{\cos 2t}.\sin 2t} = \frac{y\cos 2t}{-x\sin 2t}$$

$$= \frac{y\log x}{-x\log y} \qquad \left[\begin{array}{l}\because \cos 2t = \log x \\ \sin 2t = \log y\end{array}\right]$$

Hence, $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$.

Q50. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, show that $\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}$.

Sol. Given that: $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$.

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = a\left[\sin 2t.\frac{d}{dt}(1+\cos 2t) + (1+\cos 2t).\frac{d}{dt}\sin 2t\right]$$
$$= a[\sin 2t.(-\sin 2t).2 + (1+\cos 2t)(\cos 2t).2]$$
$$= a[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t]$$
$$= a[2(\cos^2 2t - \sin^2 2t) + 2\cos 2t]$$
$$= a[2\cos 4t + 2\cos 2t] \qquad [\because \cos 2x = \cos^2 x - \sin^2 x]$$
$$= 2a[\cos 4t + \cos 2t]$$
$$y = b\cos 2t\,(1-\cos 2t)$$
$$\frac{dy}{dt} = b\left[\cos 2t.\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t).\frac{d}{dt}(\cos 2t)\right]$$
$$= b[\cos 2t.\sin 2t.2 + (1-\cos 2t).(-\sin 2t).2]$$
$$= b[2\sin 2t.\cos 2t - 2\sin 2t + 2\sin 2t\cos 2t]$$
$$= b[\sin 4t - 2\sin 2t + \sin 4t] \quad [\because \sin 2x = 2\sin x\cos x]$$
$$= b[2\sin 4t - 2\sin 2t] = 2b(\sin 4t - \sin 2t)$$
$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b[\sin 4t - \sin 2t]}{2a[\cos 4t + \cos 2t]} = \frac{b}{a}\left[\frac{\sin 4t - \sin 2t}{\cos 4t + \cos 2t}\right]$$

Put $t = \frac{\pi}{4}$

$$\therefore \left(\frac{dy}{dx}\right)_{at\, t=\frac{\pi}{4}} = \frac{b}{a}\left[\frac{\sin 4\left(\frac{\pi}{4}\right) - \sin 2.\left(\frac{\pi}{4}\right)}{\cos 4\left(\frac{\pi}{4}\right) + \cos 2.\left(\frac{\pi}{4}\right)}\right] = \frac{b}{a}\left[\frac{\sin\pi - \sin\frac{\pi}{2}}{\cos\pi + \cos\frac{\pi}{2}}\right]$$
$$= \frac{b}{a}\left[\frac{0-1}{-1+0}\right] = \frac{b}{a}\left(\frac{-1}{-1}\right) = \frac{b}{a}$$

Hence, $\left(\frac{dy}{dx}\right)_{at\, t=\frac{\pi}{4}} = \frac{b}{a}$.

Q51. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

Sol. Given that: $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$.
Differentiating both parametric functions w.r.t. t

$$\frac{dx}{dt} = 3\cos t - \cos 3t.3 = 3(\cos t - \cos 3t)$$
$$\frac{dy}{dt} = -3\sin t + \sin 3t.3 = 3(-\sin t + \sin 3t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(-\sin t + \sin 3t)}{3(\cos t - \cos 3t)} = \frac{-\sin t + \sin 3t}{\cos t - \cos 3t}$$

Put $t = \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{-\sin\frac{\pi}{3} + \sin 3\left(\frac{\pi}{3}\right)}{\cos\frac{\pi}{3} - \cos 3\left(\frac{\pi}{3}\right)}$$

$$= \frac{-\frac{\sqrt{3}}{2} + \sin \pi}{\frac{1}{2} - \cos \pi} = \frac{-\frac{\sqrt{3}}{2} + 0}{\frac{1}{2} - (-1)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2} + 1} = \frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{-1}{\sqrt{3}}$$

Hence, $\frac{dy}{dx} = \frac{-1}{\sqrt{3}}$.

Q52. Differentiate $\frac{x}{\sin x}$ w.r.t. $\sin x$.

Sol. Let $y = \frac{x}{\sin x}$ and $z = \sin x$.

Differentiating both the parametric functions w.r.t. x,

$$\frac{dy}{dx} = \frac{\sin x . \frac{d}{dx}(x) - x . \frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{\sin x . 1 - x . \cos x}{\sin^2 x} = \frac{\sin x - x \cos x}{\sin^2 x}$$

$$\frac{dz}{dx} = \cos x$$

$$\therefore \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\frac{\sin x - x \cos x}{\sin^2 x}}{\cos x} = \frac{\sin x - x \cos x}{\sin^2 x \cos x}$$

$$= \frac{\sin x}{\sin^2 x \cos x} - \frac{x \cos x}{\sin^2 x \cos x}$$

$$= \frac{\tan x}{\sin^2 x} - \frac{x}{\sin^2 x} = \frac{\tan x - x}{\sin^2 x}$$

Hence, $\frac{dy}{dz} = \frac{\tan x - x}{\sin^2 x}$.

Q53. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$ w.r.t. $\tan^{-1} x$, when $x \neq 0$.

Sol. Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $z = \tan^{-1} x$.

Put $x = \tan\theta$.

$$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) \text{ and } z = \tan^{-1}(\tan\theta) = \theta.$$

$$\Rightarrow \quad \tan\left(\frac{\sqrt{\sec\ \theta}-1}{\tan}\right) = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right) = \tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\tan\frac{\theta}{2}\right) \Rightarrow y = \frac{\theta}{2}$$

Differentiating both parametric functions w.r.t. θ

$$\frac{dy}{d\theta} = \frac{1}{2}\cdot\frac{d}{d\theta}(\theta) \quad \text{and} \quad \frac{dz}{d\theta} = \frac{d}{d\theta}(\theta)$$

$$= \frac{1}{2}.1 = \frac{1}{2} \quad \text{and} \quad \frac{dz}{d\theta} = 1$$

$$\therefore \quad \frac{dy}{dz} = \frac{dy/d\theta}{dz/d\theta} = \frac{1/2}{1} = \frac{1}{2}.$$

Find $\dfrac{dy}{dx}$ when x *and* y are connected by the relation given in each of the Exercises 54 to 57:

Q54. $\sin xy + \dfrac{x}{y} = x^2 - y.$

Sol. Given that: $\sin xy + \dfrac{x}{y} = x^2 - y.$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\sin(xy) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}(x^2) - \frac{d}{dx}(y)$$

$$\Rightarrow \quad \cos xy.\frac{d}{dx}(xy) + \frac{y.\frac{d}{dx}x - x.\frac{dy}{dx}}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \cos xy\left[x.\frac{dy}{dx}+y.1\right]+\frac{y.1}{y^2}-\frac{x}{y^2}.\frac{dy}{dx} = 2x-\frac{dy}{dx}$$

$$\Rightarrow x\cos xy.\frac{dy}{dx}+y\cos xy+\frac{1}{y}-\frac{x}{y^2}\frac{dy}{dx} = 2x-\frac{dy}{dx}$$

$$\Rightarrow \quad x\cos xy.\frac{dy}{dx}-\frac{x}{y^2}.\frac{dy}{dx}+\frac{dy}{dx} = -y\cos xy-\frac{1}{y}+2x$$

$$\Rightarrow \quad \left[x\cos xy-\frac{x}{y^2}+1\right]\frac{dy}{dx} = 2x-y\cos xy-\frac{1}{y}$$

$$\Rightarrow \quad \frac{\left[xy^2\cos xy-x+y^2\right]}{y^2}\frac{dy}{dx} = \frac{2xy-y^2\cos xy-1}{y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy-y^2\cos xy-1}{y}\times\frac{y^2}{xy^2\cos xy-x+y^2}$$

$$= \frac{2xy^2-y^3\cos(xy)-y}{xy^2\cos(xy)-x+y^2}$$

Hence, $\dfrac{dy}{dx} = \dfrac{2xy^2-y^3\cos(xy)-y}{xy^2\cos(xy)-x+y^2}$.

Q55. $\sec(x+y)=xy$

Sol. Given that: $\sec(x+y)=xy$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\sec(x+y) = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad \sec(x+y)\tan(x+y).\frac{d}{dx}(x+y) = x.\frac{dy}{dx}+y.1$$

$$\Rightarrow \quad \sec(x+y).\tan(x+y)\left(1+\frac{dy}{dx}\right) = x.\frac{dy}{dx}+y$$

$$\Rightarrow \sec(x+y).\tan(x+y)+\sec(x+y).\tan(x+y).\frac{dy}{dx} = x.\frac{dy}{dx}+y$$

$$\Rightarrow \quad \sec(x+y).\tan(x+y).\frac{dy}{dx}-x\frac{dy}{dx} = y-\sec(x+y).\tan(x+y)$$

$$\Rightarrow \quad [\sec(x+y).\tan(x+y)-x]\frac{dy}{dx} = y-\sec(x+y).\tan(x+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-\sec(x+y).\tan(x+y)}{\sec(x+y).\tan(x+y)-x}$$

Hence, $\dfrac{dy}{dx} = \dfrac{y-\sec(x+y).\tan(x+y)}{\sec(x+y).\tan(x+y)-x}$.

Q56. $\tan^{-1}(x^2+y^2)=a$

Sol. Given that: $\tan^{-1}(x^2+y^2)=a$

$\Rightarrow \quad x^2+y^2 = \tan a.$

Differentiating both sides w.r.t. x.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(\tan a)$$

$$\Rightarrow \quad 2x+2y.\frac{dy}{dx} = 0 \Rightarrow 2y.\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Hence, $\dfrac{dy}{dx} = \dfrac{-x}{y}$.

Q57. $(x^2+y^2)^2=xy$

Sol. Given that: $(x^2+y^2)^2=xy$

$\Rightarrow \quad x^4+y^4+2x^2y^2 = xy$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(x^4)+\frac{d}{dx}(y^4)+2.\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(xy)$$

$$\Rightarrow 4x^3+4y^3.\frac{dy}{dx}+2\left[x^2.2y.\frac{dy}{dx}+y^2.2x\right] = x\frac{dy}{dx}+y.1$$

$$\Rightarrow \quad 4x^3+4y^3.\frac{dy}{dx}+4x^2y.\frac{dy}{dx}+4xy^2 = x\frac{dy}{dx}+y$$

$$\Rightarrow \quad 4y^3\frac{dy}{dx}+4x^2y\frac{dy}{dx}-x\frac{dy}{dx} = y-4x^3-4xy^2$$

$$\Rightarrow \quad (4y^3+4x^2y-x)\frac{dy}{dx} = y-4x^3-4xy^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-4x^3-4xy^2}{4y^3+4x^2y-x}$$

Hence, $\dfrac{dy}{dx} = \dfrac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$.

Q58. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then show that $\dfrac{dy}{dx}.\dfrac{dx}{dy}=1$.

Sol. Given that: $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(ax^2+2hxy+by^2+2gx+2fy+c) = \frac{d}{dx}(0)$$

$$\Rightarrow a.2x+2h\left(x.\frac{dy}{dx}+y.1\right)+b.2y.\frac{dy}{dx}+2g.1+2f.\frac{dy}{dx}+0=0$$

$$\Rightarrow 2ax+2hx.\frac{dy}{dx}+2hy+2by.\frac{dy}{dx}+2g+2f.\frac{dy}{dx}=0$$

$$\Rightarrow \quad 2hx.\frac{dy}{dx}+2by\frac{dy}{dx}+2f\frac{dy}{dx}=-2ax-2hy-2g$$

$$\Rightarrow \quad (2hx+2by+2f)\frac{dy}{dx}=-2(ax+hy+g)$$

$$\Rightarrow \quad 2(hx+by+f)\frac{dy}{dx}=-2(ax+hy+g)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-2(ax+hy+g)}{2(hx+by+f)}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-(ax+hy+g)}{(hx+by+f)}$$

Now, differentiating the given equation w.r.t. y.

$$\frac{d}{dy}(ax^2+2hxy+by^2+2gx+2fy+c)=\frac{d}{dy}(0)$$

$$\Rightarrow \quad 2ax.\frac{dx}{dy}+2h\left(y.\frac{dx}{dy}+x.1\right)+2by+2g.\frac{dx}{dy}+2f.1+0=0$$

$$\Rightarrow \quad 2ax.\frac{dx}{dy}+2hy.\frac{dx}{dy}+2hx+2by+2g.\frac{dx}{dy}+2f=0$$

$$\Rightarrow \quad 2ax\frac{dx}{dy}+2hy.\frac{dx}{dy}+2g.\frac{dx}{dy}=-2hx-2by-2f$$

$$\Rightarrow \quad (2ax+2hy+2g)\frac{dx}{dy}=-2hx-2by-2f$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{-2hx-2by-2f}{2ax+2hy+2g}$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{-2(hx+by+f)}{2(ax+hy+g)} \Rightarrow \frac{dx}{dy}=\frac{-(hx+by+f)}{(ax+hy+g)}$$

$$\therefore \quad \frac{dy}{dx}.\frac{dx}{dy}=\left[\frac{-(ax+hy+g)}{(hx+by+f)}\right]\left[\frac{-(hx+by+f)}{(ax+hy+g)}\right]=1$$

Hence, $\frac{dy}{dx}.\frac{dx}{dy}=1$. Hence, proved.

Q59. If $x = e^{x/y}$, prove that $\frac{dy}{dx} = \frac{x-y}{x \log x}$.

Sol. Given that: $x = e^{x/y}$

Taking log on both the sides,

$$\log x = \log e^{x/y}$$

$$\Rightarrow \quad \log x = \frac{x}{y}\log e \quad \Rightarrow \log x = \frac{x}{y} \quad [\because \log e = 1] \quad \ldots(i)$$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\log x = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad \frac{1}{x} = \frac{y.1 - x.\frac{dy}{dx}}{y^2}$$

$$\Rightarrow \quad y^2 = xy - x^2.\frac{dy}{dx} \quad \Rightarrow x^2.\frac{dy}{dx} = xy - y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(x-y)}{x^2} \quad \Rightarrow \frac{dy}{dx} = \frac{y}{x}.\left(\frac{x-y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\log x}.\left(\frac{x-y}{x}\right) \quad \left(\because \log x = \frac{x}{y} \text{ from eqn. } (i)\right)$$

Hence, $\frac{dy}{dx} = \frac{x-y}{x \log x}$.

Q60. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

Sol. Given that: $y^x = e^{y-x}$

Taking log on both sides $\log y^x = \log e^{y-x}$

$$\Rightarrow \quad x \log y = (y - x) \log e$$

$$\Rightarrow \quad x \log y = y - x \quad [\because \log e = 1]$$

$$\Rightarrow \quad x \log y + x = y$$

$$\Rightarrow \quad x(\log y + 1) = y$$

$$\Rightarrow \quad x = \frac{y}{\log y + 1}.$$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{y}{\log y + 1}\right)$$

$$= \frac{(\log y + 1).1 - y.\frac{d}{dy}(\log y + 1)}{(\log y + 1)^2}$$

$$= \frac{\log y + 1 - y.\frac{1}{y}}{(\log y + 1)^2} = \frac{\log y + 1 - 1}{(\log y + 1)^2} = \frac{\log y}{(\log y + 1)^2}$$

We know that

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\frac{\log y}{(\log y + 1)^2}} = \frac{(\log y + 1)^2}{\log y}$$

Hence, $\frac{dy}{dx} = \frac{(\log y + 1)^2}{\log y}$.

Q61. If $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$, show that $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$.

Sol. Given that $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$

$\Rightarrow \quad y = (\cos x)^y \quad \left[\because y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}\right]$

Taking log on both sides $\log y = y.\log(\cos x)$

Differentiating both sides w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = y.\frac{d}{dx}\log(\cos x) + \log(\cos x).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = y.\frac{1}{\cos x}.\frac{d}{dx}(\cos x) + \log(\cos x).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = y.\frac{1}{\cos x}.(-\sin x) + \log(\cos x).\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} - \log(\cos x)\frac{dy}{dx} = -y \tan x$$

$$\Rightarrow \left[\frac{1}{y} - \log(\cos x)\right]\frac{dy}{dx} = -y \tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y \tan x}{\frac{1}{y} - \log(\cos x)} = \frac{y^2 \tan x}{y \log \cos x - 1}$$

Hence, $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$. Hence, proved.

Q62. If $x \sin(a + y) + \sin a \cos(a + y) = 0$, prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$.

Sol. Given that: $x \sin(a + y) + \sin a \cos(a + y) = 0$

$\Rightarrow \quad x \sin(a+y) = -\sin a \cos(a+y)$

$\Rightarrow \quad x = \dfrac{-\sin a.\cos(a+y)}{\sin(a+y)} \Rightarrow x = -\sin a.\cot(a+y)$

Differentiating both sides w.r.t. y

$\Rightarrow \quad \dfrac{dx}{dy} = -\sin a.\dfrac{d}{dy}\cot(a+y)$

$\Rightarrow \quad \dfrac{dx}{dy} = -\sin a[-\operatorname{cosec}^2(a+y)]$

$\Rightarrow \quad \dfrac{dx}{dy} = \dfrac{\sin a}{\sin^2(a+y)}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{dx/dy} = \dfrac{1}{\dfrac{\sin a}{\sin^2(a+y)}}$

Hence, $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$. Hence proved.

Q63. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$.

Sol. Given that: $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Put $\quad x = \sin\theta$ and $y = \sin\phi$.

$\therefore \quad \theta = \sin^{-1} x$ and $\phi = \sin^{-1} y$

$\sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a(\sin\theta - \sin\phi)$

$\Rightarrow \quad \sqrt{\cos^2\theta} + \sqrt{\cos^2\phi} = a(\sin\theta - \sin\phi)$

$\Rightarrow \quad \cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$\Rightarrow \quad \dfrac{\cos\theta + \cos\phi}{\sin\theta - \sin\phi} = a \Rightarrow \dfrac{2\cos\dfrac{\theta+\phi}{2}.\cos\dfrac{\theta-\phi}{2}}{2\cos\dfrac{\theta+\phi}{2}.\sin\dfrac{\theta-\phi}{2}} = a$

$$\left[\begin{array}{l}\because \cos A + \cos B = 2\cos\dfrac{A+B}{2}.\cos\dfrac{A-B}{2} \\ \sin A - \sin B = 2\cos\dfrac{A+B}{2}.\sin\dfrac{A-B}{2}\end{array}\right]$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\sin\left(\frac{\theta-\phi}{2}\right)} = a \Rightarrow \cot\left(\frac{\theta-\phi}{2}\right) = a$$

$$\Rightarrow \quad \frac{\theta-\phi}{2} = \cot^{-1} a \Rightarrow \theta - \phi = 2\cot^{-1} a$$

$$\Rightarrow \quad \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(\sin^{-1} x) - \frac{d}{dx}(\sin^{-1} y) = 2.\frac{d}{dx}\cot^{-1} a$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-y^2}}.\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Hence, $$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.$$

Q64. If $y = \tan^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Sol. Given that: $y = \tan^{-1} x \Rightarrow x = \tan y$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

Again differentiating both sides w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cos^2 y)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos y.\frac{d}{dx}(\cos y)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos y(-\sin y).\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -2\sin y\cos y\,.\cos^2 y$$

$$\therefore \quad \frac{d^2y}{dx^2} = -2\sin y\cos^3 y$$

Verify the Rolle's Theorem for each of the functions in Exercises 65 to 69:

Q65. $f(x) = x(x-1)^2$ in $[0, 1]$

Sol. Given that: $f(x) = x(x-1)^2$ in $[0, 1]$

(*i*) $f(x) = x(x-1)^2$, being an algebraic polynomial, is continuous in $[0, 1]$.

(*ii*)
$$\begin{aligned} f'(x) &= x.2(x-1) + (x-1)^2.1 \\ &= 2x^2 - 2x + x^2 + 1 - 2x \\ &= 3x^2 - 4x + 1 \text{ which exists in } (0, 1) \end{aligned}$$

(*iii*)
$$f(x) = x(x-1)^2$$
$$f(0) = 0(0-1)^2 = 0;\ f(1) = 1(1-1)^2 = 0$$
$$\Rightarrow \quad f(0) = f(1) = 0$$

As the above conditions are satisfied, then there must exist at least one point $c \in (0, 1)$ such that $f'(c) = 0$

$\therefore \quad f'(c) = 3c^2 - 4c + 1 = 0 \Rightarrow 3c^2 - 3c - c + 1 = 0$

$\Rightarrow \quad 3c(c-1) - 1(c-1) = 0 \Rightarrow (c-1)(3c-1) = 0$

$\Rightarrow \quad c - 1 = 0 \Rightarrow c = 1$

$3c - 1 = 0 \Rightarrow 3c = 1 \quad \therefore c = \frac{1}{3} \in (0, 1)$

Hence, Rolle's Theorem is verified.

Q66. $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$.

Sol. Given that: $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$

(*i*) $f(x) = \sin^4 x + \cos^4 x$, being sine and cosine functions, $f(x)$ is continuous function in $\left[0, \frac{\pi}{2}\right]$.

(*ii*)
$$\begin{aligned} f'(x) &= 4\sin^3 x.\cos x + 4\cos^3 x(-\sin x) \\ &= 4\sin^3 x.\cos x - 4\cos^3 x.\sin x \end{aligned}$$

$= 4 \sin x \cos x (\sin^2 x - \cos^2 x)$

$= -4 \sin x \cos x (\cos^2 x - \sin^2 x)$

$= -2.2 \sin x \cos x . \cos 2x$ $\left[\begin{array}{l} \because \cos 2x = \cos^2 x - \sin^2 x \\ \sin 2x = 2 \sin x \cos x \end{array}\right]$

$= -2 \sin 2x . \cos 2x$

$= -\sin 4x$ which exists in $\left(0, \frac{\pi}{2}\right)$.

So, $f(x)$ is differentiable in $\left(0, \frac{\pi}{2}\right)$.

(*iii*) $f(0) = \sin^4(0) + \cos^4(0) = 1$

$f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^2\left(\frac{\pi}{2}\right) = 1$

$\therefore \quad f(0) = f\left(\frac{\pi}{2}\right) = 1$

As the above conditions are satisfied, there must exist at least one point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$

$\Rightarrow \quad -\sin 4c = 0$

$\Rightarrow \quad \sin 4c = 0 \quad \Rightarrow \quad \sin 4c = \sin 0$

$\Rightarrow \quad 4c = n\pi$

$\therefore \quad c = \frac{n\pi}{4}, n \in I$

For $n = 1$, $\quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Hence, the Rolle's Theorem is verified.

Q67. $f(x) = \log (x^2 + 2) - \log 3$ in $[-1, 1]$.

Sol. Given that: $f(x) = \log (x^2 + 2) - \log 3$ in $[-1, 1]$

(*i*) $f(x) = \log (x^2 + 2) - \log 3$, being a logarithm function, is continuous in $[-1, 1]$.

(*ii*) $f'(x) = \frac{1}{x^2 + 2} . 2x - 0 = \frac{2x}{x^2 + 2}$ which exists in $(-1, 1)$

So, $f(x)$ is differentiable in $(-1, 1)$.

(*iii*) $f(-1) = \log (1 + 2) - \log 3 \Rightarrow \log 3 - \log 3 = 0$

$f(1) = \log (1 + 2) - \log 3 \Rightarrow \log 3 - \log 3 = 0$

$\therefore f(-1) = f(1) = 0$

As the above conditions are satisfied, then there must exist atleast one point $c \in (-1, 1)$ such that $f'(c) = 0$.

$$\therefore \frac{2c}{c^2+2} = 0 \Rightarrow 2c = 0 \quad \therefore c = 0 \in (-1, 1)$$

Hence, Rolle's Theorem is verified.

Q68. $f(x) = x(x+3)\, e^{-x/2}$ in $[-3, 0]$.

Sol. Given that: $f(x) = x(x+3)\, e^{-x/2}$ in $[-3, 0]$

(*i*) Algebraic functions and exponential functions are continuous in their domains.

$\therefore f(x)$ is continuous in $[-3, 0]$

(*ii*)
$$f'(x) = x(x+3).\frac{d}{dx}e^{-x/2} + x.e^{-x/2}.\frac{d}{dx}(x+3) + (x+3).e^{-x/2}\frac{d}{dx}.x$$

$$= x(x+3).e^{-x/2}.\left(-\frac{1}{2}\right) + x.e^{-x/2}.1 + (x+3).e^{-x/2}.1$$

$$= e^{-x/2}\left[\frac{-x(x+3)}{2} + x + x + 3\right]$$

$$= e^{-x/2}\left[\frac{-x(x+3)}{2} + 2x + 3\right] = e^{-x/2}\left[\frac{-x^2 - 3x + 4x + 6}{2}\right]$$

$$= e^{-x/2}\left[\frac{-x^2 + x + 6}{2}\right] \text{ which exists in } (-3, 0).$$

So, $f(x)$ is differentiable in $(-3, 0)$.

(*iii*) $f(-3) = (-3)(-3+3)\, e^{-3/2} = 0$

$f(0) = (0)(0+3)\, e^{-0/2} = 0$

$\therefore f(-3) = f(0) = 0$

As the above conditions are satisfied, then there must exist atleast one point $c \in (-3, 0)$ such that

$$f'(c) = 0 \Rightarrow e^{-c/2}\left[\frac{-c^2 + c + 6}{2}\right] = 0$$

$$\Rightarrow -\frac{e^{-c/2}}{2}[c^2 - c - 6] = 0$$

$$\Rightarrow -\frac{e^{-c/2}}{2}(c-3)(c+2) = 0$$

$$\Rightarrow e^{-c/2} \neq 0 \quad \therefore (c-3)(c+2) = 0$$

Which gives $c = 3, c = -2 \in (-3, 0)$.

Hence, Rolle's Theorem is verified.

Q69. $f(x) = \sqrt{4-x^2}$ in $[-2, 2]$.

Sol. Given that: $f(x) = \sqrt{4-x^2}$ in $[-2, 2]$

(*i*) Since algebraic polynomials are continuous,

$\therefore f(x)$ is continuous in $[-2, 2]$

(*ii*) $f'(x) = \dfrac{d}{dx}\sqrt{4-x^2} = \dfrac{1}{2\sqrt{4-x^2}} \times -2x = \dfrac{-x}{\sqrt{4-x^2}}$ which exists in $(-2, 2)$

So, $f'(x)$ is differentiable in $(-2, 2)$.

(*iii*)
$$f(-2) = \sqrt{4-(-2)^2} = \sqrt{4-4} = 0$$
$$f(2) = \sqrt{4-(2)^2} = \sqrt{4-4} = 0$$

So $f(-2) = f(2) = 0$

As the above conditions are satisfied, then there must exist atleast one point $c \in (-2, 2)$ such that

$$f'(c) = 0 \Rightarrow \frac{-c}{\sqrt{4-c^2}} = 0 \Rightarrow c = 0 \in (-2, 2)$$

Hence, Rolle's Theorem is verified.

Q70. Discuss the applicability of Rolle's Theorem on the function given by

$$f(x) = \begin{cases} x^2+1, \text{ if } 0 \le x \le 1 \\ 3-x, \text{ if } 1 \le x \le 2 \end{cases}$$

Sol. (*i*) $f(x)$ being an algebraic polynomial, is continuous everywhere.

(*ii*) $f(x)$ must be differentiable at $x = 1$

$$\text{L.H.L.} = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1}$$
$$= \lim_{x \to 1} \frac{(x^2+1)-(1+1)}{x-1}$$
$$= \lim_{x \to 1} \frac{x^2+1-2}{x-1} = \lim_{x \to 1} \frac{x^2-1}{x-1}$$
$$= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1}(x+1) = (1+1) = 2$$

and $$\text{R.H.L.} = \lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x \to 1} \frac{(3-x)-(1+1)}{x-1}$$

$$= \lim_{x \to 1} \frac{(3-x)-2}{x-1} = \lim_{x \to 1} \frac{1-x}{x-1} = -1$$

$\therefore$ L.H.L. $\neq$ R.H.L.

So, $f(x)$ is not differentiable at $x = 1$.

Hence, Rolle's Theorem is not applicable in [0, 2].

Q71. Find the points on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$, where the tangent is parallel to x-axis.

Sol. Given that: $y = \cos x - 1$ on $[0, 2\pi]$

We have to find a point c on the given curve $y = \cos x - 1$ on $[0, 2\pi]$ such that the tangent at $c \in [0, 2\pi]$ is parallel to x-axis i.e., $f'(c) = 0$ where $f'(c)$ is the slope of the tangent.

So, we have to verify the Rolle's Theorem.

(i) $y = \cos x - 1$ is the combination of cosine and constant functions. So, it is continuous on $[0, 2\pi]$.

(ii) $\frac{dy}{dx} = -\sin x$ which exists in $(0, 2\pi)$.

So, it is differentiable on $(0, 2\pi)$.

(iii) Let $f(x) = \cos x - 1$

$f(0) = \cos 0 - 1 = 1 - 1 = 0; f(2\pi) = \cos 2\pi - 1 = 1 - 1 = 0$

$\therefore \quad f(0) = f(2\pi) = 0$

As the above conditions are satisfied, then there lies a point $c \in (0, 2\pi)$ such that $f'(c) = 0$.

$\therefore -\sin c = 0 \quad \Rightarrow \quad \sin c = 0$

$\therefore c = n\pi, n \in I$

$\Rightarrow c = \pi \in (0, 2\pi)$

Hence, $c = \pi$ is the point on the curve in $(0, 2\pi)$ at which the tangent is parallel to x-axis.

Q72. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0, 4]$, where the tangent is parallel to x-axis.

Sol. Given that: $y = x(x-4), x \in [0, 4]$

Let $\quad f(x) = x(x-4), x \in [0, 4]$

(i) $f(x)$ being an algebraic polynomial, is continuous function everywhere.

So, $\quad f(x) = x(x-4)$ is continuous in [0, 4].

(ii) $\quad f'(x) = 2x - 4$ which exists in (0, 4).

So, $f(x)$ is differentiable.

(*iii*) $f(0) = 0(0-4) = 0$

$f(4) = 4(4-4) = 0$

So $f(0) = f(4) = 0$

As the above conditions are satisfied, then there must exist at least one point $c \in (0, 4)$ such that $f'(c) = 0$

$\therefore 2c - 4 = 0 \Rightarrow c = 2 \in (0, 4)$

Hence, $c = 2$ is the point in $(0, 4)$ on the given curve at which the tangent is parallel to the x-axis.

Verify mean value theorem for each of the functions given in Exercises 73 to 76.

Statement of Mean Value Theorem:

Let $f(x)$ be a real valued function defined on $[a, b]$ such that if

(*i*) $f(x)$ is continuous on $[a, b]$

(*ii*) $f(x)$ is differentiable on (a, b)

Then there is some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Q73. $f(x) = \dfrac{1}{4x-1}$ in $[1, 4]$.

Sol. Given that: $f(x) = \dfrac{1}{4x-1}$ in $[1, 4]$.

(*i*) $f(x)$ is an algebraic function, so it is continuous in $[1, 4]$.

(*ii*) $f'(x) = \dfrac{-4}{(4x-1)^2}$ which exists in $(1, 4)$.

So, $f(x)$ is differentiable.

As the above conditions are satisfied then there must exist a point $c \in (1, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{-4}{(4c-1)^2} = \frac{\frac{1}{4(4)-1} - \frac{1}{4(1)-1}}{4-1}$$

$$\Rightarrow \frac{-4}{(4c-1)^2} = \frac{\frac{1}{15} - \frac{1}{3}}{3} = \frac{1-5}{15 \times 3} = \frac{-4}{45} = \frac{1}{(4c-1)^2} = \frac{1}{45}$$

$$\Rightarrow (4c-1)^2 = 45$$

$$\Rightarrow 4c - 1 = \pm 3\sqrt{5} \Rightarrow 4c = +1 \pm 3\sqrt{5}$$

$$\Rightarrow c = \frac{+1 \pm 3\sqrt{5}}{4}$$

$$\therefore \qquad c = \frac{+1 \pm 3\sqrt{5}}{4} \in (1, 4)$$

Hence, Mean Value Theorem is verified.

Q74. $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$.

Sol. Given that: $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$

(*i*) Being an algebraic polynomial, $f(x)$ is continuous in $[0, 1]$

(*ii*) $f'(x) = 3x^2 - 4x - 1$ which exists in $(0, 1)$.

So, $f(x)$ is differentiable.

As the above conditions are satisfied, then there must exist atleast one point $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \qquad 3c^2 - 4c - 1 = \frac{[(1)^3 - 2(1)^2 - (1) + 3] - [0 - 0 - 0 + 3]}{1 - 0}$$

$$\Rightarrow \qquad 3c^2 - 4c - 1 = \frac{(1 - 2 - 1 + 3) - (3)}{1}$$

$$\Rightarrow \qquad 3c^2 - 4c - 1 = 1 - 3 \Rightarrow 3c^2 - 4c - 1 = -2$$

$$\Rightarrow \qquad 3c^2 - 4c + 1 = 0 \Rightarrow 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow 3c\,(c - 1) - 1\,(c - 1) = 0 \Rightarrow (c - 1)\,(3c - 1) = 0$$

$$\Rightarrow \qquad c - 1 = 0 \quad \therefore c = 1$$

$$3c - 1 = 0 \quad \therefore c = \frac{1}{3} \in (0, 1)$$

Hence, Mean Value Theorem is verified.

Q75. $f(x) = \sin x - \sin 2x$ in $[0, \pi]$.

Sol. Given that: $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

(*i*) Since trigonometric functions are always continuous on their domain.

So, $f(x)$ is continuous on $[0, \pi]$.

(*ii*) $f'(x) = \cos x - 2\cos 2x$ which exists in $(0, \pi)$

So, $f(x)$ is differentiable on $(0, \pi)$.

Since the above conditions are satisfied, then there must exist atleast one point $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\cos c - 2\cos 2c = \frac{(\sin \pi - \sin 2\pi) - (\sin 0 - \sin 0)}{\pi - 0}$$

$$\Rightarrow \quad \cos c - 2(2\cos^2 c - 1) = 0 \Rightarrow \cos c - 4\cos^2 c + 2 = 0$$

$$\Rightarrow \qquad 4\cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \quad \cos c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 4 \times -2}}{2 \times 4}$$

$$\Rightarrow \quad \cos c = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow \quad c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi).$$

Hence, Mean Value Theorem is verified.

Q76. $f(x) = \sqrt{25 - x^2}$ in [1, 5].

Sol. Given that: $f(x) = \sqrt{25 - x^2}$ in [1, 5]

(i) $f(x)$ is continuous if $25 - x^2 \geq 0 \Rightarrow -x^2 \geq -25$

$\Rightarrow x^2 \leq 25 \Rightarrow x \leq \pm 5 \Rightarrow -5 \leq x \leq 5$

So, $f(x)$ is continuous on [1, 5].

(ii) $f'(x) = \dfrac{1}{2\sqrt{25 - x^2}} \times (-2x) = \dfrac{-x}{\sqrt{25 - x^2}}$ which exists in (1, 5).

So, $f(x)$ is differentiable in [1, 5].

Since the above conditions are satisfied then there must exist atleast one point $c \in (1, 5)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{-c}{\sqrt{25 - c^2}} = \frac{\sqrt{25 - 25} - \sqrt{25 - 1}}{5 - 1}$$

$$\Rightarrow \quad \frac{-c}{\sqrt{25 - c^2}} = \frac{0 - \sqrt{24}}{4}$$

$$\Rightarrow \quad \frac{c}{\sqrt{25 - c^2}} = \frac{2\sqrt{6}}{4} \quad \Rightarrow \quad \frac{c}{\sqrt{25 - c^2}} = \frac{\sqrt{6}}{2}$$

Squaring both sides

$$\frac{c^2}{25 - c^2} = \frac{6}{4} = \frac{3}{2}$$

$$\Rightarrow \quad 2c^2 = 75 - 3c^2 \Rightarrow 5c^2 = 75 \Rightarrow c^2 = 15$$

$$\therefore \quad c = \pm\sqrt{15} \in (1, 5)$$

Hence, Mean Value Theorem is verified.

Q77. Find a point on the curve $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Sol. Given that: $y = (x-3)^2$

Let $\quad f(x) = (x-3)^2$

(*i*) Being an algebraic polynomial, $f(x)$ is continuous at $x_1 = 3$ and $x_2 = 4$ i.e. in [3, 4].

(*ii*) $f'(x) = 2(x-3)$ which exists in (3, 4).

Hence, by mean value theorem, there must exist a point c on the curve at which the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

$$\therefore \quad f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{where } b = 4 \text{ and } a = 3$$

$$\Rightarrow \quad 2(c-3) = \frac{(4-3)^2-(3-3)^2}{4-3}$$

$$\Rightarrow \quad 2c-6 = \frac{1-0}{1} = 1 \quad \Rightarrow 2c = 6+1 = 7$$

$$\therefore \quad c = \frac{7}{2}$$

If $x = \frac{7}{2}$ $\quad \therefore y = \left(\frac{7}{2}-3\right)^2 = \frac{1}{4}$.

Hence, $\left(\frac{7}{2}, \frac{1}{4}\right)$ is the point on the curve at which the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Q78. Using Mean Value Theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points A(1, 0) and B(2, 1), where tangent is parallel to the chord AB. Also, find that point.

Sol. Given that: $y = 2x^2 - 5x + 3$

Let $\quad f(x) = 2x^2 - 5x + 3$

(*i*) Being an algebraic polynomial, $f(x)$ is continuous in [1, 2].

(*ii*) $f'(x) = 4x - 5$ which exists in (1, 2).

As per the Mean Value Theorem, there must exist a point $c \in (1, 2)$ on the curve at which the tangent is parallel to the chord joining the points A(1, 0) and B(2, 1).

$$\text{So} \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$4c-5 = \frac{(8-10+3)-(2-5+3)}{2-1}$$

$$\Rightarrow \quad 4c-5 = \frac{1-0}{1} = 1 \Rightarrow 4c = 1+5 \quad \Rightarrow \quad 4c = 6$$

$$\therefore \qquad c = \frac{6}{4} = \frac{3}{2}$$

$$\therefore \qquad y = 2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3$$

$$= 2 \times \frac{9}{4} - \frac{15}{2} + 3 = \frac{9}{2} - \frac{15}{2} + 3 = \frac{9 - 15 + 6}{2} = 0$$

Hence, $\left(\frac{3}{2}, 0\right)$ is the point on the curve at which the tangent is parallel to the chord joining the points A(1, 0) and B(2, 1).

LONG ANSWER TYPE QUESTIONS

Q79. Find the values of p and q so that

$$f(x) = \begin{cases} x^2 + 3x + p, \text{ if } x \le 1 \\ qx + 2, \text{ if } x > 1 \end{cases}$$ is differentiable at $x = 1$.

Sol. Given that:

$$f(x) = \begin{cases} x^2 + 3x + p, \text{ if } x \le 1 \\ qx + 2, \text{ if } x > 1 \end{cases} \text{ at } x = 1.$$

L.H.L. $f'(c) = \lim_{x \to 1^-} \frac{f(x) - f(c)}{x - c}$

$$\Rightarrow \quad f'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{(x^2 + 3x + p) - (1 + 3 + p)}{x - 1}$$

$$= \lim_{h \to 0} \frac{[(1-h)^2 + 3(1-h) + p] - [4 + p]}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{[1 + h^2 - 2h + 3 - 3h + p] - [4 + p]}{-h}$$

$$= \lim_{h \to 0} \frac{[h^2 - 5h + 4 + p] - [4 + p]}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 - 5h + 4 + p - 4 - p}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 - 5h}{-h} = \lim_{h \to 0} \frac{h[h - 5]}{-h} = 5$$

R.H.L. $f'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$

$$= \lim_{x \to 1^+} \frac{(qx+2)-(1+3+p)}{x-1}$$

$$= \lim_{h \to 0} \frac{[q(1+h)+2]-[4+p]}{1+h-1}$$

$$= \lim_{h \to 0} \frac{q+qh+2-4-p}{h} = \lim_{h \to 0} \frac{qh+q-2-p}{h}$$

For existing the limit

$q-2-p=0 \Rightarrow q-p=2$ ...(*i*)

$\Rightarrow \lim_{h \to 0} \frac{qh-0}{h} = q$

If L.H.L. $f'(1)$ = R.H.L. $f'(1)$ then $q = 5$.

Now putting the value of q in eqn. (*i*)

$5-p=2 \Rightarrow p=3.$

Hence, value of p is 3 and that of q is 5.

Q80. If $x^m.y^n = (x+y)^{m+n}$, prove that

(*i*) $\frac{dy}{dx} = \frac{y}{x}$ (*ii*) $\frac{d^2y}{dx^2} = 0$.

Sol. (*i*) Given that: $x^m.y^n = (x+y)^{m+n}$

Taking log on both sides

$$\log x^m.y^n = \log (x+y)^{m+n} \quad [\because \log xy = \log x + \log y]$$

$$\Rightarrow \log x^m + \log y^n = (m+n) \log (x+y)$$

$$\Rightarrow m \log x + n \log y = (m+n) \log (x+y)$$

Differentiating both sides w.r.t. x

$$\Rightarrow m.\frac{d}{dx}\log x + n.\frac{d}{dx}\log y = (m+n)\frac{d}{dx}\log (x+y)$$

$$\Rightarrow m.\frac{1}{x} + n.\frac{1}{y}.\frac{dy}{dx} = (m+n).\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{m}{x} + \frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{x+y}.\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{m}{x} + \frac{n}{y}.\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}.\frac{dy}{dx}$$

$$\Rightarrow \frac{n}{y}.\frac{dy}{dx} - \frac{m+n}{x+y}.\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \left(\frac{nx+ny-my-ny}{y(x+y)}\right)\frac{dy}{dx} = \left(\frac{mx+nx-mx-my}{x(x+y)}\right)$$

$$\Rightarrow \quad \left(\frac{nx-my}{y(x+y)}\right)\frac{dy}{dx} = \left(\frac{nx-my}{x(x+y)}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{nx-my}{x(x+y)}\times\frac{y(x+y)}{nx-my}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$ Hence proved.

(*ii*) Given that: $\frac{dy}{dx}=\frac{y}{x}$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{x.\frac{dy}{dx} \quad y.1}{} = \frac{x.\frac{y}{x}-y}{x^2} \qquad \left[\because \frac{dy}{dx}=\frac{y}{x}\right]$$

$$= \frac{y-y}{x^2}=\frac{0}{x^2}=0$$

Hence, $\frac{d^2y}{dx^2}$ = 0. Hence, proved.

Q81. If $x = \sin t$ and $y = \sin pt$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x.\frac{dy}{dx}+p^2y=0.$$

Sol. Given that: $x = \sin t$ and $y = \sin pt$

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = \cos t \quad \text{and} \quad \frac{dy}{dt} = \cos pt\,.\,p = p\cos pt$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}=\frac{p\cos pt}{\cos t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{p\cos pt}{\cos t}$$

Again differentiating w.r.t. x,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = p.\frac{d}{dx}\left(\frac{\cos pt}{\cos t}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = p\left[\frac{\cos t.\frac{d}{dx}(\cos pt) - \cos pt.\frac{d}{dx}(\cos t)}{\cos^2 t}\right]$$

$$= p\left[\frac{\cos t(-\sin pt).p\frac{dt}{dx} - \cos pt(-\sin t).\frac{dt}{dx}}{\cos^2 t}\right]$$

$$= p\left[\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^2 t}\right]\frac{dt}{dx}$$

$$= p\left[\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^2 t}\right].\frac{1}{\cos t}$$

$$= p\left[\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^3 t}\right]$$

Now we have to prove that

$$(1-x^2).\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

$$\text{L.H.S.} = (1-x^2)\left[p\left(\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^3 t}\right)\right] - x.p\frac{\cos pt}{\cos t} + p^2y$$

$$\Rightarrow (1-\sin^2 t)\left[p\left(\frac{-p\cos t\sin pt + \cos pt\sin t}{\cos^3 t}\right)\right] - \frac{p\sin t.\cos pt}{\cos t} + p^2.\sin pt$$

$$\Rightarrow \cos^2 t\left[\frac{-p^2\cos t\sin pt + p\cos pt\sin t}{\cos^3 t}\right] - \frac{p\sin t.\cos pt}{\cos t} + p^2.\sin pt$$

$$\Rightarrow \frac{-p^2\cos t\sin pt + p\cos pt\sin t}{\cos t} - \frac{p\sin t\cos pt}{\cos t} + p^2\sin pt$$

$$\Rightarrow \frac{-p^2\cos t\sin pt + p\cos pt\sin t - p\sin t\cos pt + p^2\sin pt\cos t}{\cos t}$$

$$\Rightarrow \frac{0}{\cos t} = 0 = \text{R.H.S.}$$

Hence, proved.

Q82. Find $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$.

Sol. Given that: $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$

Let $u = x^{\tan x}$ and $v = \sqrt{\frac{x^2+1}{2}}$

$\therefore \quad y = u + v$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now taking $u = x^{\tan x}$

Taking log on both sides $\log u = \log (x^{\tan x})$

$$\log u = \tan x \,.\, \log x$$

Differentiating both sides w.r.t. x

$$\frac{1}{u}.\frac{du}{dx} = \frac{d}{dx}(\tan x . \log x)$$

$$\Rightarrow \quad \frac{1}{u}.\frac{du}{dx} = \tan x.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}(\tan x)$$

$$\Rightarrow \quad \frac{1}{u}.\frac{du}{dx} = \tan x.\frac{1}{x} + \log x.\sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{\tan x}{x} + \log x.\sec^2 x\right]$$

$$\therefore \quad \frac{du}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \log x \sec^2 x\right]$$

Taking $\quad v = \sqrt{\frac{x^2+1}{2}} \quad \Rightarrow v = \frac{1}{\sqrt{2}}\sqrt{x^2+1}$

Differentiating both sides w.r.t. x

$$\frac{dv}{dx} = \frac{1}{\sqrt{2}}.\frac{1}{2\sqrt{x^2+1}}.2x = \frac{x}{\sqrt{2}\sqrt{x^2+1}}$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in eqn. (i)

$$\frac{dy}{dx} = x^{\tan x}\left[\log x \sec^2 x + \frac{\tan x}{x}\right] + \frac{x}{\sqrt{2}\sqrt{x^2+1}}$$

OBJECTIVE TYPE QUESTIONS

Choose the correct answers from the given four options in each of the Exercises 83 to 96.

Q83. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function

(*a*) $f(x) + g(x)$ (*b*) $f(x) - g(x)$ (*c*) $f(x).g(x)$ (*d*) $\frac{g(x)}{f(x)}$

Sol. We know that the algebraic polynomials are continuous functions everywhere.

$\therefore f(x) + g(x)$ is continuous [$\because$ Sum, difference and product of two continuous functions is also continuous]

$f(x) - g(x)$ is continuous

$f(x) \cdot g(x)$ is continuous

$\frac{g(x)}{f(x)}$ is only continuous if $g(x) \neq 0$

$$\therefore \boxed{\frac{f(x)}{g(x)} = \frac{2x}{\frac{x^2}{2}+1} = \frac{4x}{x^2+2}}$$

Here, $\frac{g(x)}{f(x)} = \frac{\frac{x^2}{2}+1}{2x} = \frac{x^2+2}{4x}$ which is discontinuous at $x = 0$.

Hence, the correct option is (*d*).

Q84. The function $f(x) = \frac{4-x^2}{4x-x^3}$ is

(*a*) discontinuous at only one point

(*b*) discontinuous at exactly two points

(*c*) discontinuous at exactly three points

(*d*) none of these

Sol. Given that: $f(x) = \frac{4-x^2}{4x-x^3}$

For discontinuous function

$$4x - x^3 = 0$$

$$\Rightarrow \quad x(4-x^2) = 0$$

$$\Rightarrow \quad x(2-x)(2+x) = 0$$

$$\Rightarrow \quad x = 0, x = -2, x = 2$$

Hence, the given function is discontinuous exactly at three points. Hence, the correct option is (*c*).

Q85. The set of points where the function f given by $f(x) = |2x-1| \sin x$ is differentiable is (*a*) R (*b*) $R - \left\{\frac{1}{2}\right\}$ (*c*) $(0, \infty)$ (*d*) none of these

Sol. Given that: $f(x) = |2x - 1| \sin x$

Clearly, $f(x)$ is not differentiable at $x = \frac{1}{2}$

$$\text{R.H.L.} = f'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2}+h\right) - 1\right| \sin\left(\frac{1}{2}+h\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{|2h| \sin\left(\frac{1+2h}{2}\right)}{h} = 2 \sin\left(\frac{1}{2}\right)$$

$$\text{Also L.H.L.} = f'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2}-h\right) - f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2}-h\right) - 1\right| \left[-\sin\left(\frac{1}{2}-h\right)\right] - 0}{-h}$$

$$= \frac{|-2h| \left[-\sin\left(\frac{1}{2}-h\right)\right]}{-h} = -2 \sin\left(\frac{1}{2}\right)$$

$\therefore$ R.H.L. $= f'\left(\frac{1}{2}\right) \neq$ L.H.L. $f'\left(\frac{1}{2}\right)$

So, the given function $f(x)$ is not differentiable at $x = \frac{1}{2}$.

$\therefore f(x)$ is differentiable in $R - \left\{\frac{1}{2}\right\}$.

Hence, the correct option is (*b*).

Q86. The function $f(x) = \cot x$ is discontinuous on the set

(*a*) $\{x = n\pi; n \in Z\}$ (*b*) $\{x = 2n\pi; n \in Z\}$

(*c*) $\left\{x = (2n+1)\frac{\pi}{2}; n \in Z\right\}$ (*d*) $\left\{x = \frac{n\pi}{2}; n \in Z\right\}$

Sol. Given that: $f(x) = \cot x$

$$\Rightarrow \quad f(x) = \frac{\cos x}{\sin x}$$

We know that $\sin x = 0$ if $f(x)$ is discontinuous.

$\therefore$ If $\quad \sin x = 0$

$\therefore \quad x = n\pi, n \in n\pi.$

So, the given function $f(x)$ is discontinuous on the set $\{x = n\pi; n \in Z\}$.

Hence, the correct option is (*a*).

Q87. The function $f(x) = e^{|x|}$ is

(*a*) continuous everywhere but not differentiable at $x = 0$

(*b*) continuous and differentiable everywhere.

(*c*) Not continuous at $x = 0$ (*d*) None of these

Sol. Given that: $f(x) = e^{|x|}$

We know that modulus function is continuous but not differentiable in its domain.

Let $\quad g(x) = |x|$ and $t(x) = e^x$

$\therefore \quad f(x) = got(x) = g[t(x)] = e^{|x|}$

Since $g(x)$ and $t(x)$ both are continuous at $x = 0$ but $f(x)$ is not differentiable at $x = 0$.

Hence, the correct option is (*a*).

Q88. If $f(x) = x^2 \sin \frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(*a*) 0 (*b*) -1 (*c*) 1 (*d*) none of these

Sol. Given that: $f(x) = x^2 \sin \frac{1}{x}$ where $x \neq 0$.

So, the value of the function f at $x = 0$, so that $f(x)$ is continuous is 0.

Hence, the correct option is (*a*).

Q89. If $f(x) = \begin{cases} mx + 1, \text{ if } x \leq \frac{\pi}{2} \\ \sin x + n, \text{ if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then

(*a*) $m = 1, n = 0$ (*b*) $m = \frac{n\pi}{2} + 1$ (*c*) $n = \frac{m\pi}{2}$ (*d*) $m = n = \frac{\pi}{2}$

Sol. Given that: $f(x) = \begin{cases} mx + 1, \text{ if } x \leq \frac{\pi}{2} \\ \sin x + n, \text{ if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

$$\text{L.H.L.} = \lim_{x \to \frac{\pi}{2}^-} (mx + 1) = \lim_{h \to 0}\left[m\left(\frac{\pi}{2} - h\right) + 1\right] = \frac{m\pi}{2} + 1$$

$$\text{R.H.L.} = \lim_{x \to \frac{\pi}{2}^+} (\sin x + n) = \lim_{h \to 0}\left[\sin\left(\frac{\pi}{2} + h\right) + n\right]$$

$$= \lim_{h \to 0} \cos h + n = 1 + n$$

When $f(x)$ is continuous at $x = \frac{\pi}{2}$

$\therefore$ L.H.L. = R.H.L.

$$\frac{m\pi}{2} + 1 = 1 + n \Rightarrow n = \frac{m\pi}{2}$$

Hence, the correct option is (*c*).

Q90. Let $f(x) = |\sin x|$. Then

(*a*) f is everywhere differentiable.

(*b*) f is everywhere continuous but not differentiable at $x = n\pi, n \in Z$.

(*c*) f is everywhere continuous but not differentiable at

$x = (2n+1)\frac{\pi}{2}, n \in Z$.

(*d*) none of these

Sol. Given that: $f(x) = |\sin x|$

Let $g(x) = \sin x$ and $t(x) = |x|$

$\therefore$ $f(x) = tog(x) = t[g(x)] = t(\sin x) = |\sin x|$

where $g(x)$ and $t(x)$ both are continuous.

$\therefore f(x) = got(x)$ is continuous but $t(x)$ is not differentiable at $x = 0$.

So, $f(x)$ is not continuous at $\sin x = 0 \Rightarrow x = n\pi, n \in Z$.

Hence, the correct option is (*b*).

Q91. If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(*a*) $\frac{4x^3}{1-x^4}$ (*b*) $\frac{-4x}{1-x^4}$ (*c*) $\frac{1}{4-x^4}$ (*d*) $\frac{-4x^3}{1-x^4}$

Sol. Given that: $y = \log\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow y = \log(1-x^2) - \log(1+x^2)$ $\left[\because \log\frac{x}{y} = \log x - \log y\right]$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{1-x^2}\cdot\frac{d}{dx}(1-x^2) - \frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)$$

$$= \frac{-2x}{1-x^2} - \frac{2x}{1+x^2} = \frac{-2x-2x^3-2x+2x^3}{(1-x^2)(1+x^2)} = \frac{-4x}{1-x^4}$$

Hence, the correct option is (*b*).

Q92. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to

(*a*) $\frac{\cos x}{2y-1}$ (*b*) $\frac{\cos x}{1-2y}$ (*c*) $\frac{\sin x}{1-2y}$ (*d*) $\frac{\sin x}{2y-1}$

Sol. Given that: $y = \sqrt{\sin x + y}$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\sin x + y}}\cdot\frac{d}{dx}(\sin x + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sin x + y}}\cdot\left(\cos x + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}\cdot\left[\cos x + \frac{dy}{dx}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y} + \frac{1}{2y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{2y}\cdot\frac{dy}{dx} = \frac{\cos x}{2y}$$

$$\Rightarrow \left(1 - \frac{1}{2y}\right)\frac{dy}{dx} = \frac{\cos x}{2y} \Rightarrow \left(\frac{2y-1}{2y}\right)\frac{dy}{dx} = \frac{\cos x}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y}\times\frac{2y}{2y-1} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Hence, the correct option is (*a*).

Q93. The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1} x$ is

(*a*) 2 (*b*) $\frac{-1}{2\sqrt{1-x^2}}$ (*c*) $\frac{2}{x}$ (*d*) $1-x^2$

Sol. Let $y = \cos^{-1}(2x^2-1)$ and $t = \cos^{-1} x$

Differentiating both the functions w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}(2x^2-1) \text{ and } \frac{dt}{dx} = \frac{d}{dx}\cos^{-1} x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x^2-1)^2}} \cdot \frac{d}{dx}(2x^2-1) \text{ and } \frac{dt}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-1.4x}{\sqrt{1-(4x^4+1-4x^2)}} \text{ and } \frac{dt}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-4x}{\sqrt{1-4x^4-1+4x^2}} = \frac{-4x}{\sqrt{4x^2-4x^4}} = \frac{-4x}{2x\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

$$\text{Now } \frac{dy}{dt} = \frac{dy/dx}{dt/dx} = \frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-1}{\sqrt{1-x^2}}} = 2$$

Hence, the correct option is (*a*).

Q94. If $x = t^2$ and $y = t^3$, then $\frac{d^2y}{dx^2}$ is

(*a*) $\frac{3}{2}$ (*b*) $\frac{3}{4t}$ (*c*) $\frac{3}{2t}$ (*d*) $\frac{2}{3t}$

Sol. Given that $x = t^2$ and $y = t^3$

Differentiating both the parametric functions w.r.t. t

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 3t^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t \Rightarrow \frac{dy}{dx} = \frac{3}{2}t$$

Now differentiating again w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{3}{2} \cdot \frac{dt}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2} \cdot \frac{1}{2t} = \frac{3}{4t}$$

Hence, the correct option is (*b*).

Q95. The value of '*c*' in Rolle's Theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(*a*) 1 (*b*) −1 (*c*) $\frac{3}{2}$ (*d*) $\frac{1}{3}$

Sol. Given that: $f(x) = x^3 - 3x$ in $[0, \sqrt{3}]$

We know that if $f(x) = x^3 - 3x$ satisfies the conditions of Rolle's

Theorem in $[0, \sqrt{3}]$, then

$$f'(c) = 0$$

$$\Rightarrow \quad 3c^2 - 3 = 0 \Rightarrow 3c^2 = 3 \quad \Rightarrow c^2 = 1$$

$$\therefore \quad c = \pm 1 \quad \Rightarrow \quad 1 \in \left(0, \sqrt{3}\right)$$

Hence, the correct option is (*a*).

Q96. For the function $f(x) = x + \frac{1}{x}, x \in [1, 3]$, the value of '*c*' for mean value theorem is

(*a*) 1 (*b*) $\sqrt{3}$ (*c*) 2 (*d*) none of these

Sol. Given that: $f(x) = x + \frac{1}{x}, x \in [1, 3]$

We know that if $f(x) = x + \frac{1}{x}, x \in [1, 3]$ satisfies all the conditions of mean value theorem then

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ where } a = 1 \text{ and } b = 3$$

$$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\left(3 + \frac{1}{3}\right) - \left(1 + \frac{1}{1}\right)}{3 - 1}$$

$$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\frac{10}{3} - 2}{2} \quad \Rightarrow 1 - \frac{1}{c^2} = \frac{4}{6} = \frac{2}{3} \quad \Rightarrow -\frac{1}{c^2} = \frac{2}{3} - 1$$

$$\Rightarrow \quad -\frac{1}{c^2} = -\frac{1}{3} \quad \Rightarrow \frac{1}{c^2} = \frac{1}{3} \quad \Rightarrow c = \pm\sqrt{3}.$$

Here $c = \sqrt{3} \in (1, 3)$.

Hence, the correct option is (*b*).

Fill in the blanks in each of the Exercises 97 to 101

Q97. An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is ....

Sol. $|x| + |x - 1|$ is the function which is continuous everywhere but fails to be differentiable at $x = 0$ and $x = 1$.

We can have more such examples.

Q98. Derivative of x^2 w.r.t. x^3 is ....

Sol. Let $y = x^2$ and $t = x^3$

Differentiating both the parametric functions w.r.t. x

$$\frac{dy}{dx} = 2x \text{ and } \frac{dt}{dx} = 3x^2$$

$\therefore \quad \dfrac{dy}{dt} = \dfrac{dy/dx}{dt/dx} = \dfrac{2x}{3x^2} = \dfrac{2}{3x}$

So, the derivative of x^2 w.r.t. x^3 is $\dfrac{2}{3x}$

Q99. If $f(x) = |\cos x|$, then $f'\left(\dfrac{\pi}{4}\right) = ...$

Sol. Given that: $f(x) = |\cos x|$

$\Rightarrow \quad f(x) = \cos x$ if $x \in \left(0, \dfrac{\pi}{2}\right)$

Differentiating both sides w.r.t. x, we get $f'(x) = -\sin x$

at $x = \dfrac{\pi}{4}$, $f'\left(\dfrac{\pi}{4}\right) = -\sin\dfrac{\pi}{4} = -\dfrac{1}{\sqrt{2}}$

Q100. If $f(x) = |\cos x - \sin x|$, then $f'\left(\dfrac{\pi}{3}\right) = ...$

Sol. Given that: $f(x) = |\cos x - \sin x|$

We know that $\sin x > \cos x$ if $x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$\Rightarrow \cos x - \sin x < 0$

$\therefore \quad f(x) = -(\cos x - \sin x)$

$f'(x) = -(-\sin x - \cos x) \quad \Rightarrow f'(x) = (\sin x + \cos x)$

$\therefore \quad f'\left(\dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} + \cos\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2}$

Q101. For the curve $\sqrt{x} + \sqrt{y} = 1$, $\dfrac{dy}{dx}$ at $\left(\dfrac{1}{4}, \dfrac{1}{4}\right)$ is ....

Sol. Given that: $\sqrt{x} + \sqrt{y} = 1$

Differentiating both sides w.r.t. x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{y}}\frac{dy}{dx} = \frac{-1}{\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\therefore \quad \frac{dy}{dx} \text{ at } \left(\frac{1}{4}, \frac{1}{4}\right) = -\frac{\sqrt{\frac{1}{4}}}{\sqrt{\frac{1}{4}}} = -1$$

State True or False for the statements in each of the Exercises 102 to 106.

Q102. Rolle's Theorem is applicable for the function $f(x) = |x - 1|$ in $[0, 2]$.

Sol. False. Given that $f(x) = |x - 1|$ in $[0, 2]$
We know that modulus function is not differentiable. So, it is false.

Q103. If f is continuous on its domain D, then $|f|$ is also continuous on D.

Sol. True. We know that modulus function is continuous function on its domain. So, it is true.

Q104. The composition of two continuous functions is a continuous function.

Sol. True. We know that the sum and difference of two or more functions is always continuous. So, it is true.

Q105. Trigonometric and inverse trigonometric functions are differentiable in their respective domain.

Sol. True.

Q106. If $f.g$ is continuous at $x = a$, then f and g are separately continuous at $x = a$.

Sol. False. Let us take an example: $f(x) = \sin x$ and $g(x) = \cot x$

$\therefore f(x).g(x) = \sin x.\cot x = \sin x.\dfrac{\cos x}{\sin x} = \cos x$ which is continuous at $x = 0$ but $\cot x$ is not continuous at $x = 0$.

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