

## 6

Application of  
Derivatives

## 6.3 EXERCISE

## SHORT ANSWER TYPE QUESTIONS

**Q1.** A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.

**Sol.** Ball of salt is spherical

$\therefore$  Volume of ball,  $V = \frac{4}{3}\pi r^3$ , where  $r$  = radius of the ball

As per the question,  $\frac{dV}{dt} \propto S$ , where  $S$  = surface area of the ball

$$\Rightarrow \frac{d}{dt} \left( \frac{4}{3}\pi r^3 \right) \propto 4\pi r^2 \quad [\because S = 4\pi r^2]$$

$$\Rightarrow \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \propto 4\pi r^2$$

$$\Rightarrow 4\pi r^2 \cdot \frac{dr}{dt} = K \cdot 4\pi r^2 \quad (K = \text{Constant of proportionality})$$

$$\Rightarrow \frac{dr}{dt} = K \cdot \frac{4\pi r^2}{4\pi r^2}$$

$$\therefore \frac{dr}{dt} = K \cdot 1 = K$$

Hence, the radius of the ball is decreasing at constant rate.

**Q2.** If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

**Sol.** We know that:

Area of circle,  $A = \pi r^2$ , where  $r$  = radius of the circle.

and perimeter =  $2\pi r$

As per the question,

$$\frac{dA}{dt} = K, \text{ where } K = \text{constant}$$

$$\Rightarrow \frac{d}{dt}(\pi r^2) = K \Rightarrow \pi \cdot 2r \cdot \frac{dr}{dt} = K$$

$$\therefore \frac{dr}{dt} = \frac{K}{2\pi r} \quad \dots(1)$$

Now Perimeter  $c = 2\pi r$

Differentiating both sides w.r.t.,  $t$ , we get

$$\Rightarrow \frac{dc}{dt} = \frac{d}{dt}(2\pi r) \Rightarrow \frac{dc}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dc}{dt} = 2\pi \cdot \frac{K}{2\pi r} = \frac{K}{r} \quad [\text{From (1)}]$$

$$\Rightarrow \frac{dc}{dt} \propto \frac{1}{r}$$

Hence, the perimeter of the circle varies inversely as the radius of the circle.

- Q3.** A kite is moving horizontally at a height of 151.5 metres. If the speed of the kite is 10 m/s, how fast is the string being let out; when the kite is 250 m away from the boy who is flying the kite? The height of the boy is 1.5 m.

**Sol.** Given that height of the kite ( $h$ ) = 151.5 m

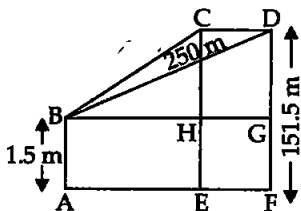
Speed of the kite ( $V$ ) = 10 m/s

Let  $FD$  be the height of the kite  
and  $AB$  be the height of the boy.

Let  $AF = x$  m

$$\therefore BG = AF = x \text{ m}$$

$$\text{and } \frac{dx}{dt} = 10 \text{ m/s}$$



From the figure, we get that

$$\begin{aligned} GD &= DF - GF \Rightarrow DF - AB \\ &= (151.5 - 1.5) \text{ m} = 150 \text{ m} \quad [\because AB = GF] \end{aligned}$$

Now in  $\triangle BGD$ ,

$$BG^2 + GD^2 = BD^2 \quad (\text{By Pythagoras Theorem})$$

$$\Rightarrow x^2 + (150)^2 = (250)^2$$

$$\Rightarrow x^2 + 22500 = 62500 \Rightarrow x^2 = 62500 - 22500$$

$$\Rightarrow x^2 = 40000 \Rightarrow x = 200 \text{ m}$$

Let initially the length of the string be  $y$  m

$\therefore$  In  $\triangle BGD$

$$BG^2 + GD^2 = BD^2 \Rightarrow x^2 + (150)^2 = y^2$$

Differentiating both sides w.r.t.,  $t$ , we get

$$\Rightarrow 2x \cdot \frac{dx}{dt} + 0 = 2y \cdot \frac{dy}{dt} \quad \left[ \because \frac{dx}{dt} = 10 \text{ m/s} \right]$$

$$\Rightarrow 2 \times 200 \times 10 = 2 \times 250 \times \frac{dy}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{2 \times 200 \times 10}{2 \times 250} = 8 \text{ m/s}$$

Hence, the rate of change of the length of the string is 8 m/s.

- Q4.** Two men A and B start with velocities  $V$  at the same time from the junction of two roads inclined at  $45^\circ$  to each other. If they travel by different roads, find the rate at which they are being separated.

**Sol.** Let  $P$  be any point at which the two roads are inclined at an angle of  $45^\circ$ .

Two men A and B are moving along the roads PA and PB respectively with the same speed ' $V$ '.

Let A and B be their final positions such that

$$AB = y$$

$\angle APB = 45^\circ$  and they move with the same speed.

$\therefore \triangle APB$  is an isosceles triangle. Draw  $PQ \perp AB$

$$AB = y \quad \therefore \quad AQ = \frac{y}{2} \quad \text{and} \quad PA = PB = x \quad (\text{let})$$

$$\angle APQ = \angle BPQ = \frac{45}{2} = 22\frac{1}{2}^\circ$$

[ $\because$  In an isosceles  $\Delta$ , the altitude drawn from the vertex, bisects the base]

Now in right  $\triangle APQ$ ,

$$\sin 22\frac{1}{2}^\circ = \frac{AQ}{AP}$$

$$\Rightarrow \sin 22\frac{1}{2}^\circ = \frac{\frac{y}{2}}{x} = \frac{y}{2x} \quad \Rightarrow y = 2x \cdot \sin 22\frac{1}{2}^\circ$$

Differentiating both sides w.r.t,  $t$ , we get

$$\begin{aligned} \frac{dy}{dt} &= 2 \cdot \frac{dx}{dt} \cdot \sin 22\frac{1}{2}^\circ \\ &= 2 \cdot V \cdot \frac{\sqrt{2-\sqrt{2}}}{2} \quad \left[ \because \sin 22\frac{1}{2}^\circ = \frac{\sqrt{2-\sqrt{2}}}{2} \right] \\ &= \sqrt{2-\sqrt{2}} \text{ V m/s} \end{aligned}$$

Hence, the rate of their separation is  $\sqrt{2-\sqrt{2}}$  V unit/s.

- Q5.** Find an angle  $\theta$ ,  $0 < \theta < \frac{\pi}{2}$ , which increases twice as fast as its sine.

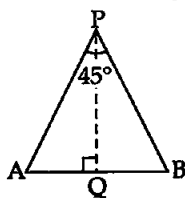
**Sol.** As per the given condition,

$$\frac{d\theta}{dt} = 2 \frac{d}{dt}(\sin \theta)$$

$$\Rightarrow \frac{d\theta}{dt} = 2 \cos \theta \cdot \frac{d\theta}{dt} \quad \Rightarrow 1 = 2 \cos \theta$$

$$\therefore \cos \theta = \frac{1}{2} \quad \Rightarrow \cos \theta = \cos \frac{\pi}{3} \quad \Rightarrow \theta = \frac{\pi}{3}$$

Hence, the required angle is  $\frac{\pi}{3}$ .



**Q6.** Find the approximate value of  $(1.999)^5$ .

**Sol.**  $(1.999)^5 = (2 - 0.001)^5$

Let  $x = 2$  and  $\Delta x = -0.001$

Let  $y = x^5$

Differentiating both sides w.r.t,  $x$ , we get

$$\frac{dy}{dx} = 5x^4 = 5(2)^4 = 80$$

$$\text{Now } \Delta y = \left(\frac{dy}{dx}\right) \cdot \Delta x = 80 \cdot (-0.001) = -0.080$$

$$\begin{aligned} \therefore (1.999)^5 &= y + \Delta y \\ &= x^5 - 0.080 = (2)^5 - 0.080 = 32 - 0.080 = 31.92 \end{aligned}$$

Hence, approximate value of  $(1.999)^5$  is 31.92.

**Q7.** Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm respectively.

**Sol.** Internal radius  $r = 3$  cm

and external radius  $R = r + \Delta r = 3.0005$  cm

$$\therefore \Delta r = 3.0005 - 3 = 0.0005 \text{ cm}$$

$$\text{Let } y = r^3 \Rightarrow y + \Delta y = (r + \Delta r)^3 = R^3 = (3.0005)^3 \quad \dots(i)$$

Differentiating both sides w.r.t,  $r$ , we get

$$\frac{dy}{dr} = 3r^2$$

$$\therefore \Delta y = \frac{dy}{dr} \times \Delta r = 3r^2 \times 0.0005$$

$$= 3 \times (3)^2 \times 0.0005 = 27 \times 0.0005 = 0.0135$$

$$\therefore (3.0005)^3 = y + \Delta y \quad [\text{From eq. (i)}]$$

$$= (3)^3 + 0.0135 = 27 + 0.0135 = 27.0135$$

$$\text{Volume of the shell} = \frac{4}{3}\pi[R^3 - r^3]$$

$$= \frac{4}{3}\pi[27.0135 - 27] = \frac{4}{3}\pi \times 0.0135$$

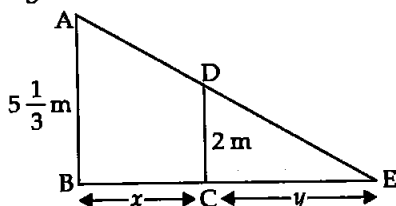
$$= 4\pi \times 0.005 = 4 \times 3.14 \times 0.0045 = 0.018\pi \text{ cm}^3$$

Hence, the approximate volume of the metal in the shell is  $0.018\pi \text{ cm}^3$ .

**Q8.** A man, 2m tall, walks at the rate of  $1\frac{2}{3}$  m/s towards a street light which is  $5\frac{1}{3}$  m above the ground. At what rate is the tip of his shadow moving? At what rate is the length of the shadow changing when he is  $3\frac{1}{3}$  m from the base of the light?

**Sol.** Let AB is the height of street light post and CD is the height of the man such that

$$AB = 5\frac{1}{3} = \frac{16}{3} \text{ m and } CD = 2 \text{ m}$$



Let  $BC = x$  length (the distance of the man from the lamp post) and  $CE = y$  is the length of the shadow of the man at any instant. From the figure, we see that

$$\triangle ABE \sim \triangle DCE \quad [\text{by AAA Similarity}]$$

$\therefore$  Taking ratio of their corresponding sides, we get

$$\begin{aligned} \frac{AB}{CD} &= \frac{BE}{CE} \Rightarrow \frac{AB}{CD} = \frac{BC + CE}{CE} \\ \Rightarrow \frac{16/3}{2} &= \frac{x + y}{y} \Rightarrow \frac{8}{3} = \frac{x + y}{y} \\ \Rightarrow 8y &= 3x + 3y \Rightarrow 8y - 3y = 3x \Rightarrow 5y = 3x \end{aligned}$$

Differentiating both sides w.r.t.  $t$ , we get

$$\begin{aligned} \frac{dy}{dt} &= 3 \cdot \frac{dx}{dt} \\ \Rightarrow \frac{dy}{dt} &= \frac{3}{5} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{3}{5} \cdot \left(-1 \frac{2}{3}\right) = \frac{3}{5} \cdot \left(\frac{-5}{3}\right) \\ &[\because \text{man is moving in opposite direction}] \\ &= -1 \text{ m/s} \end{aligned}$$

Hence, the length of shadow is decreasing at the rate of 1 m/s.

Now let  $u = x + y$

( $u$  = distance of the tip of shadow from the light post)

Differentiating both sides w.r.t.  $t$ , we get

$$\begin{aligned} \frac{du}{dt} &= \frac{dx}{dt} + \frac{dy}{dt} \\ &= \left(-1 \frac{2}{3} - 1\right) = -\left(\frac{5}{3} + 1\right) = -\frac{8}{3} = -2 \frac{2}{3} \text{ m/s} \end{aligned}$$

Hence, the tip of the shadow is moving at the rate of  $2\frac{2}{3}$  m/s towards the light post and the length of shadow decreasing at the rate of 1 m/s.

- Q9.** A swimming pool is to be drained for cleaning. If  $L$  represents the number of litres of water in the pool  $t$  seconds after the pool has been plugged off to drain and  $L = 200(10 - t)^2$ . How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?

**Sol.** Given that  $L = 200(10 - t)^2$

where  $L$  represents the number of litres of water in the pool.

Differentiating both sides w.r.t.  $t$ , we get

$$\frac{dL}{dt} = 200 \times 2(10 - t)(-1) = -400(10 - t)$$

But the rate at which the water is running out

$$= -\frac{dL}{dt} = 400(10 - t) \quad \dots(1)$$

Rate at which the water is running after 5 seconds

$$= 400 \times (10 - 5) = 2000 \text{ L/s (final rate)}$$

For initial rate put  $t = 0$

$$= 400(10 - 0) = 4000 \text{ L/s}$$

The average rate at which the water is running out

$$= \frac{\text{Initial rate} + \text{Final rate}}{2} = \frac{4000 + 2000}{2} = \frac{6000}{2} = 3000 \text{ L/s}$$

Hence, the required rate = 3000 L/s.

- Q10.** The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

**Sol.** Let  $x$  be the length of the cube

$$\therefore \text{Volume of the cube } V = x^3 \quad \dots(1)$$

Given that  $\frac{dV}{dt} = K$

Differentiating Eq. (1) w.r.t.  $t$ , we get

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} = K \text{ (constant)}$$

$$\therefore \frac{dx}{dt} = \frac{K}{3x^2}$$

Now surface area of the cube,  $S = 6x^2$

Differentiating both sides w.r.t.  $t$ , we get

$$\frac{ds}{dt} = 6 \cdot 2 \cdot x \cdot \frac{dx}{dt} = 12x \cdot \frac{K}{3x^2}$$

$$\Rightarrow \frac{ds}{dt} = \frac{4K}{x} \Rightarrow \frac{ds}{dt} \propto \frac{1}{x} \quad (4K = \text{constant})$$

Hence, the surface area of the cube varies inversely as the length of the side.

**Q11.**  $x$  and  $y$  are the sides of two squares such that  $y = x - x^2$ . Find the rate of change of the area of second square with respect to the area of first square.

**Sol.** Let area of the first square  $A_1 = x^2$   
and area of the second square  $A_2 = y^2$

Now  $A_1 = x^2$  and  $A_2 = y^2 = (x - x^2)^2$

Differentiating both  $A_1$  and  $A_2$  w.r.t.  $t$ , we get

$$\frac{dA_1}{dt} = 2x \cdot \frac{dx}{dt} \quad \text{and} \quad \frac{dA_2}{dt} = 2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}$$

$$\begin{aligned} \therefore \frac{dA_2}{dA_1} &= \frac{\frac{dA_2}{dt}}{\frac{dA_1}{dt}} = \frac{2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}}{2x \cdot \frac{dx}{dt}} \\ &= \frac{x(1 - x)(1 - 2x)}{x} = (1 - x)(1 - 2x) \\ &= 1 - 2x - x + 2x^2 = 2x^2 - 3x + 1 \end{aligned}$$

Hence, the rate of change of area of the second square with respect to first is  $2x^2 - 3x + 1$ .

**Q12.** Find the condition that the curves  $2x = y^2$  and  $2xy = k$  intersect orthogonally.

**Sol.** The two circles intersect orthogonally if the angle between the tangents drawn to the two circles at the point of their intersection is  $90^\circ$ .

Equation of the two circles are given as

$$2x = y^2 \quad \dots(i)$$

$$\text{and} \quad 2xy = k \quad \dots(ii)$$

Differentiating eq. (i) and (ii) w.r.t.  $x$ , we get

$$2.1 = 2y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{y} \Rightarrow m_1 = \frac{1}{y}$$

( $m_1$  = slope of the tangent)

$$\Rightarrow 2xy = k$$

$$\Rightarrow 2 \left[ x \cdot \frac{dy}{dx} + y \cdot 1 \right] = 0$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_2 = -\frac{y}{x}$$

( $m_2$  = slope of the other tangent)

If the two tangents are perpendicular to each other,

$$\text{then} \quad m_1 \times m_2 = -1$$

$$\Rightarrow \frac{1}{y} \times \left(-\frac{y}{x}\right) = -1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

$$\begin{array}{ll} \text{Now solving} & 2x = y^2 \quad \text{[From (i)]} \\ \text{and} & 2xy = k \quad \text{[From (ii)]} \end{array}$$

$$\text{From eq. (ii)} \quad y = \frac{k}{2x}$$

Putting the value of  $y$  in eq. (i)

$$2x = \left(\frac{k}{2x}\right)^2 \Rightarrow 2x = \frac{k^2}{4x^2}$$

$$\Rightarrow 8x^3 = k^2 \Rightarrow 8(1)^3 = k^2 \Rightarrow 8 = k^2$$

Hence, the required condition is  $k^2 = 8$ .

**Q13.** Prove that the curves  $xy = 4$  and  $x^2 + y^2 = 8$  touch each other.

$$\text{Sol. Given circles are } xy = 4 \quad \dots(i)$$

$$\text{and } x^2 + y^2 = 8 \quad \dots(ii)$$

Differentiating eq. (i) w.r.t.  $x$

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_1 = -\frac{y}{x} \quad \dots(iii)$$

where,  $m_1$  is the slope of the tangent to the curve.

Differentiating eq. (ii) w.r.t.  $x$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow m_2 = -\frac{x}{y}$$

where,  $m_2$  is the slope of the tangent to the circle.

To find the point of contact of the two circles

$$m_1 = m_2 \Rightarrow -\frac{y}{x} = -\frac{x}{y} \Rightarrow x^2 = y^2$$

Putting the value of  $y^2$  in eq. (ii)

$$x^2 + x^2 = 8 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4$$

$$\therefore x = \pm 2$$

$$\therefore x^2 = y^2 \Rightarrow y = \pm 2$$

$\therefore$  The point of contact of the two circles are  $(2, 2)$  and  $(-2, 2)$ .

**Q14.** Find the coordinates of the point on the curve  $\sqrt{x} + \sqrt{y} = 4$  at which tangent is equally inclined to the axes.

**Sol.** Equation of curve is given by  $\sqrt{x} + \sqrt{y} = 4$

Let  $(x_1, y_1)$  be the required point on the curve

$$\therefore \sqrt{x_1} + \sqrt{y_1} = 4$$

Differentiating both sides w.r.t.  $x_1$ , we get

$$\frac{d}{dx_1} \sqrt{x_1} + \frac{d}{dx_1} \sqrt{y_1} = \frac{d}{dx_1} (4)$$



$$\Rightarrow \frac{1}{2\sqrt{x_1}} + \frac{1}{2\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} = 0$$

$$\Rightarrow \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} = 0 \Rightarrow \frac{dy_1}{dx_1} = -\frac{\sqrt{y_1}}{\sqrt{x_1}} \quad \dots(i)$$

Since the tangent to the given curve at  $(x_1, y_1)$  is equally inclined to the axes.

$$\therefore \text{Slope of the tangent } \frac{dy_1}{dx_1} = \pm \tan \frac{\pi}{4} = \pm 1$$

So, from eq. (i) we get

$$-\frac{\sqrt{y_1}}{\sqrt{x_1}} = \pm 1$$

Squaring both sides, we get

$$\frac{y_1}{x_1} = 1 \Rightarrow y_1 = x_1$$

Putting the value of  $y_1$  in the given equation of the curve.

$$\sqrt{x_1} + \sqrt{y_1} = 4$$

$$\Rightarrow \sqrt{x_1} + \sqrt{x_1} = 4 \Rightarrow 2\sqrt{x_1} = 4 \Rightarrow \sqrt{x_1} = 2 \Rightarrow x_1 = 4$$

Since  $y_1 = x_1$

$\therefore y_1 = 4$

Hence, the required point is  $(4, 4)$ .

- Q15.** Find the angle of intersection of the curves  $y = 4 - x^2$  and  $y = x^2$ .

**Sol.** We know that the angle of intersection of two curves is equal to the angle between the tangents drawn to the curves at their point of intersection.

The given curves are  $y = 4 - x^2 \dots (i)$  and  $y = x^2 \dots (ii)$

Differentiating eq. (i) and (ii) with respect to  $x$ , we have

$$\frac{dy}{dx} = -2x \Rightarrow m_1 = -2x$$

$m_1$  is the slope of the tangent to the curve (i).

and  $\frac{dy}{dx} = 2x \Rightarrow m_2 = 2x$

$m_2$  is the slope of the tangent to the curve (ii).

So,  $m_1 = -2x$  and  $m_2 = 2x$

Now solving eq. (i) and (ii) we get

$$\Rightarrow 4 - x^2 = x^2 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

So,  $m_1 = -2x = -2\sqrt{2}$  and  $m_2 = 2x = 2\sqrt{2}$

Let  $\theta$  be the angle of intersection of two curves

$$\begin{aligned}\therefore \tan \theta &= \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \\ &= \left| \frac{2\sqrt{2} + 2\sqrt{2}}{1 - (2\sqrt{2})(2\sqrt{2})} \right| = \left| \frac{4\sqrt{2}}{1 - 8} \right| = \left| \frac{4\sqrt{2}}{-7} \right| = \frac{4\sqrt{2}}{7}\end{aligned}$$

$$\therefore \theta = \tan^{-1} \left( \frac{4\sqrt{2}}{7} \right)$$

Hence, the required angle is  $\tan^{-1} \left( \frac{4\sqrt{2}}{7} \right)$ .

**Q16.** Prove that the curves  $y^2 = 4x$  and  $x^2 + y^2 - 6x + 1 = 0$  touch each other at the point  $(1, 2)$ .

**Sol.** Given that the equation of the two curves are  $y^2 = 4x$  ... (i)  
and  $x^2 + y^2 - 6x + 1 = 0$  ... (ii)

$$\text{Differentiating (i) w.r.t. } x, \text{ we get } 2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

$$\text{Slope of the tangent at } (1, 2), m_1 = \frac{2}{2} = 1$$

$$\text{Differentiating (ii) w.r.t. } x \Rightarrow 2x + 2y \cdot \frac{dy}{dx} - 6 = 0$$

$$\Rightarrow 2y \cdot \frac{dy}{dx} = 6 - 2x \Rightarrow \frac{dy}{dx} = \frac{6 - 2x}{2y}$$

$\therefore$  Slope of the tangent at the same point  $(1, 2)$

$$\Rightarrow m_2 = \frac{6 - 2 \times 1}{2 \times 2} = \frac{4}{4} = 1$$

We see that  $m_1 = m_2 = 1$  at the point  $(1, 2)$ .

Hence, the given circles touch each other at the same point  $(1, 2)$ .

**Q17.** Find the equation of the normal lines to the curve  $3x^2 - y^2 = 8$  which are parallel to the line  $x + 3y = 4$ .

**Sol.** We have equation of the curve  $3x^2 - y^2 = 8$

Differentiating both sides w.r.t.  $x$ , we get

$$\Rightarrow 6x - 2y \cdot \frac{dy}{dx} = 0 \Rightarrow -2y \frac{dy}{dx} = -6x \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$$

$$\text{Slope of the tangent to the given curve} = \frac{3x}{y}$$

$$\therefore \text{Slope of the normal to the curve} = -\frac{1}{\frac{3x}{y}} = -\frac{y}{3x}$$

Now differentiating both sides the given line  $x + 3y = 4$

$$\Rightarrow 1 + 3 \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{3}$$

Since the normal to the curve is parallel to the given line  $x + 3y = 4$ .

$$\therefore -\frac{y}{3x} = -\frac{1}{3} \Rightarrow y = x$$

Putting the value of  $y$  in  $3x^2 - y^2 = 8$ , we get

$$3x^2 - x^2 = 8 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$\therefore y = \pm 2$$

$\therefore$  The points on the curve are  $(2, 2)$  and  $(-2, -2)$ .

Now equation of the normal to the curve at  $(2, 2)$  is

$$y - 2 = -\frac{1}{3}(x - 2)$$

$$\Rightarrow 3y - 6 = -x + 2 \Rightarrow x + 3y = 8$$

$$\text{at } (-2, -2) \quad y + 2 = -\frac{1}{3}(x + 2)$$

$$\Rightarrow 3y + 6 = -x - 2 \Rightarrow x + 3y = -8$$

Hence, the required equations are  $x + 3y = 8$  and  $x + 3y = -8$  or  $x + 3y = \pm 8$ .

**Q18.** At what points on the curve  $x^2 + y^2 - 2x - 4y + 1 = 0$ , the tangents are parallel to the  $y$ -axis?

**Sol.** Given that the equation of the curve is

$$x^2 + y^2 - 2x - 4y + 1 = 0 \quad \dots(i)$$

Differentiating both sides w.r.t.  $x$ , we have

$$2x + 2y \cdot \frac{dy}{dx} - 2 - 4 \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow (2y - 4) \frac{dy}{dx} = 2 - 2x \Rightarrow \frac{dy}{dx} = \frac{2 - 2x}{2y - 4} \quad \dots(ii)$$

Since the tangent to the curve is parallel to the  $y$ -axis.

$$\therefore \text{Slope } \frac{dy}{dx} = \tan \frac{\pi}{2} = \infty = \frac{1}{0}$$

So, from eq. (ii) we get

$$\frac{2 - 2x}{2y - 4} = \frac{1}{0} \Rightarrow 2y - 4 = 0 \Rightarrow y = 2$$

Now putting the value of  $y$  in eq. (i), we get

$$\Rightarrow x^2 + (2)^2 - 2x - 8 + 1 = 0$$

$$\Rightarrow x^2 - 2x + 4 - 8 + 1 = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x^2 - 3x + x - 3 = 0$$

$$\Rightarrow x(x - 3) + 1(x - 3) = 0 \Rightarrow (x - 3)(x + 1) = 0$$

$$\Rightarrow x = -1 \text{ or } 3$$

Hence, the required points are  $(-1, 2)$  and  $(3, 2)$ .

**Q19.** Show that the line  $\frac{x}{a} + \frac{y}{b} = 1$ , touches the curve  $y = b \cdot e^{-x/a}$  at the point where the curve intersects the axis of  $y$ .

**Sol.** Given that  $y = b \cdot e^{-x/a}$ , the equation of curve

and  $\frac{x}{a} + \frac{y}{b} = 1$ , the equation of line.

Let the coordinates of the point where the curve intersects the  $y$ -axis be  $(0, y_1)$

Now differentiating  $y = b \cdot e^{-x/a}$  both sides w.r.t.  $x$ , we get

$$\frac{dy}{dx} = b \cdot e^{-x/a} \left( -\frac{1}{a} \right) = -\frac{b}{a} \cdot e^{-x/a}$$

So, the slope of the tangent,  $m_1 = -\frac{b}{a} e^{-x/a}$ .

Differentiating  $\frac{x}{a} + \frac{y}{b} = 1$  both sides w.r.t.  $x$ , we get

$$\frac{1}{a} + \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

So, the slope of the line,  $m_2 = -\frac{b}{a}$ .

If the line touches the curve, then  $m_1 = m_2$

$$\Rightarrow \frac{-b}{a} \cdot e^{-x/a} = \frac{-b}{a} \Rightarrow e^{-x/a} = 1$$

$$\Rightarrow \frac{-x}{a} \log e = \log 1 \quad (\text{Taking log on both sides})$$

$$\Rightarrow \frac{-x}{a} = 0 \Rightarrow x = 0$$

Putting  $x = 0$  in equation  $y = b \cdot e^{-x/a}$

$$\Rightarrow y = b \cdot e^0 = b$$

Hence, the given equation of curve intersect at  $(0, b)$  i.e. on  $y$ -axis.

**Q20.** Show that  $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$  is increasing in  $\mathbb{R}$ .

**Sol.** Given that  $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$

Differentiating both sides w.r.t.  $x$ , we get

$$\begin{aligned} f'(x) &= 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x} \times \frac{d}{dx} (\sqrt{1+x^2} - x) \\ &= 2 - \frac{1}{1+x^2} + \frac{\left( \frac{1}{2\sqrt{1+x^2}} \times (2x-1) \right)}{\sqrt{1+x^2} - x} \end{aligned}$$

$$\begin{aligned}
 &= 2 - \frac{1}{1+x^2} + \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}(\sqrt{1+x^2} - x)} \\
 &= 2 - \frac{1}{1+x^2} - \frac{(\sqrt{1+x^2} - x)}{\sqrt{1+x^2}(\sqrt{1+x^2} - x)} \\
 &= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}
 \end{aligned}$$

For increasing function,  $f'(x) \geq 0$

$$\begin{aligned}
 \therefore 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} &\geq 0 \\
 \Rightarrow \frac{2(1+x^2) - 1 + \sqrt{1+x^2}}{(1+x^2)} &\geq 0 \Rightarrow 2 + 2x^2 - 1 + \sqrt{1+x^2} \geq 0 \\
 \Rightarrow 2x^2 + 1 + \sqrt{1+x^2} &\geq 0 \Rightarrow 2x^2 + 1 \geq -\sqrt{1+x^2}
 \end{aligned}$$

Squaring both sides, we get  $4x^4 + 1 + 4x^2 \geq 1 + x^2$

$$\Rightarrow 4x^4 + 4x^2 - x^2 \geq 0 \Rightarrow 4x^4 + 3x^2 \geq 0 \Rightarrow x^2(4x^2 + 3) \geq 0$$

which is true for any value of  $x \in \mathbb{R}$ .

Hence, the given function is an increasing function over  $\mathbb{R}$ .

**Q21.** Show that for  $a \geq 1$ ,  $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$  is decreasing in  $\mathbb{R}$ .

**Sol.** Given that:  $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ ,  $a \geq 1$

Differentiating both sides w.r.t.  $x$ , we get

$$f'(x) = \sqrt{3} \cos x + \sin x - 2a$$

For decreasing function,  $f'(x) < 0$

$$\begin{aligned}
 \therefore \sqrt{3} \cos x + \sin x - 2a &< 0 \\
 \Rightarrow 2\left(\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x\right) - 2a &< 0 \\
 \Rightarrow \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x - a &< 0 \\
 \Rightarrow \left(\cos \frac{\pi}{6} \cos x + \sin \frac{\pi}{6} \sin x\right) - a &< 0 \\
 \Rightarrow \cos\left(x - \frac{\pi}{6}\right) - a &< 0
 \end{aligned}$$

Since  $\cos x \in [-1, 1]$  and  $a \geq 1$

$$\therefore f'(x) < 0$$

Hence, the given function is decreasing in  $\mathbb{R}$ .

**Q22.** Show that  $f(x) = \tan^{-1}(\sin x + \cos x)$  is an increasing function in  $\left(0, \frac{\pi}{4}\right)$ .

**Sol.** Given that:  $f(x) = \tan^{-1}(\sin x + \cos x)$  in  $\left(0, \frac{\pi}{4}\right)$

Differentiating both sides w.r.t.  $x$ , we get

$$\begin{aligned} f'(x) &= \frac{1}{1 + (\sin x + \cos x)^2} \cdot \frac{d}{dx}(\sin x + \cos x) \\ \Rightarrow f'(x) &= \frac{1 \times (\cos x - \sin x)}{1 + (\sin x + \cos x)^2} \\ \Rightarrow f'(x) &= \frac{\cos x - \sin x}{1 + \sin^2 x + \cos^2 x + 2 \sin x \cos x} \\ \Rightarrow f'(x) &= \frac{\cos x - \sin x}{1 + 1 + 2 \sin x \cos x} \Rightarrow f'(x) = \frac{\cos x - \sin x}{2 + 2 \sin x \cos x} \end{aligned}$$

For an increasing function  $f'(x) \geq 0$

$$\begin{aligned} \therefore \frac{\cos x - \sin x}{2 + 2 \sin x \cos x} &\geq 0 \\ \Rightarrow \cos x - \sin x &\geq 0 \quad \left[ \because (2 + \sin 2x) \geq 0 \text{ in } \left(0, \frac{\pi}{4}\right) \right] \\ \Rightarrow \cos x &\geq \sin x, \text{ which is true for } \left(0, \frac{\pi}{4}\right) \end{aligned}$$

Hence, the given function  $f(x)$  is an increasing function in  $\left(0, \frac{\pi}{4}\right)$ .

**Q23.** At what point, the slope of the curve  $y = -x^3 + 3x^2 + 9x - 27$  is maximum? Also find the maximum slope.

**Sol.** Given that:  $y = -x^3 + 3x^2 + 9x - 27$

Differentiating both sides w.r.t.  $x$ , we get  $\frac{dy}{dx} = -3x^2 + 6x + 9$

Let slope of the curve  $\frac{dy}{dx} = z$

$$\therefore z = -3x^2 + 6x + 9$$

Differentiating both sides w.r.t.  $x$ , we get  $\frac{dz}{dx} = -6x + 6$

For local maxima and local minima,  $\frac{dz}{dx} = 0$

$$\therefore -6x + 6 = 0 \Rightarrow x = 1$$

$$\Rightarrow \frac{d^2z}{dx^2} = -6 < 0 \quad \text{Maxima}$$

$$\begin{aligned} \text{Put } x = 1 \text{ in equation of the curve } y &= (-1)^3 + 3(1)^2 + 9(1) - 27 \\ &= -1 + 3 + 9 - 27 = -16 \end{aligned}$$

$$\text{Maximum slope} = -3(1)^2 + 6(1) + 9 = 12$$

Hence,  $(1, -16)$  is the point at which the slope of the given curve is maximum and maximum slope = 12.

**Q24.** Prove that  $f(x) = \sin x + \sqrt{3} \cos x$  has maximum value at  $x = \frac{\pi}{6}$ .

$$\begin{aligned} \text{Sol. We have: } f(x) &= \sin x + \sqrt{3} \cos x = 2 \left( \frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x \right) \\ &= 2 \left( \cos \frac{\pi}{3} \sin x + \sin \frac{\pi}{3} \cos x \right) = 2 \sin \left( x + \frac{\pi}{3} \right) \end{aligned}$$

$$f'(x) = 2 \cos \left( x + \frac{\pi}{3} \right); f''(x) = -2 \sin \left( x + \frac{\pi}{3} \right)$$

$$\begin{aligned} f''(x)_{x=\frac{\pi}{6}} &= -2 \sin \left( \frac{\pi}{6} + \frac{\pi}{3} \right) \\ &= -2 \sin \frac{\pi}{2} = -2.1 = -2 < 0 \text{ (Maxima)} \\ &= -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \text{ (Maxima)} \end{aligned}$$

Maximum value of the function at  $x = \frac{\pi}{6}$  is

$$\sin \frac{\pi}{6} + \sqrt{3} \cos \frac{\pi}{6} = \frac{1}{2} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} = 2$$

Hence, the given function has maximum value at  $x = \frac{\pi}{6}$  and the maximum value is 2.

### LONG ANSWERTYPE QUESTIONS

**Q25.** If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is  $\frac{\pi}{3}$ .

**Sol.** Let  $\Delta ABC$  be the right angled triangle in which  $\angle B = 90^\circ$

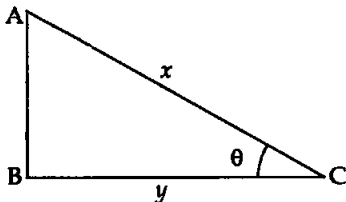
Let  $AC = x$ ,  $BC = y$

$$\therefore AB = \sqrt{x^2 - y^2}$$

$$\angle ACB = \theta$$

Let  $Z = x + y$  (given)

Now area of  $\Delta ABC$ ,  $A = \frac{1}{2} \times AB \times BC$



$$\Rightarrow A = \frac{1}{2}y \cdot \sqrt{x^2 - y^2} \Rightarrow A = \frac{1}{2}y \cdot \sqrt{(Z-y)^2 - y^2}$$

Squaring both sides, we get

$$A^2 = \frac{1}{4}y^2 [(Z-y)^2 - y^2] \Rightarrow A^2 = \frac{1}{4}y^2 [Z^2 + y^2 - 2Zy - y^2]$$

$$\Rightarrow P = \frac{1}{4}y^2 [Z^2 - 2Zy] \Rightarrow P = \frac{1}{4}[y^2Z^2 - 2Zy^3] \quad [A^2 = P]$$

Differentiating both sides w.r.t.  $y$  we get

$$\frac{dP}{dy} = \frac{1}{4}[2yZ^2 - 6Zy^2] \quad \dots(i)$$

For local maxima and local minima,  $\frac{dP}{dy} = 0$

$$\therefore \frac{1}{4}(2yZ^2 - 6Zy^2) = 0$$

$$\Rightarrow \frac{2yZ}{4}(Z - 3y) = 0 \Rightarrow yZ(Z - 3y) = 0$$

$$\Rightarrow yZ \neq 0 \quad (\because y \neq 0 \text{ and } Z \neq 0)$$

$$\therefore Z - 3y = 0$$

$$\Rightarrow y = \frac{Z}{3} \Rightarrow y = \frac{x+y}{3} \quad (\because Z = x+y)$$

$$\Rightarrow 3y = x+y \Rightarrow 3y - y = x \Rightarrow 2y = x$$

$$\Rightarrow \frac{y}{x} = \frac{1}{2} \Rightarrow \cos \theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$

Differentiating eq. (i) w.r.t.  $y$ , we have  $\frac{d^2P}{dy^2} = \frac{1}{4}[2Z^2 - 12Zy]$

$$\begin{aligned} \frac{d^2P}{dy^2} \text{ at } y = \frac{Z}{3} &= \frac{1}{4} \left[ 2Z^2 - 12Z \cdot \frac{Z}{3} \right] \\ &= \frac{1}{4} [2Z^2 - 4Z^2] = \frac{-Z^2}{2} < 0 \text{ Maxima} \end{aligned}$$

Hence, the area of the given triangle is maximum when the angle between its hypotenuse and a side is  $\frac{\pi}{3}$ .

**Q26.** Find the points of local maxima, local minima and the points of inflection of the function  $f(x) = x^5 - 5x^4 + 5x^3 - 1$ . Also find the corresponding local maximum and local minimum values.

**Sol.** We have  $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$$\Rightarrow f'(x) = 5x^4 - 20x^3 + 15x^2$$

For local maxima and local minima,  $f'(x) = 0$



$$\Rightarrow 5x^4 - 20x^3 + 15x^2 = 0 \Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow 5x^2(x^2 - 3x - x + 3) = 0 \Rightarrow x^2(x-3)(x-1) = 0$$

$$\therefore x = 0, x = 1 \text{ and } x = 3$$

$$\text{Now } f''(x) = 20x^3 - 60x^2 + 30x$$

$$\Rightarrow f''(x)_{\text{at } x=0} = 20(0)^3 - 60(0)^2 + 30(0) = 0 \text{ which is neither maxima nor minima.}$$

$$\therefore f(x) \text{ has the point of inflection at } x = 0$$

$$\begin{aligned} f''(x)_{\text{at } x=1} &= 20(1)^3 - 60(1)^2 + 30(1) \\ &= 20 - 60 + 30 = -10 < 0 \text{ Maxima} \end{aligned}$$

$$\begin{aligned} f''(x)_{\text{at } x=3} &= 20(3)^3 - 60(3)^2 + 30(3) \\ &= 540 - 540 + 90 = 90 > 0 \text{ Minima} \end{aligned}$$

The maximum value of the function at  $x = 1$

$$\begin{aligned} f(x) &= (1)^5 - 5(1)^4 + 5(1)^3 - 1 \\ &= 1 - 5 + 5 - 1 = 0 \end{aligned}$$

The minimum value at  $x = 3$  is

$$\begin{aligned} f(x) &= (3)^5 - 5(3)^4 + 5(3)^3 - 1 \\ &= 243 - 405 + 135 - 1 = 378 - 406 = -28 \end{aligned}$$

Hence, the function has its maxima at  $x = 1$  and the maximum value = 0 and it has minimum value at  $x = 3$  and its minimum value is -28.

$x = 0$  is the point of inflection.

- Q27.** A telephone company in a town has 500 subscribers on its list and collects fixed charges of ₹ 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of ₹ 1.00, one subscriber will discontinue the service. Find what increase will bring maximum profit?

**Sol.** Let us consider that the company increases the annual subscription by ₹  $x$ .

So,  $x$  is the number of subscribers who discontinue the services.

$$\begin{aligned} \therefore \text{Total revenue, } R(x) &= (500 - x)(300 + x) \\ &= 150000 + 500x - 300x - x^2 \\ &= -x^2 + 200x + 150000 \end{aligned}$$

Differentiating both sides w.r.t.  $x$ , we get  $R'(x) = -2x + 200$

For local maxima and local minima,  $R'(x) = 0$

$$-2x + 200 = 0 \Rightarrow x = 100$$

$$R''(x) = -2 < 0 \text{ Maxima}$$

So,  $R(x)$  is maximum at  $x = 100$

Hence, in order to get maximum profit, the company should increase its annual subscription by ₹ 100.

**Q28.** If the straight line  $x \cos \alpha + y \sin \alpha = p$  touches the curve  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , then prove that  $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$ .

**Sol.** The given curve is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  ... (i)

and the straight line  $x \cos \alpha + y \sin \alpha = p$  ... (ii)

Differentiating eq. (i) w.r.t.  $x$ , we get

$$\frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

$$\text{So the slope of the curve} = \frac{-b^2}{a^2} \cdot \frac{x}{y}$$

Now differentiating eq. (ii) w.r.t.  $x$ , we have

$$\cos \alpha + \sin \alpha \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\cos \alpha}{\sin \alpha} = -\cot \alpha$$

So, the slope of the straight line =  $-\cot \alpha$

If the line is the tangent to the curve, then

$$\frac{-b^2}{a^2} \cdot \frac{x}{y} = -\cot \alpha \Rightarrow \frac{x}{y} = \frac{a^2}{b^2} \cdot \cot \alpha \Rightarrow x = \frac{a^2}{b^2} \cot \alpha \cdot y$$

Now from eq. (ii) we have  $x \cos \alpha + y \sin \alpha = p$

$$\Rightarrow \frac{a^2}{b^2} \cdot \cot \alpha \cdot y \cdot \cos \alpha + y \sin \alpha = p$$

$$\Rightarrow a^2 \cot \alpha \cdot \cos \alpha y + b^2 \sin \alpha y = b^2 p$$

$$\Rightarrow a^2 \frac{\cos \alpha}{\sin \alpha} \cdot \cos \alpha y + b^2 \sin \alpha y = b^2 p$$

$$\Rightarrow a^2 \cos^2 \alpha y + b^2 \sin^2 \alpha y = b^2 \sin \alpha p$$

$$\Rightarrow a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = \frac{b^2}{y} \cdot \sin \alpha \cdot p$$

$$\Rightarrow a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p \cdot p \quad \left[ \because \frac{b^2}{y} \sin \alpha = p \right]$$

$$\text{Hence, } a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$$

**Alternate method:**

We know that  $y = mx + c$  will touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ if } c^2 = a^2m^2 + b^2$$

Here equation of straight line is  $x \cos \alpha + y \sin \alpha = p$  and that

$$\text{of ellipse is } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x \cos \alpha + y \sin \alpha = p$$

$$\Rightarrow y \sin \alpha = -x \cos \alpha + p$$

$$\Rightarrow y = -x \frac{\cos \alpha}{\sin \alpha} + \frac{p}{\sin \alpha} \Rightarrow y = -x \cot \alpha + \frac{p}{\sin \alpha}$$

Comparing with  $y = mx + c$ , we get

$$m = -\cot \alpha \quad \text{and} \quad c = \frac{p}{\sin \alpha}$$

So, according to the condition, we get  $c^2 = a^2m^2 + b^2$

$$\frac{p^2}{\sin^2 \alpha} = a^2(-\cot \alpha)^2 + b^2$$

$$\Rightarrow \frac{p^2}{\sin^2 \alpha} = \frac{a^2 \cos^2 \alpha}{\sin^2 \alpha} + b^2 \Rightarrow p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

Hence,  $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$  Hence proved.

- Q29.** An open box with square base is to be made of a given quantity of card board of area  $c^2$ . Show that the maximum volume of the box is  $\frac{c^3}{6\sqrt{3}}$  cubic units.

**Sol.** Let  $x$  be the length of the side of the square base of the cubical open box and  $y$  be its height.

$\therefore$  Surface area of the open box

$$c^2 = x^2 + 4xy \Rightarrow y = \frac{c^2 - x^2}{4x} \quad \dots(i)$$

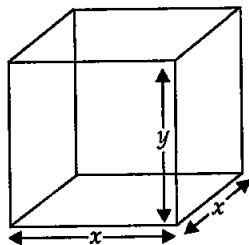
Now volume of the box,  $V = x \times x \times y$

$$\Rightarrow V = x^2y$$

$$\Rightarrow V = x^2 \left( \frac{c^2 - x^2}{4x} \right)$$

$$\Rightarrow V = \frac{1}{4}(c^2x - x^3)$$

Differentiating both sides w.r.t.  $x$ , we get



$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2) \quad \dots(ii)$$

For local maxima and local minima,  $\frac{dV}{dx} = 0$

$$\therefore \frac{1}{4}(c^2 - 3x^2) = 0 \Rightarrow c^2 - 3x^2 = 0$$

$$\Rightarrow x^2 = \frac{c^2}{3}$$

$$\therefore x = \sqrt{\frac{c^2}{3}} = \frac{c}{\sqrt{3}}$$

Now again differentiating eq. (ii) w.r.t.  $x$ , we get

$$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2} \cdot \frac{c}{\sqrt{3}} < 0 \quad (\text{maxima})$$

Volume of the cubical box (V) =  $x^2 y$

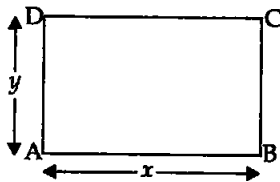
$$= x^2 \left( \frac{c^2 - x^2}{4x} \right) = \frac{c}{\sqrt{3}} \left[ \frac{c^2 - \frac{c^2}{3}}{4} \right] = \frac{c}{\sqrt{3}} \times \frac{2c^2}{3 \times 4} = \frac{c^3}{6\sqrt{3}}$$

Hence, the maximum volume of the open box is

$$\frac{c^3}{6\sqrt{3}} \text{ cubic units.}$$

- Q30.** Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

**Sol.** Let  $x$  and  $y$  be the length and breadth of a given rectangle ABCD as per question, the rectangle be revolved about side AD which will make a cylinder with radius  $x$  and height  $y$ .



$$\therefore \text{Volume of the cylinder } V = \pi r^2 h$$

$$\Rightarrow V = \pi x^2 y \quad \dots(i)$$

$$\text{Now perimeter of rectangle } P = 2(x + y) \Rightarrow 36 = 2(x + y)$$

$$\Rightarrow x + y = 18 \Rightarrow y = 18 - x \quad \dots(ii)$$

Putting the value of  $y$  in eq. (i) we get

$$V = \pi x^2(18 - x)$$

$$\Rightarrow V = \pi(18x^2 - x^3)$$

Differentiating both sides w.r.t.  $x$ , we get

$$\frac{dV}{dx} = \pi(36x - 3x^2) \quad \dots(iii)$$

For local maxima and local minima  $\frac{dV}{dx} = 0$

$$\therefore \pi(36x - 3x^2) = 0 \Rightarrow 36x - 3x^2 = 0$$

$$\Rightarrow 3x(12 - x) = 0$$

$$\Rightarrow x \neq 0 \quad \therefore 12 - x = 0 \Rightarrow x = 12$$

From eq. (ii)  $y = 18 - 12 = 6$

Differentiating eq. (iii) w.r.t.  $x$ , we get  $\frac{d^2V}{dx^2} = \pi(36 - 6x)$

$$\text{at } x = 12 \quad \frac{d^2V}{dx^2} = \pi(36 - 6 \times 12) \\ = \pi(36 - 72) = -36\pi < 0 \text{ maxima}$$

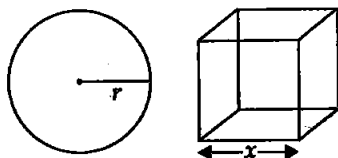
Now volume of the cylinder so formed =  $\pi x^2 y$

$$= \pi \times (12)^2 \times 6 = \pi \times 144 \times 6 = 864\pi \text{ cm}^3$$

Hence, the required dimensions are 12 cm and 6 cm and the maximum volume is  $864\pi \text{ cm}^3$ .

**Q31.** If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

**Sol.** Let  $x$  be the edge of the cube and  $r$  be the radius of the sphere.  
Surface area of cube =  $6x^2$



and surface area of the sphere =  $4\pi r^2$

$$\therefore 6x^2 + 4\pi r^2 = K(\text{constant}) \Rightarrow r = \sqrt{\frac{K - 6x^2}{4\pi}} \quad \dots(i)$$

Volume of the cube =  $x^3$  and the volume of sphere =  $\frac{4}{3}\pi r^3$

$\therefore$  Sum of their volumes ( $V$ ) = Volume of cube  
+ Volume of sphere

$$\Rightarrow V = x^3 + \frac{4}{3}\pi r^3$$

$$\Rightarrow V = x^3 + \frac{4}{3}\pi \times \left(\frac{K - 6x^2}{4\pi}\right)^{3/2}$$

Differentiating both sides w.r.t.  $x$ , we get

$$\frac{dV}{dx} = 3x^2 + \frac{4\pi}{3} \times \frac{3}{2} (K - 6x^2)^{1/2} (-12x) \times \frac{1}{(4\pi)^{3/2}}$$

$$= 3x^2 + \frac{2\pi}{(4\pi)^{3/2}} \times (-12x)(K - 6x^2)^{1/2}$$

$$= 3x^2 + \frac{1}{4\pi^{1/2}} \times (-12x)(K - 6x^2)^{1/2}$$

$$\therefore \frac{dV}{dx} = 3x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2} \quad \dots(ii)$$

For local maxima and local minima,  $\frac{dV}{dx} = 0$

$$\therefore 3x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2} = 0$$

$$\Rightarrow 3x \left[ x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}} \right] = 0$$

$$x \neq 0 \quad \therefore x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}} = 0$$

$$\Rightarrow x = \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}}$$

Squaring both sides, we get

$$x^2 = \frac{K - 6x^2}{\pi} \Rightarrow \pi x^2 = K - 6x^2$$

$$\Rightarrow \pi x^2 + 6x^2 = K \Rightarrow x^2(\pi + 6) = K \Rightarrow x^2 = \frac{K}{\pi + 6}$$

$$\therefore x = \sqrt{\frac{K}{\pi + 6}}$$

Now putting the value of K in eq. (i), we get

$$6x^2 + 4\pi r^2 = x^2(\pi + 6)$$

$$\Rightarrow 6x^2 + 4\pi r^2 = \pi x^2 + 6x^2 \Rightarrow 4\pi r^2 = \pi x^2 \Rightarrow 4r^2 = x^2$$

$$\therefore 2r = x$$

$$\therefore x:2r = 1:1$$

Now differentiating eq. (ii) w.r.t x, we have

$$\frac{d^2V}{dx^2} = 6x - \frac{3}{\sqrt{\pi}} \frac{d}{dx} [x(K - 6x^2)^{1/2}]$$

$$= 6x - \frac{3}{\sqrt{\pi}} \left[ x \cdot \frac{1}{2\sqrt{K - 6x^2}} \times (-12x) + (K - 6x^2)^{1/2} \cdot 1 \right]$$

$$= 6x - \frac{3}{\sqrt{\pi}} \left[ \frac{-6x^2}{\sqrt{K - 6x^2}} + \sqrt{K - 6x^2} \right]$$

$$= 6x - \frac{3}{\sqrt{\pi}} \left[ \frac{-6x^2 + K - 6x^2}{\sqrt{K - 6x^2}} \right] = 6x + \frac{3}{\sqrt{\pi}} \left[ \frac{12x^2 - K}{\sqrt{K - 6x^2}} \right]$$

$$\begin{aligned} \text{Put } x &= \sqrt{\frac{K}{\pi+6}} = 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}} \left[ \frac{\frac{12K}{\pi+6} - K}{\sqrt{K - \frac{6K}{\pi+6}}} \right] \\ &= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}} \left[ \frac{12K - \pi K - 6K}{\sqrt{\frac{\pi K + 6K - 6K}{\pi+6}}} \right] \\ &= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}} \left[ \frac{6K - \pi K}{\sqrt{\frac{\pi K}{\pi+6}}} \right] \\ &= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\pi\sqrt{K}} [(6K - \pi K)\sqrt{\pi+6}] > 0 \end{aligned}$$

So it is minima.

Hence, the required ratio is 1 : 1 when the combined volume is minimum.

- Q32.** AB is a diameter of a circle and C is any point on the circle. Show that the area of  $\Delta ABC$  is maximum, when it is isosceles.

- Sol.** Let AB be the diameter and C be any point on the circle with radius  $r$ .

$\angle ACB = 90^\circ$  [angle in the semi circle is  $90^\circ$ ]

Let  $AC = x$

$$\therefore BC = \sqrt{AB^2 - AC^2}$$

$$\Rightarrow BC = \sqrt{(2r)^2 - x^2} \Rightarrow BC = \sqrt{4r^2 - x^2} \quad \dots(i)$$

Now area of  $\Delta ABC$ ,  $A = \frac{1}{2} \times AC \times BC$

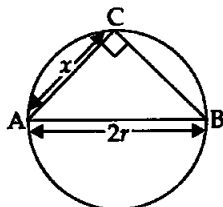
$$\Rightarrow A = \frac{1}{2} x \cdot \sqrt{4r^2 - x^2}$$

Squaring both sides, we get

$$A^2 = \frac{1}{4} x^2 (4r^2 - x^2)$$

Let  $A^2 = Z$

$$\therefore Z = \frac{1}{4} x^2 (4r^2 - x^2) \Rightarrow Z = \frac{1}{4} (4x^2 r^2 - x^4)$$



Differentiating both sides w.r.t.  $x$ , we get

$$\frac{dZ}{dx} = \frac{1}{4} [8xr^2 - 4x^3] \quad \dots(ii)$$

For local maxima and local minima  $\frac{dZ}{dx} = 0$

$$\therefore \frac{1}{4} [8xr^2 - 4x^3] = 0 \Rightarrow x[2r^2 - x^2] = 0$$

$$x \neq 0 \quad \therefore \quad 2r^2 - x^2 = 0$$

$$\Rightarrow \quad x^2 = 2r^2 \Rightarrow x = \sqrt{2}r = AC$$

Now from eq. (i) we have

$$BC = \sqrt{4r^2 - 2r^2} \Rightarrow BC = \sqrt{2r^2} \Rightarrow BC = \sqrt{2}r$$

So  $AC = BC$

Hence,  $\Delta ABC$  is an isosceles triangle.

Differentiating eq. (ii) w.r.t.  $x$ , we get  $\frac{d^2Z}{dx^2} = \frac{1}{4} [8r^2 - 12x^2]$

Put  $x = \sqrt{2}r$

$$\begin{aligned} \therefore \quad \frac{d^2Z}{dx^2} &= \frac{1}{4} [8r^2 - 12 \times 2r^2] = \frac{1}{4} [8r^2 - 24r^2] \\ &= \frac{1}{4} \times (-16r^2) = -4r^2 < 0 \quad \text{maxima} \end{aligned}$$

Hence, the area of  $\Delta ABC$  is maximum when it is an isosceles triangle.

- Q33.** A metal box with a square base and vertical sides is to contain  $1024 \text{ cm}^3$ . The material for the top and bottom costs ₹  $5/\text{cm}^2$  and the material for the sides costs ₹  $2.50/\text{cm}^2$ . Find the least cost of the box.

**Sol.** Let  $x$  be the side of the square base and  $y$  be the length of the vertical sides.

Area of the base and bottom =  $2x^2 \text{ cm}^2$

$$\begin{aligned} \therefore \text{Cost of the material required} &= ₹ 5 \times 2x^2 \\ &= ₹ 10x^2 \end{aligned}$$

Area of the 4 sides =  $4xy \text{ cm}^2$

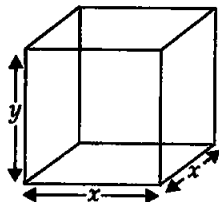
$$\begin{aligned} \therefore \text{Cost of the material for the four sides} \\ &= ₹ 2.50 \times 4xy = ₹ 10xy \end{aligned}$$

$$\text{Total cost} \quad C = 10x^2 + 10xy \quad \dots(i)$$

New volume of the box =  $x \times x \times y$

$$\Rightarrow \quad 1024 = x^2y$$

$$\therefore \quad y = \frac{1024}{x^2} \quad \dots(ii)$$





Putting the value of  $y$  in eq. (i) we get

$$C = 10x^2 + 10x \times \frac{1024}{x^2} \Rightarrow C = 10x^2 + \frac{10240}{x}$$

Differentiating both sides w.r.t.  $x$ , we get

$$\frac{dC}{dx} = 20x - \frac{10240}{x^2} \quad \dots(iii)$$

For local maxima and local minima  $\frac{dC}{dx} = 0$

$$20 - \frac{10240}{x^2} = 0$$

$$\Rightarrow 20x^3 - 10240 = 0 \Rightarrow x^3 = 512 \Rightarrow x = 8 \text{ cm}$$

Now from eq. (ii)

$$y = \frac{10240}{(8)^2} = \frac{10240}{64} = 16 \text{ cm}$$

$$\therefore \text{Cost of material used } C = 10x^2 + 10xy \\ = 10 \times 8 \times 8 + 10 \times 8 \times 16 = 640 + 1280 = 1920$$

Now differentiating eq. (iii) we get

$$\frac{d^2C}{dx^2} = 20 + \frac{20480}{x^3}$$

Put  $x = 8$

$$= 20 + \frac{20480}{(8)^3} = 20 + \frac{20480}{512} = 20 + 40 = 60 > 0 \text{ minima}$$

Hence, the required cost is ₹ 1920 which is the minimum.

**Q34.** The sum of the surface areas of a rectangular parallelepiped with sides  $x$ ,  $2x$  and  $\frac{x}{3}$  and a sphere is given to be constant.

Prove that the sum of their volumes is minimum, if  $x$  is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.

**Sol.** Let ' $r$ ' be the radius of the sphere.

$$\therefore \text{Surface area of the sphere} = 4\pi r^2$$

$$\text{Volume of the sphere} = \frac{4}{3}\pi r^3$$

The sides of the parallelepiped are  $x$ ,  $2x$  and  $\frac{x}{3}$

$$\therefore \text{Its surface area} = 2 \left[ x \times 2x + 2x \times \frac{x}{3} + x \times \frac{x}{3} \right] \\ = 2 \left[ 2x^2 + \frac{2x^2}{3} + \frac{x^2}{3} \right] = 2[2x^2 + x^2] \\ = 2[3x^2] = 6x^2$$

$$\text{Volume of the parallelepiped} = x \times 2x \times \frac{x}{3} = \frac{2}{3}x^3$$

As per the conditions of the question,

Surface area of the parallelepiped

+ Surface area of the sphere = constant

$$\Rightarrow 6x^2 + 4\pi r^2 = K \text{ (constant)} \Rightarrow 4\pi r^2 = K - 6x^2$$

$$\therefore r^2 = \frac{K - 6x^2}{4\pi} \quad \dots(i)$$

Now let  $V =$  Volume of parallelepiped  
+ Volume of the sphere

$$\Rightarrow V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$\Rightarrow V = \frac{2}{3}x^3 + \frac{4}{3}\pi \left[ \frac{K - 6x^2}{4\pi} \right]^{3/2} \quad [\text{from eq. (i)}]$$

$$\Rightarrow V = \frac{2}{3}x^3 + \frac{4}{3}\pi \times \frac{1}{(4)^{3/2} \pi^{3/2}} [K - 6x^2]^{3/2}$$

$$\Rightarrow V = \frac{2}{3}x^3 + \frac{4}{3}\pi \times \frac{1}{8 \times \pi^{3/2}} [K - 6x^2]^{3/2}$$

$$\Rightarrow V = \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}} [K - 6x^2]^{3/2}$$

Differentiating both sides w.r.t.  $x$ , we have

$$\begin{aligned} \frac{dV}{dx} &= \frac{2}{3} \cdot 3x^2 + \frac{1}{6\sqrt{\pi}} \left[ \frac{3}{2} (K - 6x^2)^{1/2} (-12x) \right] \\ &= 2x^2 + \frac{1}{6\sqrt{\pi}} \times \frac{3}{2} \times (-12x) (K - 6x^2)^{1/2} \\ &= 2x^2 - \frac{3x}{\sqrt{\pi}} [K - 6x^2]^{1/2} \end{aligned}$$

For local maxima and local minima, we have  $\frac{dV}{dx} = 0$

$$\therefore 2x^2 - \frac{3x}{\sqrt{\pi}} (K - 6x^2)^{1/2} = 0$$

$$\Rightarrow 2\sqrt{\pi}x^2 - 3x(K - 6x^2)^{1/2} = 0$$

$$\Rightarrow x[2\sqrt{\pi}x - 3(K - 6x^2)^{1/2}] = 0$$

Here  $x \neq 0$  and  $2\sqrt{\pi}x - 3(K - 6x^2)^{1/2} = 0$

$$\Rightarrow 2\sqrt{\pi}x = 3(K - 6x^2)^{1/2}$$

Squaring both sides, we get

$$4\pi x^2 = 9(K - 6x^2) \Rightarrow 4\pi x^2 = 9K - 54x^2$$

$$\begin{aligned} \Rightarrow 4\pi x^2 + 54x^2 &= 9K \\ \Rightarrow K &= \frac{4\pi x^2 + 54x^2}{9} \quad \dots(ii) \\ \Rightarrow 2x^2(2\pi + 27) &= 9K \\ \therefore x^2 &= \frac{9K}{2(2\pi + 27)} = 3\sqrt{\frac{K}{4\pi + 54}} \end{aligned}$$

Now from eq. (i) we have

$$\begin{aligned} r^2 &= \frac{K - 6x^2}{4\pi} \\ \Rightarrow r^2 &= \frac{\frac{4\pi x^2 + 54x^2}{9} - 6x^2}{4\pi} \\ \Rightarrow r^2 &= \frac{4\pi x^2 + 54x^2 - 54x^2}{9 \times 4\pi} = \frac{4\pi x^2}{9 \times 4\pi} \\ \Rightarrow r^2 &= \frac{x^2}{9} \Rightarrow r = \frac{x}{3} \quad \therefore x = 3r \end{aligned}$$

Now we have  $\frac{dV}{dx} = 2x^2 - \frac{3x}{\sqrt{\pi}} (K - 6x^2)^{1/2}$

Differentiating both sides w.r.t.  $x$ , we get

$$\begin{aligned} \frac{d^2V}{dx^2} &= 4x - \frac{3}{\sqrt{\pi}} \left[ x \cdot \frac{d}{dx} (K - 6x^2)^{1/2} + (K - 6x^2)^{1/2} \cdot \frac{d}{dx} \cdot x \right] \\ &= 4x - \frac{3}{\sqrt{\pi}} \left[ x \cdot \frac{1 \times (-12x)}{2\sqrt{K - 6x^2}} + (K - 6x^2)^{1/2} \cdot 1 \right] \\ &= 4x - \frac{3}{\sqrt{\pi}} \left[ \frac{-6x^2}{(K - 6x^2)^{1/2}} + (K - 6x^2)^{1/2} \right] \\ &= 4x - \frac{3}{\sqrt{\pi}} \left[ \frac{-6x^2 + K - 6x^2}{(K - 6x^2)^{1/2}} \right] = 4x - \frac{3}{\sqrt{\pi}} \left[ \frac{K - 12x^2}{(K - 6x^2)^{1/2}} \right] \end{aligned}$$

Put  $x = 3 \cdot \sqrt{\frac{K}{4\pi + 54}}$

$$\frac{d^2V}{dx^2} = 4 \cdot 3 \sqrt{\frac{K}{4\pi + 54}} - \frac{3}{\sqrt{\pi}} \left[ \frac{K - 12 \cdot \frac{9K}{4\pi + 54}}{\sqrt{(K - 6 \cdot \frac{9K}{4\pi + 54})}} \right]$$

$$\begin{aligned}
&= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}} \frac{4K\pi+54K-108K}{4\pi+54} \\
&= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}} \frac{4K\pi-54K}{4\pi+54} \\
&= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}} \left[ \frac{4K\pi-54K}{\sqrt{4K\pi} \cdot \sqrt{4\pi+54}} \right] \\
&= 12\sqrt{\frac{K}{4\pi+54}} - \frac{6K}{\sqrt{\pi}} \left( \frac{2\pi-27}{\sqrt{4K\pi} \cdot \sqrt{4\pi+54}} \right) \\
&= 12\sqrt{\frac{K}{4\pi+54}} + \frac{6K}{\sqrt{\pi}} \left[ \frac{27-2\pi}{\sqrt{4K\pi} \cdot \sqrt{4\pi+54}} \right] > 0
\end{aligned}$$

$[\because 27-2\pi > 0]$

$\therefore \frac{d^2V}{dx^2} > 0$  so, it is minima.

Hence, the sum of volume is minimum for  $x = 3\sqrt{\frac{K}{4\pi+54}}$

$\therefore$  Minimum volume,

$$\begin{aligned}
V \text{ at } \left( x = 3\sqrt{\frac{K}{4\pi+54}} \right) &= \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \left(\frac{x}{3}\right)^3 \\
&= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \frac{x^3}{27} = \frac{2}{3}x^3 + \frac{4}{81}\pi x^3 \\
&= \frac{2}{3}x^3 \left( 1 + \frac{2\pi}{27} \right)
\end{aligned}$$

Hence, the required minimum volume is  $\frac{2}{3}x^3 \left( 1 + \frac{2\pi}{27} \right)$  and  $x = 3r$ .

### OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the following questions 35 to 59:

- Q35.** The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is:

- (a)  $10 \text{ cm}^2/\text{s}$  (b)  $\sqrt{3} \text{ cm}^2/\text{s}$   
 (c)  $10\sqrt{3} \text{ cm}^2/\text{s}$  (d)  $\frac{10}{3} \text{ cm}^2/\text{s}$

**Sol.** Let the length of each side of the given equilateral triangle be  $x$  cm.

$$\therefore \frac{dx}{dt} = 2 \text{ cm/sec}$$

$$\text{Area of equilateral triangle } A = \frac{\sqrt{3}}{4} x^2$$

$$\therefore \frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot 2x \cdot \frac{dx}{dt} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{sec}$$

Hence, the rate of increasing of area =  $10\sqrt{3} \text{ cm}^2/\text{sec}$ .

Hence, the correct option is (c).

**Q36.** A ladder, 5 m long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is:

- (a)  $\frac{1}{10}$  radian/sec (b)  $\frac{1}{20}$  radian/sec  
 (c) 20 radian/sec  
 (d) 10 radian/sec

**Sol.** Length of ladder = 5 m  
 Let  $AB = y$  m and  $BC = x$  m  
 $\therefore$  In right  $\triangle ABC$ ,

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow x^2 + y^2 = (5)^2 \Rightarrow x^2 + y^2 = 25$$

Differentiating both sides w.r.t  $x$ , we have

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

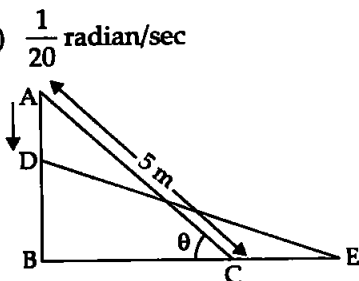
$$\Rightarrow x \frac{dx}{dt} + y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow 2 \cdot \frac{dx}{dt} + y \times (-0.1) = 0 \quad [\because x = 2\text{m}]$$

$$\Rightarrow 2 \cdot \frac{dx}{dt} + \sqrt{25 - x^2} \times (-0.1) = 0$$

$$\Rightarrow 2 \cdot \frac{dx}{dt} + \sqrt{25 - 4} \times (-0.1) = 0$$

$$\Rightarrow 2 \cdot \frac{dx}{dt} - \frac{\sqrt{21}}{10} = 0 \Rightarrow \frac{dx}{dt} = \frac{\sqrt{21}}{20}$$





$\frac{3x}{y}$  is the slope of the tangent

$$\therefore \text{Slope of the normal} = \frac{-1}{dy/dx} = \frac{-y}{3x}$$

Now  $x + 3y = 8$  is parallel to the normal

Differentiating both sides w.r.t.  $x$ , we have

$$1 + 3 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{3}$$

$$\therefore \frac{-y}{3x} = -\frac{1}{3} \Rightarrow y = x$$

Putting  $y = x$  in eq. (i) we get

$$3x^2 - x^2 = 8 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4$$

$$\therefore x = \pm 2 \text{ and } y = \pm 2$$

So the points are  $(2, 2)$  and  $(-2, -2)$ .

Equation of normal to the given curve at  $(2, 2)$  is

$$y - 2 = -\frac{1}{3}(x - 2)$$

$$\Rightarrow 3y - 6 = -x + 2 \Rightarrow x + 3y - 8 = 0$$

Equation of normal at  $(-2, -2)$  is

$$y + 2 = -\frac{1}{3}(x + 2)$$

$$\Rightarrow 3y + 6 = -x - 2 \Rightarrow x + 3y + 8 = 0$$

$\therefore$  The equations of the normals to the curve are

$$x + 3y \pm 8 = 0$$

Hence, the correct option is (c).

**Q39.** If the curve  $ay + x^2 = 7$  and  $x^3 = y$ , cut orthogonally at  $(1, 1)$ , then the value of 'a' is:

- (a) 1                      (b) 0                      (c) -6                      (d) 6

**Sol.** Equation of the given curves are  $ay + x^2 = 7$  ... (i)  
and  $x^3 = y$  ... (ii)

Differentiating eq. (i) w.r.t.  $x$ , we have

$$a \frac{dy}{dx} + 2x = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{a}$$

$$\therefore m_1 = -\frac{2x}{a} \quad \left( m_1 = \frac{dy}{dx} \right)$$

Now differentiating eq. (ii) w.r.t.  $x$ , we get

$$3x^2 = \frac{dy}{dx} \Rightarrow m_2 = 3x^2 \quad \left( m_2 = \frac{dy}{dx} \right)$$

The two curves are said to be orthogonal if the angle between the tangents at the point of intersection is  $90^\circ$ .

$$\therefore m_1 \times m_2 = -1$$

$$\Rightarrow \frac{-2x}{a} \times 3x^2 = -1 \Rightarrow \frac{-6x^3}{a} = -1 \Rightarrow 6x^3 = a$$

(1, 1) is the point of intersection of two curves.

$$\therefore 6(1)^3 = a$$

$$\text{So } a = 6$$

Hence, the correct option is (d).

**Q40.** If  $y = x^4 - 10$  and if  $x$  changes from 2 to 1.99, what is the change in  $y$ ?

(a) 0.32      (b) 0.032      (c) 5.68      (d) 5.968

**Sol.** Given that  $y = x^4 - 10$

$$\frac{dy}{dx} = 4x^3$$

$$\Delta x = 2.00 - 1.99 = 0.01$$

$$\therefore \Delta y = \frac{dy}{dx} \cdot \Delta x = 4x^3 \times \Delta x$$

$$= 4 \times (2)^3 \times 0.01 = 32 \times 0.01 = 0.32$$

Hence, the correct option is (a).

**Q41.** The equation of tangent to the curve  $y(1 + x^2) = 2 - x$ , where it crosses  $x$ -axis is:

(a)  $x + 5y = 2$       (b)  $x - 5y = 2$

(c)  $5x - y = 2$       (d)  $5x + y = 2$

**Sol.** Given that  $y(1 + x^2) = 2 - x$       ... (i)

If it cuts  $x$ -axis, then  $y$ -coordinate is 0.

$$\therefore 0(1 + x^2) = 2 - x \Rightarrow x = 2$$

Put  $x = 2$  in equation (i)

$$y(1 + 4) = 2 - 2 \Rightarrow y(5) = 0 \Rightarrow y = 0$$

Point of contact = (2, 0)

Differentiating eq. (i) w.r.t.  $x$ , we have

$$y \times 2x + (1 + x^2) \frac{dy}{dx} = -1$$

$$\Rightarrow 2xy + (1 + x^2) \frac{dy}{dx} = -1 \Rightarrow (1 + x^2) \frac{dy}{dx} = -1 - 2xy$$

$$\therefore \frac{dy}{dx} = \frac{-(1 + 2xy)}{(1 + x^2)} \Rightarrow \frac{dy}{dx}_{(2,0)} = \frac{-1}{(1 + 4)} = \frac{-1}{5}$$

$$\text{Equation of tangent is } y - 0 = -\frac{1}{5}(x - 2)$$

$$\Rightarrow 5y = -x + 2 \Rightarrow x + 5y = 2$$

Hence, the correct option is (a).



**Q42.** The points at which the tangents to the curve  $y = x^3 - 12x + 18$  are parallel to  $x$ -axis are:

- (a)  $(2, -2), (-2, -34)$       (b)  $(2, 34), (-2, 0)$   
 (c)  $(0, 34), (-2, 0)$       (d)  $(2, 2), (-2, 34)$

**Sol.** Given that  $y = x^3 - 12x + 18$

Differentiating both sides w.r.t.  $x$ , we have

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12$$

Since the tangents are parallel to  $x$ -axis, then  $\frac{dy}{dx} = 0$

$$\therefore 3x^2 - 12 = 0 \Rightarrow x = \pm 2$$

$$\therefore y_{x=2} = (2)^3 - 12(2) + 18 = 8 - 24 + 18 = 8$$

$$y_{x=-2} = (-2)^3 - 12(-2) + 18 = -8 + 24 + 18 = 34$$

$\therefore$  Points are  $(2, 2)$  and  $(-2, 34)$

Hence, the correct option is (d).

**Q43.** The tangent to the curve  $y = e^{2x}$  at the point  $(0, 1)$  meets  $x$ -axis at:

- (a)  $(0, 1)$       (b)  $\left(-\frac{1}{2}, 0\right)$       (c)  $(2, 0)$       (d)  $(0, 2)$

**Sol.** Equation of the curve is  $y = e^{2x}$

$$\text{Slope of the tangent } \frac{dy}{dx} = 2e^{2x} \Rightarrow \frac{dy}{dx}_{(0,1)} = 2 \cdot e^0 = 2$$

$\therefore$  Equation of tangent to the curve at  $(0, 1)$  is

$$y - 1 = 2(x - 0)$$

$$\Rightarrow y - 1 = 2x \Rightarrow y - 2x = 1$$

Since the tangent meets  $x$ -axis where  $y = 0$

$$\therefore 0 - 2x = 1 \Rightarrow x = -\frac{1}{2}$$

So the point is  $\left(-\frac{1}{2}, 0\right)$

Hence, the correct option is (b).

**Q44.** The slope of tangent to the curve  $x = t^2 + 3t - 8$  and  $y = 2t^2 - 2t - 5$  at the point  $(2, -1)$  is:

- (a)  $\frac{22}{7}$       (b)  $\frac{6}{7}$       (c)  $-\frac{6}{7}$       (d)  $-6$

**Sol.** The given curve is  $x = t^2 + 3t - 8$  and  $y = 2t^2 - 2t - 5$

$$\frac{dx}{dt} = 2t + 3 \quad \text{and} \quad \frac{dy}{dt} = 4t - 2$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3}$$

Now  $(2, -1)$  lies on the curve

$$\begin{aligned} \therefore 2 &= t^2 + 3t - 8 \Rightarrow t^2 + 3t - 10 = 0 \\ &\Rightarrow t^2 + 5t - 2t - 10 = 0 \\ &\Rightarrow t(t+5) - 2(t+5) = 0 \\ &\Rightarrow (t+5)(t-2) = 0 \end{aligned}$$

$$\therefore t = 2, t = -5 \quad \text{and} \quad -1 = 2t^2 - 2t - 5$$

$$\Rightarrow 2t^2 - 2t - 4 = 0$$

$$\Rightarrow t^2 - t - 2 = 0 \Rightarrow t^2 - 2t + t - 2 = 0$$

$$\Rightarrow t(t-2) + 1(t-2) = 0 \Rightarrow (t+1)(t-2) = 0$$

$$\Rightarrow t = -1 \quad \text{and} \quad t = 2$$

So  $t = 2$  is common value

$$\therefore \text{Slope } \frac{dy}{dx}_{x=2} = \frac{4 \times 2 - 2}{2 \times 2 + 3} = \frac{6}{7}$$

Hence, the correct option is (b).

**Q45.** The two curves  $x^3 - 3xy^2 + 2 = 0$  and  $3x^2y - y^3 - 2 = 0$  intersect at an angle of:

(a)  $\frac{\pi}{4}$       (b)  $\frac{\pi}{3}$       (c)  $\frac{\pi}{2}$       (d)  $\frac{\pi}{6}$

**Sol.** The given curves are  $x^3 - 3xy^2 + 2 = 0$  ... (i)

and  $3x^2y - y^3 - 2 = 0$  ... (ii)

Differentiating eq. (i) w.r.t.  $x$ , we get

$$3x^2 - 3\left(x \cdot 2y \frac{dy}{dx} + y^2 \cdot 1\right) = 0$$

$$\Rightarrow x^2 - 2xy \frac{dy}{dx} - y^2 = 0 \Rightarrow 2xy \frac{dy}{dx} = x^2 - y^2$$

$$\therefore \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

So slope of the curve  $m_1 = \frac{x^2 - y^2}{2xy}$

Differentiating eq. (ii) w.r.t.  $x$ , we get

$$3\left[x^2 \frac{dy}{dx} + y \cdot 2x\right] - 3y^2 \cdot \frac{dy}{dx} = 0$$

$$x^2 \frac{dy}{dx} + 2xy - y^2 \frac{dy}{dx} = 0 \Rightarrow (x^2 - y^2) \frac{dy}{dx} = -2xy$$

$$\therefore \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2}$$

So the slope of the curve  $m_2 = \frac{-2xy}{x^2 - y^2}$

$$\text{Now } m_1 \times m_2 = \frac{x^2 - y^2}{2xy} \times \frac{-2xy}{x^2 - y^2} = -1$$

So the angle between the curves is  $\frac{\pi}{2}$ .

Hence, the correct option is (c).

**Q46.** The interval on which the function  $f(x) = 2x^3 + 9x^2 + 12x - 1$  is decreasing is:

- (a)  $[-1, \infty)$  (b)  $[-2, -1]$   
 (c)  $(-\infty, -2]$  (d)  $[-1, 1]$

**Sol.** The given function is  $f(x) = 2x^3 + 9x^2 + 12x - 1$

$$f'(x) = 6x^2 + 18x + 12$$

For increasing and decreasing  $f'(x) = 0$

$$\therefore 6x^2 + 18x + 12 = 0$$

$$\Rightarrow x^2 + 3x + 2 = 0 \Rightarrow x^2 + 2x + x + 2 = 0$$

$$\Rightarrow x(x+2) + 1(x+2) = 0 \Rightarrow (x+2)(x+1) = 0$$

$$\Rightarrow x = -2, x = -1$$

The possible intervals are  $(-\infty, -2)$ ,  $(-2, -1)$ ,  $(-1, \infty)$

Now  $f'(x) = (x+2)(x+1)$

$$\Rightarrow f'(x)_{(-\infty, -2)} = (-)(-) = (+) \text{ increasing}$$

$$\Rightarrow f'(x)_{(-2, -1)} = (+)(-) = (-) \text{ decreasing}$$

$$\Rightarrow f'(x)_{(-1, \infty)} = (+)(+) = (+) \text{ increasing}$$

Hence, the correct option is (b).

**Q47.** Let the  $f: \mathbf{R} \rightarrow \mathbf{R}$  be defined by  $f(x) = 2x + \cos x$ , then  $f$ :

- (a) has a minimum at  $x = \pi$  (b) has a maximum at  $x = 0$   
 (c) is a decreasing function (d) is an increasing function

**Sol.** Given that  $f(x) = 2x + \cos x$

$$f'(x) = 2 - \sin x$$

Since  $f'(x) > 0 \forall x$

So  $f(x)$  is an increasing function.

Hence, the correct option is (d).

**Q48.**  $y = x(x-3)^2$  decreases for the values of  $x$  given by:

- (a)  $1 < x < 3$  (b)  $x < 0$  (c)  $x > 0$  (d)  $0 < x < \frac{3}{2}$

**Sol.** Here  $y = x(x-3)^2$

$$\frac{dy}{dx} = x \cdot 2(x-3) + (x-3)^2 \cdot 1 \Rightarrow \frac{dy}{dx} = 2x(x-3) + (x-3)^2$$

For increasing and decreasing  $\frac{dy}{dx} = 0$

$$\therefore 2x(x-3) + (x-3)^2 = 0 \Rightarrow (x-3)(2x+x-3) = 0$$

$$\Rightarrow (x-3)(3x-3) = 0 \Rightarrow 3(x-3)(x-1) = 0$$

$$\therefore x = 1, 3$$

$\therefore$  Possible intervals are  $(-\infty, 1)$ ,  $(1, 3)$ ,  $(3, \infty)$

$$\frac{dy}{dx} = (x-3)(x-1)$$

For  $(-\infty, 1) = (-)(-) = (+)$  increasing

For  $(1, 3) = (-)(+) = (-)$  decreasing

For  $(3, \infty) = (+)(+) = (+)$  increasing

So the function decreases in  $(1, 3)$  or  $1 < x < 3$

Hence, the correct option is (a).

**Q49.** The function  $f(x) = 4 \sin^3 x - 6 \sin^2 x + 12 \sin x + 100$  is strictly

(a) increasing in  $\left(\pi, \frac{3\pi}{2}\right)$  (b) decreasing in  $\left(\frac{\pi}{2}, \pi\right)$

(c) decreasing in  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  (d) decreasing in  $\left[0, \frac{\pi}{2}\right]$

**Sol.** Here,

$$f(x) = 4 \sin^3 x - 6 \sin^2 x + 12 \sin x + 100$$

$$f'(x) = 12 \sin^2 x \cdot \cos x - 12 \sin x \cos x + 12 \cos x$$

$$= 12 \cos x [\sin^2 x - \sin x + 1]$$

$$= 12 \cos x [\sin^2 x + (1 - \sin x)]$$

$$\therefore 1 - \sin x \geq 0 \text{ and } \sin^2 x \geq 0$$

$$\therefore \sin^2 x + 1 - \sin x \geq 0 \quad (\text{when } \cos x > 0)$$

Hence,  $f'(x) > 0$ , when  $\cos x > 0$  i.e.,  $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So,  $f(x)$  is increasing where  $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  and  $f'(x) < 0$

when  $\cos x < 0$  i.e.  $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence,  $f(x)$  is decreasing when  $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$$\text{As } \left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

So  $f(x)$  is decreasing in  $\left(\frac{\pi}{2}, \pi\right)$

Hence, the correct option is (b).

**Q50.** Which of the following functions is decreasing in  $\left(0, \frac{\pi}{2}\right)$ ?

- (a)  $\sin 2x$       (b)  $\tan x$       (c)  $\cos x$       (d)  $\cos 3x$

**Sol.** Here, Let  $f(x) = \cos x$ ; So,  $f'(x) = -\sin x$

$$f'(x) < 0 \text{ in } \left(0, \frac{\pi}{2}\right)$$

So  $f(x) = \cos x$  is decreasing in  $\left(0, \frac{\pi}{2}\right)$

Hence, the correct option is (c).

**Q51.** The function  $f(x) = \tan x - x$

- (a) always increases      (b) always decreases

(c) never increases

(d) sometimes increases and sometimes decreases.

**Sol.** Here,  $f(x) = \tan x - x$  So,  $f'(x) = \sec^2 x - 1$   
 $f'(x) > 0 \forall x \in \mathbb{R}$

So  $f(x)$  is always increasing

Hence, the correct option is (a).

**Q52.** If  $x$  is real, the minimum value of  $x^2 - 8x + 17$  is

- (a)  $-1$       (b)  $0$       (c)  $1$       (d)  $2$

**Sol.** Let  $f(x) = x^2 - 8x + 17$   
 $f'(x) = 2x - 8$

For local maxima and local minima,  $f'(x) = 0$

$$\therefore 2x - 8 = 0 \Rightarrow x = 4$$

So,  $x = 4$  is the point of local maxima and local minima.

$$f''(x) = 2 > 0 \text{ minima at } x = 4$$

$$\therefore f(x)_{x=4} = (4)^2 - 8(4) + 17 \\ = 16 - 32 + 17 = 33 - 32 = 1$$

So the minimum value of the function is 1

Hence, the correct option is (c).

**Q53.** The smallest value of the polynomial  $x^3 - 18x^2 + 96x$  in  $[0, 9]$  is:

- (a) 126      (b) 0      (c) 135      (d) 160

**Sol.** Let  $f(x) = x^3 - 18x^2 + 96x$ ; So,  $f'(x) = 3x^2 - 36x + 96$

For local maxima and local minima  $f'(x) = 0$

$$\therefore 3x^2 - 36x + 96 = 0$$

$$\Rightarrow x^2 - 12x + 32 = 0 \Rightarrow x^2 - 8x - 4x + 32 = 0$$

$$\Rightarrow x(x - 8) - 4(x - 8) = 0 \Rightarrow (x - 8)(x - 4) = 0$$

$$\therefore x = 8, 4 \in [0, 9]$$

So,  $x = 4, 8$  are the points of local maxima and local minima.

Now we will calculate the absolute maxima or absolute minima at  $x = 0, 4, 8, 9$

$$\therefore f(x) = x^3 - 18x^2 + 96x$$

$$f(x)_{x=0} = 0 - 0 + 0 = 0$$

$$f(x)_{x=4} = (4)^3 - 18(4)^2 + 96(4)$$

$$= 64 - 288 + 384 = 448 - 288 = 160$$

$$f(x)_{x=8} = (8)^3 - 18(8)^2 + 96(8)$$

$$= 512 - 1152 + 768 = 1280 - 1152 = 128$$

$$f(x)_{x=9} = (9)^3 - 18(9)^2 + 96(9)$$

$$= 729 - 1458 + 864 = 1593 - 1458 = 135$$

So, the absolute minimum value of  $f$  is 0 at  $x = 0$

Hence, the correct option is (b).

**Q54.** The function  $f(x) = 2x^3 - 3x^2 - 12x + 4$ , has

- (a) two points of local maximum  
 (b) two points of local minimum  
 (c) one maxima and one minima  
 (d) no maxima or minima

**Sol.** We have  $f(x) = 2x^3 - 3x^2 - 12x + 4$   
 $f'(x) = 6x^2 - 6x - 12$

For local maxima and local minima  $f'(x) = 0$

$$\therefore 6x^2 - 6x - 12 = 0$$

$$\Rightarrow x^2 - x - 2 = 0 \Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow x(x-2) + 1(x-2) = 0 \Rightarrow (x+1)(x-2) = 0$$

$$\Rightarrow x = -1, 2 \text{ are the points of local maxima and local minima}$$

Now  $f''(x) = 12x - 6$

$$f''(x)_{x=-1} = 12(-1) - 6 = -12 - 6 = -18 < 0, \text{ maxima}$$

$$f''(x)_{x=2} = 12(2) - 6 = 24 - 6 = 18 > 0 \text{ minima}$$

So, the function is maximum at  $x = -1$  and minimum at  $x = 2$

Hence, the correct option is (c).

**Q55.** The maximum value of  $\sin x \cos x$  is

- (a)  $\frac{1}{4}$       (b)  $\frac{1}{2}$       (c)  $\sqrt{2}$       (d)  $2\sqrt{2}$

**Sol.** We have  $f(x) = \sin x \cos x$

$$\Rightarrow f(x) = \frac{1}{2} \cdot 2 \sin x \cos x = \frac{1}{2} \sin 2x$$

$$f'(x) = \frac{1}{2} \cdot 2 \cos 2x$$

$$\Rightarrow f'(x) = \cos 2x$$

Now for local maxima and local minima  $f'(x) = 0$

$$\therefore \cos 2x = 0$$

$$2x = (2n+1) \frac{\pi}{2}, \quad n \in I$$

$$\Rightarrow x = (2n+1) \frac{\pi}{4}$$

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4} \dots$$

$$f''(x) = -2 \sin 2x$$

$$f''(x)_{x=\frac{\pi}{4}} = -2 \sin 2 \cdot \frac{\pi}{4} = -2 \sin \frac{\pi}{2} = -2 < 0 \text{ maxima}$$

$$f''(x)_{x=\frac{3\pi}{4}} = -2 \sin 2 \cdot \frac{3\pi}{4} = -2 \sin \frac{3\pi}{2} = 2 > 0 \text{ minima}$$

So  $f(x)$  is maximum at  $x = \frac{\pi}{4}$

$$\therefore \text{Maximum value of } f(x) = \sin \frac{\pi}{4} \cdot \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

Hence, the correct option is (b).

**Q56.** At  $x = \frac{5\pi}{6}$ ,  $f(x) = 2 \sin 3x + 3 \cos 3x$  is:

- (a) maximum                      (b) minimum  
(c) zero                              (d) neither maximum nor minimum.

**Sol.** We have  $f(x) = 2 \sin 3x + 3 \cos 3x$

$$f'(x) = 2 \cos 3x \cdot 3 - 3 \sin 3x \cdot 3 = 6 \cos 3x - 9 \sin 3x$$

$$f''(x) = -6 \sin 3x \cdot 3 - 9 \cos 3x \cdot 3$$

$$= -18 \sin 3x - 27 \cos 3x$$

$$f''\left(\frac{5\pi}{6}\right) = -18 \sin 3\left(\frac{5\pi}{6}\right) - 27 \cos 3\left(\frac{5\pi}{6}\right)$$

$$= -18 \sin \left(\frac{5\pi}{2}\right) - 27 \cos \left(\frac{5\pi}{2}\right)$$

$$= -18 \sin \left(2\pi + \frac{\pi}{2}\right) - 27 \cos \left(2\pi + \frac{\pi}{2}\right)$$

$$= -18 \sin \frac{\pi}{2} - 27 \cos \frac{\pi}{2} = -18 \cdot 1 - 27 \cdot 0$$

$$= -18 < 0 \text{ maxima}$$

Maximum value of  $f(x)$  at  $x = \frac{5\pi}{6}$

$$f\left(\frac{5\pi}{6}\right) = 2 \sin 3\left(\frac{5\pi}{6}\right) + 3 \cos 3\left(\frac{5\pi}{6}\right) = 2 \sin \frac{5\pi}{2} + 3 \cos \frac{5\pi}{2}$$

$$= 2 \sin \left(2\pi + \frac{\pi}{2}\right) + 3 \cos \left(2\pi + \frac{\pi}{2}\right) = 2 \sin \frac{\pi}{2} + 3 \cos \frac{\pi}{2} = 2$$

Hence, the correct option is (a).

**Q57.** Maximum slope of the curve  $y = -x^3 + 3x^2 + 9x - 27$  is:

- (a) 0                      (b) 12                      (c) 16                      (d) 32

**Sol.** Given that  $y = -x^3 + 3x^2 + 9x - 27$

$$\frac{dy}{dx} = -3x^2 + 6x + 9$$

∴ Slope of the given curve,

$$m = -3x^2 + 6x + 9 \quad \left(\frac{dy}{dx} = m\right)$$

$$\frac{dm}{dx} = -6x + 6$$

For local maxima and local minima,  $\frac{dm}{dx} = 0$

$$\therefore -6x + 6 = 0 \Rightarrow x = 1$$

Now  $\frac{d^2m}{dx^2} = -6 < 0$  maxima

∴ Maximum value of the slope at  $x = 1$  is

$$m_{x=1} = -3(1)^2 + 6(1) + 9 = -3 + 6 + 9 = 12$$

Hence, the correct option is (b).

**Q58.**  $f(x) = x^x$  has a stationary point at

(a)  $x = e$       (b)  $x = \frac{1}{e}$       (c)  $x = 1$       (d)  $x = \sqrt{e}$

**Sol.** We have  $f(x) = x^x$

Taking log of both sides, we have

$$\log f(x) = x \log x$$

Differentiating both sides w.r.t.  $x$ , we get

$$\frac{1}{f(x)} \cdot f'(x) = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow f'(x) = f(x) [1 + \log x] = x^x [1 + \log x]$$

To find stationary point,  $f'(x) = 0$

$$\therefore x^x [1 + \log x] = 0$$

$$x^x \neq 0 \quad \therefore 1 + \log x = 0$$

$$\Rightarrow \log x = -1 \Rightarrow x = e^{-1} \Rightarrow x = \frac{1}{e}$$

Hence, the correct option is (b).

**Q59.** The maximum value of  $\left(\frac{1}{x}\right)^x$  is:

(a)  $e$       (b)  $e^e$       (c)  $e^{1/e}$       (d)  $\left(\frac{1}{e}\right)^{1/e}$

**Sol.** Let  $f(x) = \left(\frac{1}{x}\right)^x$

Taking log on both sides, we get

$$\log [f(x)] = x \log \frac{1}{x}$$

$$\Rightarrow \log [f(x)] = x \log x^{-1} \Rightarrow \log [f(x)] = -[x \log x]$$



Differentiating both sides w.r.t.  $x$ , we get

$$\frac{1}{f(x)} \cdot f'(x) = -\left[x \cdot \frac{1}{x} + \log x \cdot 1\right] = -f(x) [1 + \log x]$$

$$\Rightarrow f'(x) = -\left(\frac{1}{x}\right)^x [1 + \log x]$$

For local maxima and local minima  $f'(x) = 0$

$$-\left(\frac{1}{x}\right)^x [1 + \log x] = 0 \Rightarrow \left(\frac{1}{x}\right)^x [1 + \log x] = 0$$

$$\left(\frac{1}{x}\right)^x \neq 0$$

$$\therefore 1 + \log x = 0 \Rightarrow \log x = -1 \Rightarrow x = e^{-1}$$

So,  $x = \frac{1}{e}$  is the stationary point.

$$\text{Now } f'(x) = -\left(\frac{1}{x}\right)^x [1 + \log x]$$

$$f''(x) = -\left[\left(\frac{1}{x}\right)^x \left(\frac{1}{x}\right) + (1 + \log x) \cdot \frac{d}{dx} (x)^x\right]$$

$$f''(x) = -\left[(e)^{1/e} (e) + \left(1 + \log \frac{1}{e}\right) \frac{d}{dx} \left(\frac{1}{e}\right)^{1/e}\right]$$

$$x = \frac{1}{e} = -e^{-1+1} < 0 \text{ maxima}$$

$\therefore$  Maximum value of the function at  $x = \frac{1}{e}$  is

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{1/e}\right)^{1/e} = e^{1/e}$$

Hence, the correct option is (c).

**Fill in the blanks in each of the following exercises 60 to 64.**

**Q60.** The curves  $y = 4x^2 + 2x - 8$  and  $y = x^3 - x + 13$  touch each other at the point \_\_\_\_\_.

**Sol.** We have

$$y = 4x^2 + 2x - 8 \quad \dots(i)$$

$$\text{and } y = x^3 - x + 13 \quad \dots(ii)$$

Differentiating eq. (i) w.r.t.  $x$ , we have

$$\frac{dy}{dx} = 8x + 2 \Rightarrow m_1 = 8x + 2$$

[ $m$  is the slope of curve (i)]

Differentiating eq. (ii) w.r.t.  $x$ , we get

$$\frac{dy}{dx} = 3x^2 - 1 \Rightarrow m_2 = 3x^2 - 1$$

[ $m_2$  is the slope of curve (ii)]

If the two curves touch each other, then  $m_1 = m_2$

$$\therefore 8x + 2 = 3x^2 - 1$$

$$\Rightarrow 3x^2 - 8x - 3 = 0 \Rightarrow 3x^2 - 9x + x - 3 = 0$$

$$\Rightarrow 3x(x - 3) + 1(x - 3) = 0 \Rightarrow (x - 3)(3x + 1) = 0$$

$$\therefore x = 3, \quad \frac{-1}{3}$$

Putting  $x = 3$  in eq. (i), we get

$$y = 4(3)^2 + 2(3) - 8 = 36 + 6 - 8 = 34$$

So, the required point is  $(3, 34)$

Now for  $x = -\frac{1}{3}$

$$\begin{aligned} y &= 4\left(\frac{-1}{3}\right)^2 + 2\left(\frac{-1}{3}\right) - 8 = 4 \times \frac{1}{9} - \frac{2}{3} - 8 \\ &= \frac{4}{9} - \frac{2}{3} - 8 = \frac{4 - 6 - 72}{9} = \frac{-74}{9} \end{aligned}$$

$\therefore$  Other required point is  $\left(-\frac{1}{3}, \frac{-74}{9}\right)$ .

Hence, the required points are  $(3, 34)$  and  $\left(-\frac{1}{3}, \frac{-74}{9}\right)$ .

**Q61.** The equation of normal to the curve  $y = \tan x$  at  $(0, 0)$  is \_\_\_\_\_.

**Sol.** We have  $y = \tan x$ . So,  $\frac{dy}{dx} = \sec^2 x$

$$\therefore \text{Slope of the normal} = \frac{-1}{\sec^2 x} = -\cos^2 x$$

at the point  $(0, 0)$  the slope  $= -\cos^2(0) = -1$

So the equation of normal at  $(0, 0)$  is  $y - 0 = -1(x - 0)$

$$\Rightarrow y = -x \Rightarrow y + x = 0$$

Hence, the required equation is  $y + x = 0$ .

**Q62.** The values of  $a$  for which the function  $f(x) = \sin x - ax + b$  increases on  $\mathbf{R}$  are \_\_\_\_\_.

**Sol.** We have  $f(x) = \sin x - ax + b \Rightarrow f'(x) = \cos x - a$

For increasing the function  $f'(x) > 0$

$$\therefore \cos x - a > 0$$

Since  $\cos x \in [-1, 1]$

$$\therefore a < -1 \Rightarrow a \in (-\infty, -1)$$

Hence, the value of  $a$  is  $(-\infty, -1)$ .

**Q63.** The function  $f(x) = \frac{2x^2 - 1}{x^4}$ ,  $x > 0$ , decreases in the interval \_\_\_\_\_.

**Sol.** We have  $f(x) = \frac{2x^2 - 1}{x^4}$

$$f'(x) = \frac{x^4(4x) - (2x^2 - 1) \cdot 4x^3}{x^8}$$

$$\Rightarrow f'(x) = \frac{4x^5 - (2x^2 - 1) \cdot 4x^3}{x^8} = \frac{4x^3[x^2 - 2x^2 + 1]}{x^8} = \frac{4(-x^2 + 1)}{x^5}$$

For decreasing the function  $f'(x) < 0$

$$\therefore \frac{4(-x^2 + 1)}{x^5} < 0 \Rightarrow -x^2 + 1 < 0 \Rightarrow x^2 > 1$$

$$\therefore x > \pm 1 \Rightarrow x \in (1, \infty)$$

Hence, the required interval is  $(1, \infty)$ .

**Q64.** The least value of the function  $f(x) = ax + \frac{b}{x}$  (where  $a > 0$ ,  $b > 0$ ,  $x > 0$ ) is \_\_\_\_\_.

**Sol.** Here,  $f(x) = ax + \frac{b}{x} \Rightarrow f'(x) = a - \frac{b}{x^2}$

For maximum and minimum value  $f'(x) = 0$

$$\therefore a - \frac{b}{x^2} = 0 \Rightarrow x^2 = \frac{b}{a} \Rightarrow x = \pm \sqrt{\frac{b}{a}}$$

Now  $f''(x) = \frac{2b}{x^3}$

$$f''(x)_{x=\sqrt{\frac{b}{a}}} = \frac{2b}{\left(\frac{b}{a}\right)^{3/2}} = 2 \frac{a^{3/2}}{b^{1/2}} > 0 \quad (\because a, b > 0)$$

Hence, minima

So the least value of the function at  $x = \sqrt{\frac{b}{a}}$  is

$$f\left(\sqrt{\frac{b}{a}}\right) = a \cdot \sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\frac{b}{a}}} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

Hence, least value is  $2\sqrt{ab}$ .

□□□