7 Integrals

7.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Verify the following:

Q1. $\int \frac{2x-1}{2x+3}\,dx = x - \log\left|(2x+3)^2\right| + C$

Sol. L.H.S. $= \int \frac{2x-1}{2x+3}\,dx$

$\Rightarrow \int \left(1 - \frac{4}{2x+3}\right) dx$ [Dividing the numerator by the denominator]

$\Rightarrow \int 1.dx - 4\int \frac{1}{2x+3}\,dx \Rightarrow \int 1.dx - \frac{4}{2}\int \frac{1}{x+\frac{3}{2}}\,dx$

$\Rightarrow \int 1.dx - 2\int \frac{1}{x+\frac{3}{2}}\,dx \Rightarrow x - 2\log\left|x+\frac{3}{2}\right| + C$

$\Rightarrow x - 2\log\left|\frac{2x+3}{2}\right| + C \Rightarrow x - \log\left|\left(\frac{2x+3}{2}\right)^2\right| + C$

[$\because n \log m = \log m^n$]

$\Rightarrow x - \log\left|(2x+3)^2\right| - \log 2^2 + C$

$\Rightarrow x - \log\left|(2x+3)^2\right| + C_1 \Rightarrow$ R.H.S. [where $C_1 = C - \log 2^2$]

L.H.S. = R.H.S.

Hence proved.

Q2. $\int \frac{2x+3}{x^2+3x}\,dx = \log\left|x^2+3x\right| + C$

Sol. L.H.S. $= \int \frac{2x+3}{x^2+3x}\,dx$

Put $x^2 + 3x = t$

$\therefore$ $(2x+3)\,dx = dt$

$\Rightarrow$ $\int \frac{dt}{t} = \log|t| \Rightarrow \log\left|x^2+3x\right| + C =$ R.H.S.

L.H.S. = R.H.S.

Hence verified.

Evaluate the following:

Q3. $\int \frac{(x^2+2)}{x+1}dx$

Sol. Let $I = \int \frac{x^2+2}{x+1}dx$

$$x+1\overline{)\,x^2+2\,}(x-1$$
$$\frac{x^2+x}{\quad(-)\;(-)}$$
$$-x+2$$
$$\frac{-x-1}{\quad(+)\;(+)}$$
$$3$$

$\therefore$ $I = \int\left[(x-1)+\frac{3}{x+1}\right]dx$

$= \int(x-1)\,dx + 3\int\frac{1}{x+1}dx$

$= \frac{x^2}{2} - x + 3\log|x+1| + C$

Hence, the required solution is $\frac{x^2}{2} - x + 3\log|x+1| + C$.

Q4. $\int \frac{e^{6\log x}-e^{5\log x}}{e^{4\log x}-e^{3\log x}}dx$

Sol. Let $I = \int \frac{e^{6\log x}-e^{5\log x}}{e^{4\log x}-e^{3\log x}}dx = \int \frac{e^{\log x^6}-e^{\log x^5}}{e^{\log x^4}-e^{\log x^3}}dx$

$= \int \frac{x^6-x^5}{x^4-x^3}dx = \int \frac{x^2(x^4-x^3)}{x^4-x^3}dx = \int x^2dx = \frac{1}{3}x^3 + C$

Hence, the required solution is $\frac{1}{3}x^3 + C$.

Q5. $\int \frac{(1+\cos x)}{x+\sin x}dx$

Sol. Let $I = \int \frac{1+\cos x}{x+\sin x}dx$

Put $x + \sin x = t \Rightarrow (1+\cos x)\,dx = dt$

$\therefore$ $I = \int \frac{dt}{t} = \log|t| = \log|x+\sin x| + C$

Hence, the required solution is $\log|x+\sin x| + C$.

Q6. $\int \frac{dx}{1+\cos x}$

Sol. Let $I = \int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2 x/2}$ $\left[\because 1+\cos x = 2\cos^2\frac{x}{2}\right]$

$= \frac{1}{2}\int \sec^2\frac{x}{2}dx = \frac{1}{2}.2\tan\frac{x}{2} + C = \tan\frac{x}{2} + C$

Hence, the required solution is $\tan\frac{x}{2} + C$.

Q7. $\int \tan^2 x . \sec^4 x \, dx$

Sol. Let $I = \int \tan^2 x . \sec^4 x \, dx$

$$= \int \tan^2 x \sec^2 x . \sec^2 x \, dx = \int \tan^2 x (1 + \tan^2 x) . \sec^2 x \, dx$$

Put $\tan x = t$, $\therefore \sec^2 x \, dx = dt$

$$\therefore \quad I = \int t^2 (1 + t^2) \, dt = \int (t^2 + t^4) \, dt = \int t^2 \, dt + \int t^4 \, dt$$

$$= \frac{1}{3} t^3 + \frac{1}{5} t^5 = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$$

Hence, the required solution is $\frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$.

Q8. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

Sol. Let $I = \int \frac{\sin x + \cos x}{\sqrt{1 + 2 \sin x \cos x}} dx$

$$= \int \frac{(\sin x + \cos x)}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}} dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx = \int \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$= \int 1 \, dx$$

$$= x + C$$

Hence, the required solution is $x + C$.

Q9. $\int \sqrt{1 + \sin x} \, dx$

Sol. Let $I = \int \sqrt{1 + \sin x} \, dx$

$$= \int \sqrt{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} \right)} dx$$

$$= \int \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2} dx = \int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx$$

$$= \int \sin \frac{x}{2} dx + \int \cos \frac{x}{2} dx = -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} + C$$

$$= 2 \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right) + C$$

Hence, the required solution is $2 \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right) + C$.

Q10. $\int \frac{x}{\sqrt{x} + 1} dx$ (Hint: Put $\sqrt{x} = z$)

Sol. $I = \int \frac{x}{\sqrt{x}+1} dx$

Put $\sqrt{x} = t \Rightarrow x = t^2 \quad \therefore dx = 2t \, . \, dt$

$$\therefore I = \int \frac{t^2 . 2t . dt}{t+1} = 2\int \frac{t^3}{t+1} dt = 2\int \frac{t^3+1-1}{t+1} dt$$

$$= 2\int \frac{t^3+1}{t+1} dt - 2\int \frac{1}{t+1} dt$$

$$= 2\int \frac{(t+1)(t^2-t+1)}{t+1} dt - 2\int \frac{1}{t+1} dt$$

$$= 2\int (t^2-t+1)\, dt - 2\int \frac{1}{t+1} dt$$

$$= 2\left[\frac{t^3}{3} - \frac{t^2}{2} + t\right] - 2\log|t+1|$$

$$= 2\left[\frac{x^{3/2}}{3} - \frac{x}{2} + \sqrt{x}\right] - 2\log|\sqrt{x}+1| + C$$

$$= 2\left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log|\sqrt{x}+1|\right] + C$$

Hence, $I = 2\left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log|\sqrt{x}+1|\right] + C$

Q11. $\int \sqrt{\frac{a+x}{a-x}} dx$

Sol. Let $I = \int \sqrt{\frac{a+x}{a-x}} dx$

$$= \int \sqrt{\frac{a+x}{a-x} \times \frac{a+x}{a+x}} dx = \int \frac{a+x}{\sqrt{(a-x)(a+x)}} dx$$

$$= \int \frac{a+x}{\sqrt{a^2-x^2}} dx = \int \frac{a}{\sqrt{a^2-x^2}} dx + \int \frac{x}{\sqrt{a^2-x^2}} dx$$

Let $I = I_1 + I_2$

Now $I_1 = \int \frac{a}{\sqrt{a^2-x^2}} dx = a . \sin^{-1}\frac{x}{a} + C_1$

and $I_2 = \int \frac{x}{\sqrt{a^2-x^2}} dx$

Put $a^2 - x^2 = t \Rightarrow -2x\, dx = dt$

$$x\, dx = \frac{dt}{-2}$$

$$\therefore \quad I_2 = -\frac{1}{2}\int\frac{dt}{\sqrt{t}} = -\frac{1}{2}\times 2\sqrt{t} = -\sqrt{a^2-x^2}+C_2$$

Since $\quad I = I_1 + I_2$

$$= a\sin^{-1}\frac{x}{a} + C_1 - \sqrt{a^2-x^2} + C_2$$

$$\therefore \quad I = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + (C_1+C_2)$$

Hence, $\quad I = a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C \qquad [C = C_1 + C_2]$

Alternate method:

$$I = \int\sqrt{\frac{a+x}{a-x}}\,dx$$

Put $\quad x = a\cos 2\theta$

$\therefore \quad dx = a(-2\sin 2\theta)\,d\theta = -2a\sin 2\theta\,d\theta$

$$\therefore \quad I = \int\sqrt{\frac{a+a\cos 2\theta}{a-a\cos 2\theta}}.(-2a\sin 2\theta)\,d\theta$$

$$= \int\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}.(-2a\sin 2\theta)\,d\theta$$

$$= -2a\int\sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}.\sin 2\theta\,d\theta$$

$$= -2a\int\sqrt{\frac{\cos^2\theta}{\sin^2\theta}}.2\sin\theta\cos\theta\,d\theta$$

$$= -2a\int\frac{\cos\theta}{\sin\theta}.2\sin\theta\cos\theta\,d\theta$$

$$= -4a\int\cos\theta\cos\theta\,d\theta = -4a\int\cos^2\theta\,d\theta$$

$$= -4a\int\frac{1+\cos 2\theta}{2}\,d\theta = -2a\int(1+\cos 2\theta)\,d\theta$$

$$= -2a\left[\int 1\,d\theta + \int\cos 2\theta\,d\theta\right] = -2a\left[\theta + \frac{1}{2}\sin 2\theta\right]$$

Now $\quad x = a\cos 2\theta$

$$\frac{x}{a} = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}\frac{x}{a} \Rightarrow \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}$$

$$\sin 2\theta = \sqrt{1-\cos^2 2\theta} = \sqrt{1-\frac{x^2}{a^2}} = \frac{\sqrt{a^2-x^2}}{a}$$

$$\therefore \quad I = -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} + \frac{1}{2}\frac{\sqrt{a^2-x^2}}{a}\right] + C_1$$

$$= -a\cos^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C_1$$

$$= -a\left[\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right] - \sqrt{a^2 - x^2} + C_1$$

$$= \frac{-\pi a}{2} + a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C_1$$

$$= a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + \left(C_1 - \frac{\pi a}{2}\right)$$

$$= a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C \qquad \left[C = \left(C_1 - \frac{\pi a}{2}\right)\right]$$

Hence, $I = a\sin^{-1}\frac{x}{a} - \sqrt{a^2 - x^2} + C$

Q12. $\int \frac{x^{1/2}}{1 + x^{3/4}}\,dx$ (Hint: Put $x = z^4$)

Sol. Let $I = \int \frac{x^{1/2}}{1 + x^{3/4}}\,dx$

Put $x = t^4 \Rightarrow dx = 4t^3\,dt$

$$= \int \frac{t^2 . 4t^3}{1 + t^3}\,dt = 4\int \frac{t^5}{1 + t^3}\,dt$$

$$\begin{array}{r|l} t^3 + 1 & t^5 \quad (t^2 \\ & \underline{t^5 + t^2} \\ & (-)\ (-) \\ & -t^2 \end{array}$$

$$= 4\int \left(t^2 - \frac{t^2}{t^3 + 1}\right)dt = 4\int t^2\,dt - 4\int \frac{t^2}{t^3 + 1}.dt$$

$$I = I_1 - I_2$$

Now $I_1 = 4\int t^2\,dt = 4.\frac{t^3}{3} + C_1 = \frac{4}{3}x^{3/4} + C_1$

$$I_2 = 4\int \frac{t^2}{t^3 + 1}\,dt$$

Put $t^3 + 1 = z \Rightarrow 3t^2\,dt = dz$

$$t^2\,dt = \frac{1}{3}dz$$

$\therefore$ $I_2 = \frac{4}{3}\int \frac{dz}{z} = \frac{4}{3}\log|z| + C_2 = \frac{4}{3}\log\left|t^3 + 1\right| + C_2$

$$= \frac{4}{3}\log\left|(x)^{3/4} + 1\right| + C_2$$

$\therefore$ $I = I_1 - I_2$

$$= \frac{4}{3}x^{3/4} + C_1 - \frac{4}{3}\log\left|(x)^{3/4} + 1\right| - C_2$$

$$= \frac{4}{3}\left[x^{3/4} - \log\left|(x)^{3/4} + 1\right|\right] + C_1 - C_2$$

Hence, $I = \frac{4}{3}\left[x^{3/4} - \log\left|(x)^{3/4} + 1\right|\right] + C$ $[\because C = C_1 - C_2]$

Q13. $\int \frac{\sqrt{1+x^2}}{x^4}\,dx$

Sol. Let $I = \int \frac{\sqrt{1+x^2}}{x^4}\,dx = \int \sqrt{\frac{1+x^2}{x^2}}\cdot\frac{1}{x^3}\,dx = \int \sqrt{\frac{1}{x^2}+1}\cdot\frac{1}{x^3}\,dx$

Put $\frac{1}{x^2} + 1 = t^2$

$\frac{-2}{x^3}\,dx = 2t\,dt \Rightarrow \frac{dx}{x^3} = -t\,dt$

$\therefore$ $I = \int t(\ \ t\,dt) = -\int t^2\,dt = -\frac{1}{3}t^3 + C$

Hence, $I = -\frac{1}{3}\left(\frac{1}{x^2}+1\right)^{3/2} + C$

Q14. $\int \frac{dx}{\sqrt{16-9x^2}}$

Sol. Let $I = \int \frac{dx}{\sqrt{16-9x^2}}$

$= \frac{1}{3}\int \frac{dx}{\sqrt{\frac{16}{9} - x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{4}{3}\right)^2 - x^2}}$

$= \frac{1}{3}\sin^{-1}\frac{x}{4/3} + C$ $\left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C\right]$

$= \frac{1}{3}\sin^{-1}\frac{3x}{4} + C$

Hence, $I = \frac{1}{3}\sin^{-1}\frac{3x}{4} + C.$

Q15. $\int \frac{dt}{\sqrt{3t-2t^2}}$

Sol. Let $I = \int \frac{dt}{\sqrt{3t-2t^2}} = \int \frac{dt}{\sqrt{-2\left(t^2 - \frac{3}{2}t\right)}}$

$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left(t^2 - \frac{3}{2}t + \frac{9}{16} - \frac{9}{16}\right)}}$ [Making perfect square]

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t-\frac{3}{4}\right)^2 - \frac{9}{16}\right]}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\frac{9}{16} - \left(t-\frac{3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2 - \left(t-\frac{3}{4}\right)^2}} = \frac{1}{\sqrt{2}}.\sin^{-1}\frac{t-\frac{3}{4}}{\frac{3}{4}} + C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{4t-3}{3} + C$$

Hence, $I = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4t-3}{3}\right) + C.$

Q16. $\int \frac{3x-1}{\sqrt{x^2+9}}dx$

Sol. Let $I = \int \frac{3x-1}{\sqrt{x^2+9}}dx = \int \frac{3x}{\sqrt{x^2+9}}dx - \int \frac{1}{\sqrt{x^2+9}}dx$

$I = I_1 - I_2$

Now $I_1 = \int \frac{3x}{\sqrt{x^2+9}}dx$

Put $x^2 + 9 = t \Rightarrow 2x\,dx = dt$

$x\,dx = -dt$

$$\therefore I_1 = \frac{3}{2}\int \frac{dt}{\sqrt{t}} = \frac{3}{2}.2\sqrt{t} + C_1 = 3\sqrt{x^2+9} + C_1$$

$$I_2 = \int \frac{1}{\sqrt{x^2+9}}dx = \int \frac{1}{\sqrt{x^2+(3)^2}}dx = \log\left|x + \sqrt{x^2+(3)^2}\right| + C_2$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left|x+\sqrt{x^2+a^2}\right| + C\right]$$

$$= \log\left|x+\sqrt{x^2+9}\right| + C_2$$

$\therefore I = I_1 - I_2$

$$= 3\sqrt{x^2+9} + C_1 - \log\left|x+\sqrt{x^2+9}\right| - C_2$$

$$= 3\sqrt{x^2+9} - \log\left|x+\sqrt{x^2+9}\right| + (C_1 - C_2)$$

Hence, $I = 3\sqrt{x^2+9} - \log\left|x+\sqrt{x^2+9}\right| + C$

Q17. $\int \sqrt{5-2x+x^2}\,dx$

Sol. Let $I = \int \sqrt{5-2x+x^2}\,dx = \int \sqrt{x^2-2x+5}\,dx$

$= \int \sqrt{x^2-2x+1-1+5}\,dx$ (Making perfect square)

$= \int \sqrt{(x-1)^2+4}\,dx = \int \sqrt{(x-1)^2+(2)^2}\,dx$

$= \frac{x-1}{2}\sqrt{(x-1)^2+(2)^2} + \frac{4}{2}\log\left|(x-1)+\sqrt{(x-1)^2+(2)^2}\right| + C$

$\left[\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\left\{\log\left|x+\sqrt{x^2+a^2}\right|\right\} + C\right]$

$= \frac{x-1}{2}\sqrt{x^2+1-2x+4} + 2\log\left|(x-1)+\sqrt{x^2+1-2x+4}\right| + C$

$= \frac{x-1}{2}\sqrt{x^2-2x+5} + 2\log\left|(x-1)+\sqrt{x^2-2x+5}\right| + C$

Hence,

$I = \frac{x-1}{2}\sqrt{x^2-2x+5} + 2\log\left|(x-1)+\sqrt{x^2-2x+5}\right| + C$

Q18. $\int \frac{x}{x^4-1}\,dx$

Sol. Let $I = \int \frac{x}{x^4-1}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$\frac{1}{2}\int \frac{dt}{t^2-1} = \frac{1}{2}\int \frac{dt}{t^2-(1)^2} = \frac{1}{2}\cdot\frac{1}{2\cdot 1}\log\left|\frac{t-1}{t+1}\right| + C$

$\left[\because \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$

$= \frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right| + C$

Hence, $I = \frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right| + C.$

Q19. $\int \frac{x^2}{1-x^4}\,dx$ (Put $x^2 = t$)

Sol. Let $I = \int \frac{x^2}{1-x^4}\,dx = \int \frac{x^2}{(1-x^2)(1+x^2)}\,dx$

Put $x^2 = t$ for the purpose of partial fractions.

We get $\dfrac{t}{(1-t)(1+t)}$

Resolving into partial fractions we put

$$\frac{t}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t}$$

[where A and B are arbitrary constants]

$$\Rightarrow \quad \frac{t}{(1-t)(1+t)} = \frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$$

$$\Rightarrow \quad t = A + At + B - Bt$$

Comparing the like terms, we get $A - B = 1$ and $A + B = 0$

Solving the above equations, we have $A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$

$$\therefore \quad I = \int \frac{1/2}{1-x^2}\,dx + \int \frac{-1/2}{1+x^2}\,dx \qquad \text{(Putting } t = x^2\text{)}$$

$$= \frac{1}{2}\cdot\frac{1}{2.1}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C$$

Hence, $\quad I = \dfrac{1}{4}\log\left|\dfrac{1+x}{1-x}\right| - \dfrac{1}{2}\tan^{-1}x + C.$

Q20. $\int\sqrt{2ax - x^2}\,dx$

Sol. Let $\quad I = \int\sqrt{2ax - x^2}\,dx$

$$= \int\sqrt{-(x^2 - 2ax)}\,dx = \int\sqrt{-(x^2 - 2ax + a^2 - a^2)}\,dx$$

$$= \int\sqrt{-\left[(x-a)^2 - a^2\right]}\,dx = \int\sqrt{a^2 - (x-a)^2}\,dx$$

$$= \frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C\right]$$

$$= \frac{x-a}{2}\sqrt{a^2-(x^2-2ax+a^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

$$= \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

Hence, $I = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C.$

Q21. $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

Sol. Let $I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$I = \int \frac{\sin^{-1}(\sin\theta)}{(1-\sin^2\theta)^{3/2}}.\cos\theta\, d\theta$$

$$= \int \frac{\theta.\cos\theta\, d\theta}{(\cos^2\theta)^{3/2}} = \int \frac{\theta.\cos\theta}{\cos^3\theta} d\theta$$

$$= \int \frac{\theta}{\cos^2\theta} d\theta = \int \underset{\text{I}}{\theta} \underset{\text{II}}{\sec^2\theta}\, d\theta$$

$$= \theta.\int \sec^2\theta\, d\theta - \int \left(D(\theta).\int \sec^2\theta\, d\theta\right) d\theta$$

$$\left[\because \int \underset{\text{I}}{u}.\underset{\text{II}}{v}\, dx = u.\int v\, dx - \int \left(D(u)\int v\, dv\right) dv + C\right]$$

$$= \theta.\tan\theta - \int 1.\tan\theta\, d\theta$$

$$= \theta.\tan\theta - \log\sec\theta + C$$

$$= \sin^{-1} x.\frac{x}{\sqrt{1-x^2}} - \log\left|\sqrt{1-x^2}\right| + C$$

$$\left[\begin{array}{l}\text{when } x = \sin\theta \\ \therefore \tan\theta = \frac{x}{\sqrt{1-x^2}} \text{ and } \sec\theta = \sqrt{1-x^2}\end{array}\right]$$

Hence, $I = \frac{x\sin^{-1} x}{\sqrt{1-x^2}} - \log\left|\sqrt{1-x^2}\right| + C$

Q22. $\int \frac{(\cos 5x + \cos 4x)}{1-2\cos 3x} dx$

Sol. Let $I = \int \frac{\cos 5x + \cos 4x}{1-2\cos 3x} dx = \int \frac{2\cos\frac{5x+4x}{2}.\cos\frac{5x-4x}{2}}{1-2\left(2\cos^2\frac{3x}{2}-1\right)} dx$

$$= \int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{1-4\cos^2\frac{3x}{2}+2} dx = \int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{3-4\cos^2\frac{3x}{2}} dx$$

$$= -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{4\cos^2\frac{3x}{2}-3}\,dx = -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}.\cos\frac{3x}{2}}{4\cos^3\frac{3x}{2}-3\cos\frac{3x}{2}}\,dx$$

$\left[\text{Multiplying and dividing by } \cos\frac{3x}{2}\right]$

$$= -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}.\cos\frac{3x}{2}}{\cos 3.\frac{3x}{2}}\,dx$$

$[\because \cos 3x = 4\cos^3 x - 3\cos x]$

$$= -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}.\cos\frac{3x}{2}}{\cos\frac{9x}{2}}\,dx = -\int 2\cos\frac{3x}{2}.\cos\frac{x}{2}\,dx$$

$$= -\int\left[\cos\left(\frac{3x}{2}+\frac{x}{2}\right)+\cos\left(\frac{3x}{2}-\frac{x}{2}\right)\right]dx$$

$$= -\int(\cos 2x + \cos x)\,dx$$

$[\because 2\cos A\cos B = \cos(A+B)+\cos(A-B)]$

$$= -\int \cos 2x\,dx - \int \cos x\,dx = -\frac{1}{2}\sin 2x - \sin x + C$$

Hence, $I = -\left[\frac{1}{2}\sin 2x + \sin x\right] + C.$

Q23. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}\,dx$

Sol. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}\,dx = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x.\cos^2 x}\,dx$

$$= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x.\cos^2 x}\,dx$$

$[\because a^3 + b^3 = (a+b)^3 - 3ab\,(a+b)]$

$$= \int \frac{(1)^3 - 3\sin^2 x\cos^2 x.(1)}{\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{1 - 3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\,dx$$

$$= \int\left(\frac{1}{\sin^2 x\cos^2 x} - \frac{3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$

$$= \int\left(\frac{1}{\sin^2 x \cos^2 x} - 3\right)dx = \int\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - 3\right)dx$$

$$= \int\left[\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) - 3\right]dx$$

$$= \int(\sec^2 x + \operatorname{cosec}^2 x - 3)\,dx$$

$$= \int \sec^2 x\,dx + \int \operatorname{cosec}^2 x\,dx - 3\int 1\,dx$$

$$= \tan x - \cot x - 3x + C$$

Hence, $I = \tan x - \cot x - 3x + C$.

Q24. $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\,dx$

Sol. Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\,dx = \int \frac{x^{1/2}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}\,dx$

Put $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2}\,dx = dt \Rightarrow x^{1/2}\,dx = \frac{2}{3}dt$

$\therefore$ $I = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2 - (t)^2}}$

$$= \frac{2}{3}\sin^{-1}\frac{t}{a^{3/2}} + C = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

Hence, $I = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C.$

Q25. $\int \frac{\cos x - \cos 2x}{1 - \cos x}\,dx$

Sol. Let $I = \int \frac{\cos x - \cos 2x}{1 - \cos x}\,dx$

$$= \int \frac{2\sin\frac{x + 2x}{2}.\sin\left(\frac{2x - x}{2}\right)}{2\sin^2 x/2}\,dx$$

$$\left[\because \cos C - \cos D = 2\sin\frac{C + D}{2}.\sin\frac{D - C}{2}\right]$$

$$= \int \frac{2\sin\frac{3x}{2}.\sin\frac{x}{2}}{2\sin^2\frac{x}{2}}\,dx = \int \frac{\sin\frac{3x}{2}}{\sin\frac{x}{2}}\,dx = \int \frac{\sin 3\left(\frac{x}{2}\right)}{\sin\frac{x}{2}}\,dx$$

$$= \int \frac{3\sin\frac{x}{2} - 4\sin^3\frac{x}{2}}{\sin\frac{x}{2}}\,dx \quad [\sin 3x = 3\sin x - 4\sin^3 x]$$

$$= \int \frac{\sin\frac{x}{2}\left(3 - 4\sin^2\frac{x}{2}\right)}{\sin\frac{x}{2}} dx = \int \left(3 - 4\sin^2\frac{x}{2}\right) dx$$

$$= \int [3 - 2(1 - \cos x)]\, dx \qquad \left[\because 2\sin^2\frac{x}{2} = 1 - \cos x\right]$$

$$= \int (3 - 2 + 2\cos x)\, dx = \int (1 + 2\cos x)\, dx$$

$$= x + 2\sin x + C$$

Hence, $I = x + 2\sin x + C.$

Q26. $\int \frac{dx}{x\sqrt{x^4 - 1}}$ (Hint: Put $x^2 = \sec\theta$)

Sol. Let $I = \int \frac{dx}{x\sqrt{x^4 - 1}} = \int \frac{x\, dx}{x^2\sqrt{x^4 - 1}}$

Put $x^2 = \sec\theta$

$\therefore$ $2x\, dx = \sec\theta \tan\theta\, d\theta$

$x\, dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$\therefore$ $I = \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta - 1}} d\theta$

$$= \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sec\theta.\tan\theta} d\theta = \frac{1}{2}\int 1\, d\theta = \frac{1}{2}\theta + C$$

So $I = \frac{1}{2}\sec^{-1} x^2 + C$

Hence, $I = \frac{1}{2}\sec^{-1} x^2 + C.$

Evaluate the following as a limit of sum:

Q27. $\int_0^2 (x^2 + 3)\, dx$

Sol. Let $I = \int_0^2 (x^2 + 3)\, dx$

Using the formula,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + \overline{n-1}h)\right]$$

where $h = \frac{b-a}{n}$

Here, $a = 0$ and $b = 2$

$\therefore \quad h = \dfrac{2-0}{n} \quad \therefore nh = 2$

Here,
$$f(x) = x^2 + 3$$
$$f(0) = 0 + 3 = 3$$
$$f(0+h) = (0+h)^2 + 3 = h^2 + 3$$
$$f(0+2h) = (0+2h)^2 + 3 = 4h^2 + 3$$

....

....

$$f(0+\overline{n-1}h) = (0+\overline{n-1}h)^2 + 3(n-1)^2h^2 + 3$$

Now

$$\int_0^2 (x^2+3)\,dx$$

$$= \lim_{h\to 0} h\left[3 + h^2 + 3 + 4h^2 + 3 + \ldots + (n-1)^2h^2 + 3\right]$$

$$= \lim_{h\to 0} h\left[(3+3+3+\ldots+n) + \left\{h^2 + 4h^2 + \ldots + (n-1)^2h^2\right\}\right]$$

$$= \lim_{h\to 0} h\left[3n + h^2\left\{1 + 4 + \ldots + (n-1)^2\right\}\right]$$

$$= \lim_{h\to 0} h\left[3n + h^2\frac{n(n-1)(2n-1)}{6}\right]$$

$$\left[\because 1 + 4 + 9 + \ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[3nh + \frac{h^3n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[3nh + \frac{nh(nh-h)(2nh-h)}{6}\right]$$

$$= \left[3\times 2 + \frac{2(2-0)(2\times 2-0)}{6}\right] \qquad \left[\begin{matrix}\because nh = 2\\ h = 0\end{matrix}\right]$$

$$= \left[6 + \frac{2\times 2\times 4}{6}\right] = 6 + \frac{8}{3} = \frac{26}{3}$$

Hence, $\displaystyle\int_0^2 (x^2+3)\,dx = \frac{26}{3}$.

Q28. $\displaystyle\int_0^2 e^x\,dx$

Sol. Let $\quad \displaystyle I = \int_0^2 e^x\,dx$

Here, $a = 0$ and $b = 2$ $\therefore h = \dfrac{b-a}{n} \Rightarrow h = \dfrac{2-0}{n}$ $\therefore nh = 2$

Here $\quad f(x) = e^x$

$f(0) = e^0 = 1$

$f(0+h) = e^{0+h} = e^h$

$f(0+2h) = e^{0+2h} = e^{2h}$

....

....

$f(0+\overline{n-1}h) = e^{0+(n-1)h} = e^{(n-1)h}$

Using $\int_a^b f(x)\,dx$

$$= \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+\overline{n-1}h)\right]$$

$$\therefore \int_0^2 e^x\,dx = \lim_{h\to 0} h\left[1+e^h+e^{2h}+\ldots+e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} h\left[\frac{1(e^{nh}-1)}{e^h-1}\right]$$

$$\left[\because a+ar+ar^2+\ldots+ar^{n-1} = \frac{a(r^n-1)}{r-1}\right]$$

$$= \lim_{h\to 0} \frac{e^{nh}-1}{\frac{e^h-1}{h}} = \frac{e^2-1}{1} = e^2-1 \left[\because \lim_{x\to 0}\frac{e^x-1}{x} = 1\right]$$

Hence, $\mathrm{I} = e^2 - 1$.

Evaluate the following:

Q29. $\displaystyle\int_0^1 \frac{dx}{e^x+e^{-x}}$

Sol. Let $\quad \mathrm{I} = \displaystyle\int_0^1 \frac{dx}{e^x+e^{-x}}$

$$= \int_0^1 \frac{dx}{e^x+\frac{1}{e^x}} = \int_0^1 \frac{dx}{\frac{e^{2x}+1}{e^x}} = \int_0^1 \frac{e^x\,dx}{e^{2x}+1}$$

Put $\quad e^x = t \Rightarrow e^x\,dx = dt$

Changing the limit, we have

When $x = 0$ $\quad \therefore t = e^0 = 1$

When $x = 1$ $\quad \therefore t = e^1 = e$

$$\therefore \text{I} = \int_1^e \frac{dt}{t^2+1} = [\tan^{-1} t]_1^e = \left[\tan^{-1} e - \tan^{-1}(1)\right] = \tan^{-1} e - \frac{\pi}{4}$$

Hence, $\text{I} = \tan^{-1} e - \frac{\pi}{4}$.

Q30. $\displaystyle\int_0^{\pi/2} \frac{\tan x}{1+m^2\tan^2 x}\,dx$

Sol. Let $\displaystyle \text{I} = \int_0^{\pi/2} \frac{\tan x}{1+m^2\tan^2 x}\,dx$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2\frac{\sin^2 x}{\cos^2 x}}\,dx = \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x + m^2\sin^2 x}{\cos^2 x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x\cos x}{\cos^2 x + m^2\sin^2 x}\,dx = \int_0^{\pi/2} \frac{\sin x\cos x}{1-\sin^2 x + m^2\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x\cos x}{1-\sin^2 x\,(1-m^2)}\,dx$$

Put $\sin^2 x = t$

$2\sin x\cos x\,dx = dt$

$\sin x\cos x\,dx = \frac{dt}{2}$

Changing the limits we get,

When $x = 0 \;\therefore\; t = \sin^2 0 = 0$; When $x = \frac{\pi}{2} \;\therefore\; t = \sin^2\frac{\pi}{2} = 1$

$$\therefore \text{I} = \frac{1}{2}\int_0^1 \frac{dt}{1-(1-m^2)t}$$

$$\text{I} = \frac{1}{2}\int_0^1 \frac{dt}{1+(m^2-1)t} = \frac{1}{2}\left[\frac{\log[1+(m^2-1)t]}{m^2-1}\right]_0^1$$

$$= \frac{1}{2(m^2-1)}\left[\log(1+m^2-1) - \log(1)\right] = \frac{\log|m^2|}{2(m^2-1)}$$

Hence, $\text{I} = \frac{\log|m^2|}{2(m^2-1)} = \frac{\log|m|}{m^2-1}$.

Q31. $\displaystyle\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

Sol. Let $\displaystyle \text{I} = \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

$$= \int_1^2 \frac{dx}{\sqrt{2x - x^2 - 2 + x}} = \int_1^2 \frac{dx}{\sqrt{-x^2 + 3x - 2}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-(x^2 - 3x + 2)}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2\right)}}$$ [Making perfect square]

$$= \int_1^2 \frac{dx}{\sqrt{-\left[\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}\right]}} = \int_1^2 \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{3}{2}\right)^2}}$$

$$= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{1}{2}}\right)\right]_1^2$$

$$= \left[\sin^{-1}\left(\frac{2x - 3}{1}\right)\right]_1^2 = \sin^{-1}(4 - 3) - \sin^{-1}(2 - 3)$$

$$= \sin^{-1}(1) - \sin^{-1}(-1) = \sin^{-1}(1) + \sin^{-1}(1)$$

$$= 2\sin^{-1}(1) = 2 \times \frac{\pi}{2} = \pi$$

Hence, $I = \pi$.

Q32. $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1 + x^2}}$

Sol. Let $\displaystyle I = \int_0^1 \frac{x\,dx}{\sqrt{1 + x^2}}$

Put $1 + x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

Changing the limits, we have

When $x = 0 \quad \therefore t = 1$

When $x = 1 \quad \therefore t = 2$

$$\therefore \quad I = \frac{1}{2}\int_1^2 \frac{dt}{\sqrt{t}} = \frac{1}{2} \cdot 2\left[t^{1/2}\right]_1^2 = \sqrt{2} - 1$$

Hence, $I = \sqrt{2} - 1$.

Q33. $\displaystyle\int_0^{\pi} x \sin x \cos^2 x\,dx$

Sol. Let $$I = \int_0^{\pi} x \sin x \cos^2 x \, dx \quad ...(i)$$

$$I = \int_0^{\pi} (\pi - x) \sin(\pi - x) \cos^2(\pi - x) \, dx$$

$$I = \int_0^{\pi} (\pi - x) \sin x \cos^2 x \, dx \quad ...(ii)$$

Adding (*i*) and (*ii*) we get,

$$2I = \int_0^{\pi} \left[x \sin x \cos^2 x + (\pi - x) \sin x \cos^2 x\right] dx$$

$$2I = \int_0^{\pi} \sin x \cos^2 x . (x + \pi - x) \, dx$$

$$2I = \int_0^{\pi} \pi \sin x \cos^2 x \, dx = \pi \int_0^{\pi} \sin x \cos^2 x \, dx$$

Put $\cos x = t \Rightarrow -\sin x \, dx = dt \Rightarrow \sin x \, dx = -dt$

Changing the limits, we have

When $x = 0$, $t = \cos 0 = 1$; When $x = \pi$, $t = \cos \pi = -1$

$$2I = \pi \int_1^{-1} -t^2 \, dt = -\pi \int_1^{-1} t^2 \, dt$$

$$2I = \pi \int_{-1}^{1} t^2 \, dt \qquad \left[\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx\right]$$

$$2I = \pi \left[\frac{t^3}{3}\right]_{-1}^{1} = \pi \left[\frac{1}{3} + \frac{1}{3}\right] = \pi \left(\frac{2}{3}\right)$$

$$\therefore \quad I = \frac{\pi}{3}$$

Q34. $\displaystyle\int_0^{1/2} \frac{dx}{(1 + x^2)\sqrt{1 - x^2}}$ (Hint: Let $x = \sin \theta$)

Sol. Let $$I = \int_0^{1/2} \frac{dx}{(1 + x^2)\sqrt{1 - x^2}}$$

Put $x = \sin \theta$

$\therefore \quad dx = \cos \theta \, d\theta$

Changing the limits, we get

When $x = 0 \quad \therefore \sin \theta = 0 \quad \therefore \theta = 0$

When $x = \frac{1}{2} \quad \therefore \sin \theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{6}$

$$\therefore \quad I = \int_0^{\pi/6} \frac{\cos\theta \, d\theta}{(1+\sin^2\theta)\sqrt{1-\sin^2\theta}}$$

$$= \int_0^{\pi/6} \frac{\cos\theta \, d\theta}{(1+\sin^2\theta)\cos\theta} = \int_0^{\pi/6} \frac{1}{1+\sin^2\theta} d\theta$$

Now, dividing the numerator and denominator by $\cos^2\theta$, we get

$$= \int_0^{\pi/6} \frac{\frac{1}{\cos^2\theta}}{\frac{1}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}} d\theta = \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta+\tan^2\theta} d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+\tan^2\theta+\tan^2\theta} d\theta = \int_0^{\pi/6} \frac{\sec^2\theta}{2\tan^2\theta+1} d\theta$$

Put $\tan\theta = t$

$\therefore$ $\sec^2\theta \, d\theta = dt$

Changing the limits, we get

When $\theta = 0$ $\therefore t = \tan 0 = 0$

When $\theta = \frac{\pi}{6}$ $\therefore t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$$\therefore \quad I = \int_0^{1/\sqrt{3}} \frac{dt}{2t^2+1} = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dt}{t^2+\frac{1}{2}} = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dt}{t^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{2}\times\frac{1}{1/\sqrt{2}}\left[\tan^{-1}\frac{t}{1/\sqrt{2}}\right]_0^{1/\sqrt{3}} = \frac{1}{\sqrt{2}}\tan^{-1}\left[\sqrt{2}\,t\right]_0^{1/\sqrt{3}}$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{\sqrt{2}}{\sqrt{3}} - \tan^{-1}0\right] = \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{2}{3}}$$

LONG ANSWER TYPE QUESTIONS

Q35. $\int \frac{x^2}{x^4-x^2-12} dx$

Sol. Let $I = \int \frac{x^2}{x^4-x^2-12} dx = \int \frac{x^2}{x^4-4x^2+3x^2-12} dx$

$$= \int \frac{x^2}{x^2(x^2-4)+3(x^2-4)} dx = \int \frac{x^2}{(x^2-4)(x^2+3)} dx$$

Put $x^2 = t$ for the purpose of partial fraction.

We get $\frac{t}{(t-4)(t+3)}$

Let $\frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$

[where A and B are arbitrary constants]

$$\frac{t}{(t-4)(t+3)} = \frac{A(t+3)+B(t-4)}{(t-4)(t+3)}$$

$\Rightarrow \quad t = At + 3A + Bt - 4B$

Comparing the like terms, we get

$A + B = 1$ and $3A - 4B = 0$

$\Rightarrow \quad 3A = 4B$

$\therefore \quad A = \frac{4}{3}B$

Now $\frac{4}{3}B + B = 1$

$\frac{7}{3}B = 1 \quad \therefore B = \frac{3}{7}$ and $A = \frac{4}{3} \times \frac{3}{7} = \frac{4}{7}$

So, $A = \frac{4}{7}$ and $B = \frac{3}{7}$

$$\therefore \int \frac{x^2}{(x^2-4)(x^2+3)} dx$$

$$= \frac{4}{7}\int \frac{1}{x^2-4} dx + \frac{3}{7}\int \frac{1}{x^2+3} dx$$

$$= \frac{4}{7}\int \frac{1}{x^2-(2)^2} dx + \frac{3}{7}\int \frac{1}{x^2+(\sqrt{3})^2} dx$$

$$= \frac{4}{7} \times \frac{1}{2\times 2} \log\left|\frac{x-2}{x+2}\right| + \frac{3}{7} \times \frac{1}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}}$$

$$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

Hence, $I = \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C.$

Q36. Evaluate: $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

Sol. Let $I = \int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

Put $x^2 = t$ for the purpose of partial fraction.

We get $\frac{t}{(t+a^2)(t+b^2)}$

Put $\dfrac{t}{(t+a^2)(t+b^2)} = \dfrac{A}{t+a^2} + \dfrac{B}{t+b^2}$

$\Rightarrow \dfrac{t}{(t+a^2)(t+b^2)} = \dfrac{A(t+b^2)+B(t+a^2)}{(t+a^2)(t+b^2)}$

$\Rightarrow \quad t = At + Ab^2 + Bt + Ba^2$

Comparing the like terms, we get

$A + B = 1 \quad$ and $\quad Ab^2 + Ba^2 = 0$

$$A = \frac{-a^2}{b^2}B$$

$\therefore \quad \dfrac{-a^2}{b^2}B + B = 1$

$B\left(\dfrac{-a^2}{b^2}+1\right) = 1 \Rightarrow B\left(\dfrac{-a^2+b^2}{b^2}\right) = 1$

$\Rightarrow \quad B = \dfrac{b^2}{b^2-a^2}$ and $A = \dfrac{-a^2}{b^2} \times \dfrac{b^2}{b^2-a^2} = \dfrac{a^2}{a^2-b^2}$

So $\quad A = \dfrac{a^2}{a^2-b^2}$ and $B = \dfrac{-b^2}{a^2-b^2}$

$\therefore \displaystyle\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$

$$= \frac{a^2}{a^2-b^2}\int \frac{1}{x^2+a^2}dx - \frac{b^2}{a^2-b^2}\int \frac{1}{x^2+b^2}dx$$

$$= \frac{a^2}{a^2-b^2} \times \frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{b^2}{a^2-b^2}.\frac{1}{b}.\tan^{-1}\frac{x}{b}$$

$$= \frac{a}{a^2-b^2}\tan^{-1}\frac{x}{a} - \frac{b}{a^2-b^2}\tan^{-1}\frac{x}{b} + C$$

Hence, $\quad I = \dfrac{1}{a^2-b^2}\left[a\tan^{-1}\dfrac{x}{a} - b\tan^{-1}\dfrac{x}{b}\right] + C$

Q37. Evaluate: $\displaystyle\int_0^{\pi} \frac{x}{1+\sin x}dx$

Sol. Let $\quad I = \displaystyle\int_0^{\pi} \frac{x}{1+\sin x}dx \quad$ ...(*i*)

$= \displaystyle\int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}dx \quad \left[\text{using} \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \displaystyle\int_0^{\pi} \frac{\pi - x}{1+\sin x}dx \quad$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi}\left(\frac{x}{1+\sin x}+\frac{\pi-x}{1+\sin x}\right)dx$$

$$= \int_0^{\pi}\left(\frac{x+\pi-x}{1+\sin x}\right)dx = \int_0^{\pi}\frac{\pi}{1+\sin x}dx$$

$$= \pi\int_0^{\pi}\frac{1}{1+\sin x}dx = \pi\int_0^{\pi}\frac{1.(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$$

$$= \pi\int_0^{\pi}\frac{1-\sin x}{1-\sin^2 x}dx = \pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$= \pi\int_0^{\pi}\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)dx = \pi\int_0^{\pi}(\sec^2 x-\sec x\tan x)\,dx$$

$$= \pi\left[\tan x-\sec x\right]_0^{\pi} = \pi\left[(\tan\pi-\tan 0)-(\sec\pi-\sec 0)\right]$$

$$2I = \pi\left[0-(-1-1)\right] = \pi(2)$$

$\therefore \quad I = \pi$

Hence, $I = \pi$.

Q38. Evaluate: $\displaystyle\int\frac{2x-1}{(x-1)(x+2)(x-3)}dx$

Sol. Let $\quad I = \displaystyle\int\frac{2x-1}{(x-1)(x+2)(x-3)}dx$

Resolving into partial fraction, we put

$$\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-3}$$

$\Rightarrow \quad 2x-1 = A(x+2)(x-3)+B(x-1)(x-3)+C(x-1)(x+2)$

put $x=1$, $\quad 1 = A(3)(-2) \quad \Rightarrow A = -\frac{1}{6}$

put $x=-2$, $\quad -5 = B(-3)(-5) \quad \Rightarrow B = -\frac{1}{3}$

put $x=3$, $\quad 5 = C(2)(5) \quad \Rightarrow C = \frac{1}{2}$

$\therefore \displaystyle\int\frac{2x-1}{(x-1)(x+2)(x-3)}dx$

$$= -\frac{1}{6}\int\frac{1}{x-1}dx-\frac{1}{3}\int\frac{1}{x+2}dx+\frac{1}{2}\int\frac{1}{x-3}dx$$

$$= -\frac{1}{6}\log|x-1|-\frac{1}{3}\log|x+2|+\frac{1}{2}\log|x-3|+C$$

$$= -\log|x-1|^{1/6} - \log(x+2)^{1/3} + \log(x-3)^{1/2} + C$$

Hence, $\int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx = \log\left[\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right] + C.$

Q39. Evaluate: $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

Sol. Let $\quad I = \int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

Put $\quad \tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}.dx = dt$

$$= \int e^t(1+\tan t+\tan^2 t)\,dt = \int e^t(\sec^2 t+\tan t)\,dt$$

Here $\quad f(t) = \tan t$

$\therefore \quad f'(t) = \sec^2 t$

$$= e^t.f(t) = e^t\tan t = e^{\tan^{-1}x}.x + C$$

$$\left[\because \int e^x[f(x)+f'(x)]\,dx = e^x f(x) + C\right]$$

Hence, $\quad I = e^{\tan^{-1}x}.x + C.$

Q40. Evaluate: $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$ (Hint: Put $x = a\tan^2\theta$)

Sol. Let $\quad I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Put $\quad x = a\tan^2\theta$

$dx = 2a\tan\theta.\sec^2\theta.d\theta$

$$\therefore \quad I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}.2a\tan\theta.\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}\frac{\sqrt{a}\tan\theta}{\sqrt{a}\sec\theta}.2a\tan\theta.\sec\theta\,d\theta$$

$$= \int \sin^{-1}\left(\frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}\right).2a\tan\theta.\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}(\sin\theta).2a\tan\theta.\sec^2\theta\,d\theta$$

$$= 2a\int \theta\tan\theta.\sec^2\theta\,d\theta$$

$$= 2a\left[\theta\int \tan\theta.\sec^2\theta\, d\theta - \int\left[D(\theta).\int \tan\theta.\sec^2\theta\, d\theta\right]\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \int\frac{1.\tan^2\theta}{2}\, d\theta\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}\int(\sec^2\theta - 1)\, d\theta\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}(\tan\theta - \theta)\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}\tan\theta + \frac{1}{2}\theta\right]$$

$$= 2a\left[\tan^{-1}\sqrt{\frac{x}{a}}.\frac{x}{2a} - \frac{1}{2}\sqrt{\frac{x}{a}} + \frac{1}{2}\tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

Hence, $I = a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C.$

Q41. Evaluate: $\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx$

Sol. Let $I = \int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sqrt{2\cos^2 x/2}}{(2\sin^2 x/2)^{5/2}}\,dx$

$$= \int_{\pi/3}^{\pi/2}\frac{\sqrt{2}\cos x/2}{(2)^{5/2}\sin^5 x/2}\,dx = \frac{1}{4}\int_{\pi/3}^{\pi/2}\frac{\cos x/2}{\sin^5 x/2}\,dx$$

Put $\sin\frac{x}{2} = t \Rightarrow \frac{1}{2}\cos\frac{x}{2}\,dx = dt \Rightarrow \cos\frac{x}{2}\,dx = 2dt$

Changing the limits, we have

When $x = \frac{\pi}{3}$, $\sin\frac{\pi}{6} = t \quad \therefore t = \frac{1}{2}$

When $x = \frac{\pi}{2}$, $\sin\frac{\pi}{4} = t \quad \therefore t = \frac{1}{\sqrt{2}}$

$\therefore \quad I = \frac{1}{4}\times 2\int_{1/2}^{1/\sqrt{2}}\frac{dt}{t^5} = \frac{1}{2}\times\left(-\frac{1}{4}\right)[t^{-4}]_{1/2}^{1/\sqrt{2}} = -\frac{1}{8}\left[\frac{1}{t^4}\right]_{1/2}^{1/\sqrt{2}}$

$$= -\frac{1}{8}\left[\frac{1}{(1/\sqrt{2})^4} - \frac{1}{(1/2)^4}\right] = -\frac{1}{8}[4 - 16]$$

$$= -\frac{1}{8}\times(-12) = \frac{3}{2}$$

Hence, $I = \frac{3}{2}$.

Q42. Evaluate: $\int e^{-3x}\cos^3 x\,dx$

Sol. Let $I = \int \underset{\text{II}}{e^{-3x}}\,\underset{\text{I}}{\cos^3} x\,dx$

$$= \cos^3 x.\int e^{-3x}\,dx - \int\left(D(\cos^3 x).\int e^{-3x}dx\right)dx$$

$$= \cos^3 x.\frac{e^{-3x}}{-3} - \int\left(3\cos^2 x.(-\sin x).\frac{e^{-3x}}{-3}\right)dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \cos^2 x\sin x.e^{-3x}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int (1-\sin^2 x)\sin x.e^{-3x}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}\,dx + \int \underset{\text{I}}{\sin^3} x.\underset{\text{II}}{e^{-3x}}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}\,dx + \sin^3 x\int e^{-3x}\,dx - \int\left(D(\sin^3 x)\int e^{-3x}dx\right)dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}dx + \sin^3 x.\frac{e^{-3x}}{-3} - \int 3\sin^2 x.\cos x.\frac{e^{-3x}}{-3}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}dx - \frac{1}{3}e^{-3x}.\sin^3 x + \int \sin^2 x\cos x\, e^{-3x}\,dx$$

$$= -\frac{1}{3}e^{-3x}\cos^3 x - \int \sin x.e^{-3x}\,dx - \frac{1}{3}e^{-3x}\sin^3 x + \int (1-\cos^2 x)\cos x.e^{-3x}\,dx$$

$$I = -\frac{1}{3}e^{-3x}\cos^3 x - \left[\sin x.\frac{e^{-3x}}{-3} - \int \cos x.\frac{e^{-3x}}{-3}\right] - \frac{1}{3}e^{-3x}.\sin^3 x + \int \cos x.e^{-3x} - \int \cos^3 x.e^{-3x}\,dx$$

$$I = -\frac{1}{3}e^{-3x}\cos^3 x + \sin x.\frac{e^{-3x}}{3} - \int \cos x.\frac{e^{-3x}}{3}\,dx - \frac{1}{3}e^{-3x}\sin^3 x + \int \cos x.e^{-3x} - I$$

$$2I = \frac{e^{-3x}}{-3}[\cos^3 x + \sin^3 x] - \left[\sin x.\frac{e^{-3x}}{-3} - \int \cos x.\frac{e^{-3x}}{-3}dx\right] + \int \cos x.e^{-3x}\,dx$$

$$= \frac{e^{-3x}}{-3}\left[\cos^3 x + \sin^3 x\right] + \frac{1}{3}\sin x.e^{-3x} - \frac{1}{3}\int \cos x.e^{-3x}\,dx + \int \cos x.e^{-3x}\,dx$$

$$\therefore \quad 2I = \frac{e^{-3x}}{-3}\left[\cos^3 x + \sin^3 x\right] + \frac{1}{3}\sin x.e^{-3x} + \frac{2}{3}\int \cos x.e^{-3x}\,dx$$

Now, put

$$I_1 = \frac{2}{3}\int \underset{\text{I}}{\cos x}.\underset{\text{II}}{e^{-3x}}\,dx$$

$$= \frac{2}{3}\left[\cos x.\int e^{-3x}\,dx - \int\left(D(\cos x).\int e^{-3x}\,dx\right)dx\right]$$

$$= \frac{2}{3}\left[\cos x.\frac{e^{-3x}}{-3} - \int -\sin x.\frac{e^{-3x}}{-3}dx\right]$$

$$= \frac{2}{3}\left[\cos x.\frac{e^{-3x}}{-3} - \frac{1}{3}\int \sin x.e^{-3x}\,dx\right]$$

$$= -\frac{2}{9}\cos x.e^{-3x} - \frac{2}{9}\int \sin x.e^{-3x}\,dx$$

$$= -\frac{2}{9}\cos x.e^{-3x} - \frac{2}{9}\left[\sin x.\frac{e^{-3x}}{-3} - \int \cos x.\frac{e^{-3x}}{-3}dx\right]$$

$$I_1 = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x} - \frac{2}{27}\int \cos x.e^{-3x}\,dx$$

$$I_1 = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x} - \frac{1}{9}.\frac{2}{3}\int \cos x.e^{-3x}\,dx$$

$$I_1 = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x} - \frac{1}{9}.I_1$$

$$I_1 + \frac{1}{9}I_1 = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x}$$

$$\Rightarrow \frac{10I_1}{9} = -\frac{2}{9}\cos x.e^{-3x} + \frac{2}{27}\sin x.e^{-3x}$$

$$\therefore \quad I_1 = -\frac{1}{10}\cos x.e^{-3x} + \frac{1}{15}\sin x.e^{-3x}$$

So $\quad 2I = -\frac{1}{3}e^{-3x}[\sin^3 x + \cos^3 x] + \frac{1}{3}\sin x.e^{-3x} - \frac{1}{10}\cos x.e^{-3x}$

$$+\frac{1}{15}\sin x . e^{-3x}$$

$$\therefore \quad I = -\frac{1}{6}e^{-3x}[\sin^3 x+\cos^3 x]+\frac{1}{6}\sin x . e^{-3x}-\frac{1}{20}\cos x . e^{-3x} + \frac{1}{30}\sin x . e^{-3x}$$

$$= -\frac{1}{6}e^{-3x}[\sin^3 x+\cos^3 x]+\frac{1}{5}\sin x . e^{-3x}-\frac{1}{20}\cos x . e^{-3x}$$

$$= \frac{e^{-3x}}{24}[\sin 3x-\cos 3x]+\frac{3e^{-3x}}{40}[\sin x-3\cos x]+C$$

$$\left[\because \begin{array}{l}\sin 3x = 3\sin x - 4\sin^3 x \\ \cos 3x = 4\cos^3 x - 3\cos x\end{array}\right]$$

Hence, $I = \frac{e^{-3x}}{24}[\sin 3x-\cos 3x]+\frac{3e^{-3x}}{40}[\sin x-3\cos x]+C.$

Q43. Evaluate: $\int\sqrt{\tan x}\,dx$ (Hint: Put $\tan x = t^2$)

Sol. Let $I = \int\sqrt{\tan x}\,dx$

Put $\tan x = t^2$

$$\sec^2 x\,dx = 2t\,dt \Rightarrow dx = \frac{2t\,dt}{\sec^2 x} = \frac{2t\,dt}{1+\tan^2 x} \Rightarrow dx = \frac{2t\,dt}{1+t^4}$$

$$\therefore \quad I = \int\frac{t.2t}{1+t^4}dt = \int\frac{2t^2}{1+t^4}dt = \int\frac{2}{t^2+\frac{1}{t^2}}dt$$

[Dividing the numerator and denominator by t^2]

$$= \int\frac{1+\frac{1}{t^2}+1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+2-2}dt$$

$$= \int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}dt + \int\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+2-2}dt$$

$$= \int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}dt + \int\frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}dt$$

Put $I_1 = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}\,dt$ and $I_2 = \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}\,dt$

$\therefore \quad I = I_1 + I_2 \qquad \ldots(i)$

Now $I_1 = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}\,dt$

Put $t-\frac{1}{t} = u$

$\therefore \quad \left(1+\frac{1}{t^2}\right)dt = du$

$$I_1 = \int \frac{du}{u^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C_1$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t-\frac{1}{t}}{\sqrt{2}} + C_1 = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t^2-1}{\sqrt{2}t} + C_1$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\sqrt{\tan x}}\right) + C_1$$

Now $I_2 = \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}\,dt$

Put $t+\frac{1}{t} = v \Rightarrow \left(1-\frac{1}{t^2}\right)dt = dv$

$$= \int \frac{dv}{v^2-(\sqrt{2})^2} = \frac{1}{2.\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C_2$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + C_2$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + C_2$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x-\sqrt{2}\sqrt{\tan x}+1}{\tan x+\sqrt{2}\sqrt{\tan x}+1}\right| + C_2$$

So $\quad I = I_1 + I_2$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{\tan x - 1}{\sqrt{2\tan x}}\right] + \frac{1}{2\sqrt{2}} \log \left|\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right| + C_1 + C_2$$

Hence, $$I = \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{\tan x - 1}{\sqrt{2\tan x}}\right] + \frac{1}{2\sqrt{2}} \log \left|\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right| + C.$$

Q44. Evaluate: $\displaystyle\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

(Hint: Divide Numerator and Denominator by $\cos^4 x$)

Sol. Let $\quad I = \displaystyle\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

Dividing the numerator and denominator by $\cos^4 x$, we have

$$I = \int_0^{\pi/2} \frac{\sec^4 x}{\left(\frac{a^2\cos^2 x}{\cos^2 x} + \frac{b^2\sin^2 x}{\cos^2 x}\right)^2} dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x . \sec^2 x}{(a^2 + b^2\tan^2 x)^2} dx = \int_0^{\pi/2} \frac{(1+\tan^2 x)\sec^2 x}{(a^2 + b^2\tan^2 x)^2} dx$$

Put $\quad \tan x = t \;\Rightarrow\; \sec^2 x\, dx = dt$

Changing the limits, we get

When $x = 0$, $\quad t = \tan 0 = 0$

When $x = \frac{\pi}{2}$, $\quad t = \tan\frac{\pi}{2} = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{1+t^2}{(a^2 + b^2t^2)^2} dt$$

Put $t^2 = u$ only for the purpose of partial fraction

$$\therefore \quad \frac{1+u}{(a^2+b^2u)^2} = \frac{A}{(a^2+b^2u)} + \frac{B}{(a^2+b^2u)^2}$$

$$1 + u = A(a^2 + b^2u) + B$$

Comparing the coefficients of like terms, we get

$a^2A + B = 1$ and $b^2A = 1 \Rightarrow A = \frac{1}{b^2}$

Now $a^2 . \frac{1}{b^2} + B = 1 \quad \Rightarrow B = 1 - \frac{a^2}{b^2} = \frac{b^2 - a^2}{b^2}$

$$\therefore \quad I = \int_0^\infty \frac{1+t^2}{(a^2+b^2t^2)^2} = \frac{1}{b^2}\int_0^\infty \frac{dt}{a^2+b^2t^2} + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{dt}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{b^2}\int_0^\infty \frac{dt}{b^2\left(\frac{a^2}{b^2}+t^2\right)} + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{dt}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{ab^3}\left[\tan^{-1}\frac{t}{a/b}\right]_0^\infty + \frac{b^2-a^2}{b^2}\left(\frac{\pi}{4}.\frac{1}{a^3b}\right)$$

$$= \frac{1}{ab^3}\left[\tan^{-1}\infty - \tan 0\right] + \frac{b^2-a^2}{b^2}\left(\frac{\pi}{4a^3b}\right)$$

$$= \frac{1}{ab^3}.\frac{\pi}{2} + \frac{\pi}{4}.\frac{b^2-a^2}{a^3b^3} = \frac{\pi}{2ab^3} + \frac{\pi}{4}.\frac{b^2-a^2}{a^3b^3}$$

$$= \pi\left[\frac{2a^2+b^2-a^2}{4a^3b^3}\right] = \frac{\pi}{4}\left(\frac{a^2+b^2}{a^3b^3}\right)$$

Hence, $I = \frac{\pi}{4}\left(\frac{a^2+b^2}{a^3b^3}\right)$

Q45. Evaluate: $\int_0^1 x \log|1+2x|\, dx$

Sol. Let $I = \int_0^1 \underset{\text{II}}{x} \log \underset{\text{I}}{|1+2x|}\, dx$

$$= \left[\log|1+2x|.\left(\frac{x^2}{2}\right)\right]_0^1 - \int_0^1\left(\frac{1.2}{1+2x}.\frac{x^2}{2}\right)dx$$

$$= \frac{1}{2}\left[x^2 \log(1+2x)\right]_0^1 - \int_0^1 \frac{x^2}{1+2x}\,dx$$

$$= \frac{1}{2}[\log 3 - 0] - \int_0^1\left(\frac{x}{2} - \frac{x/2}{1+2x}\right)dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_0^1 \frac{x}{1+2x}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}.\frac{1}{2}\int_0^1 \frac{(2x+1-1)}{2x+1}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4}[1-0] + \frac{1}{4}\int_0^1 1\,dx - \frac{1}{4}\int_0^1 \frac{1}{2x+1}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{4}.\frac{1}{2}\left[\log|2x+1|\right]_0^1$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - 0]$$

$$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3 = \frac{3}{8}\log 3$$

Hence, $I = \frac{3}{8}\log 3$.

Q46. Evaluate: $\int_0^{\pi} x \log \sin x \, dx$

Sol. Let $$I = \int_0^{\pi} x \log \sin x \, dx \quad ...(i)$$

$$= \int_0^{\pi} (\pi - x) \log \sin (\pi - x) \, dx$$

$$\left[\text{using } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$I = \int_0^{\pi} (\pi - x) \log \sin x \, dx \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} [(\pi - x)\log \sin x + x \log \sin x]\, dx$$

$$2I = \int_0^{\pi} \pi \log \sin x \, dx$$

$$2I = 2\pi \int_0^{\pi/2} \log \sin x \, dx \qquad \left[\because \int_0^a f(x)\, dx = 2\int_0^{a/2} f(x)\, dx\right]$$

$$\therefore \quad I = \pi \int_0^{\pi/2} \log \sin x \, dx \quad ...(iii)$$

$$I = \pi \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$$I = \pi \int_0^{\pi/2} \log \cos x \, dx \quad ...(iv)$$

On adding (*iii*) and (*iv*), we get

$$2I = \pi \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$2I = \pi \int_0^{\pi/2} \log \sin x \cos x \, dx = \pi \int_0^{\pi/2} \log \frac{2 \sin x \cos x}{2}\, dx$$

$$2I = \pi\int_0^{\pi/2} \log \sin 2x\, dx - \pi\int_0^{\pi/2} \log 2\, dx$$

Put $2x = t \Rightarrow 2\,dx = dt \Rightarrow dx = \frac{dt}{2}$

$$2I = \pi\int_0^{\pi} \log \sin t\, dt - \pi.\log 2 \int_0^{\pi/2} 1\, dx \text{ [Changing the limit]}$$

$$2I = I - \pi.\log 2\, [x]_0^{\pi/2} \qquad \text{[from eqn. } (iii)]$$

$$2I - I = -\frac{\pi^2}{2}\log 2$$

So $I = \frac{\pi^2}{2}\log\left(\frac{1}{2}\right)$

Q47. Evaluate: $\int_{-\pi/4}^{\pi/4} \log|\sin x + \cos x|\, dx$

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \log|\sin x + \cos x|\, dx \qquad ...(i)$

$$= \int_{-\pi/4}^{\pi/4} \log\left|\sin\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right)\right| dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$$

$$= \int_{-\pi/4}^{\pi/4} \log|\sin(-x) + \cos x|\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \log|\cos x - \sin x|\, dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_{-\pi/4}^{\pi/4} \log|\cos x + \sin x|\, dx + \int_{-\pi/4}^{\pi/4} \log|\cos x - \sin x|\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \log|(\cos x + \sin x)(\cos x - \sin x)|\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \log|\cos^2 x - \sin^2 x|\, dx$$

$$\therefore \quad 2I = \int_{-\pi/4}^{\pi/4} \log \cos 2x\, dx$$

$$2I = 2\int_0^{\pi/4} \log \cos 2x \, dx$$

$$\left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$\therefore \quad I = \int_0^{\pi/4} \log \cos 2x \, dx$$

Put $2x = t \Rightarrow dx = \dfrac{dt}{2}$

Changing the limits we get

When $x = 0 \quad \therefore t = 0$; When $x = \dfrac{\pi}{4} \quad \therefore t = \dfrac{\pi}{2}$

$$I = \frac{1}{2}\int_0^{\pi/2} \log \cos t \, dt \qquad \text{...(iii)}$$

$$I = \frac{1}{2}\int_0^{\pi/2} \log \cos\left(\frac{\pi}{2} - t\right) dt$$

$$I = \frac{1}{2}\int_0^{\pi/2} \log \sin t \, dt \qquad \text{...(iv)}$$

On adding (*iii*) and (*iv*), we get,

$$2I = \frac{1}{2}\int_0^{\pi/2} (\log \cos t + \log \sin t) \, dt$$

$$\Rightarrow \quad 2I = \frac{1}{2}\int_0^{\pi/2} \log \sin t \cos t \, dt$$

$$\Rightarrow \quad 2I = \frac{1}{2}\int_0^{\pi/2} \log \frac{2 \sin t \cos t}{2} \, dt$$

$$\Rightarrow \quad 2I = \frac{1}{2}\int_0^{\pi/2} (\log \sin 2t - \log 2) \, dt$$

$$\Rightarrow \quad 4I = \int_0^{\pi/2} \log \sin 2t \, dt - \int_0^{\pi/2} \log 2 \, dt$$

Put $2t = u \Rightarrow 2\,dt = du \Rightarrow dt = \dfrac{du}{2}$

$$\therefore \quad 4I = \frac{1}{2}\int_0^{\pi} \log \sin u \, du - \int_0^{\pi/2} \log 2 \,.\, dt$$

$$\Rightarrow \quad 4I = \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin u \, du - \log 2 [t]_0^{\pi/2}$$

$$\Rightarrow \quad 4I = \int_0^{\pi/2} \log \sin u \, du - \log 2 . \frac{\pi}{2}$$

$$\Rightarrow \quad 4I = 2I - \frac{\pi}{2} \log 2 \qquad \text{[From eq. (ii)]}$$

$$\Rightarrow \quad 2I = -\frac{\pi}{2} \log 2 \quad \Rightarrow \quad I = \frac{\pi}{4} \log \frac{1}{2}$$

Hence, $I = \frac{\pi}{4} \log \frac{1}{2}$.

OBJECTIVE TYPE QUESTIONS

Choose the correct option from given four options in each of the Exercises from 48 to 58.

Q48. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$ is equal to

(a) $2(\sin x + x \cos \theta) + C$ (b) $2(\sin x - x \cos \theta) + C$
(c) $2(\sin x + 2x \cos \theta) + C$ (d) $2(\sin x - 2x \cos \theta) + C$

Sol. Let $I = \int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \theta - 1)}{\cos x - \cos \theta} dx$$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2 \theta + 1}{\cos x - \cos \theta} dx$$

$$= \int \frac{2\cos^2 x - 2\cos^2 \theta}{\cos x - \cos \theta} dx = 2 \int \frac{\cos^2 x - \cos^2 \theta}{\cos x - \cos \theta} dx$$

$$= 2 \int \frac{(\cos x + \cos \theta)(\cos x - \cos \theta)}{(\cos x - \cos \theta)} dx$$

$$= 2 \int (\cos x + \cos \theta) \, dx$$

$$\therefore \quad I = 2(\sin x + \cos \theta . x) + C.$$

Hence, correct option is (a).

Q49. $\int \frac{dx}{\sin (x-a) . \sin (x-b)}$ is equal to—

(a) $\sin (b-a) \log \left| \frac{\sin(x-b)}{\sin(x-a)} \right| + C$

(b) $\operatorname{cosec} (b-a) \log \left| \frac{\sin(x-a)}{\sin(x-b)} \right| + C$

(*c*) $\text{cosec}\,(b-a)\log\left|\dfrac{\sin(x-b)}{\sin(x-a)}\right|+C$

(*d*) $\sin(b-a)\log\left|\dfrac{\sin(x-a)}{\sin(x-b)}\right|+C$

Sol. Let $\quad I = \displaystyle\int \frac{dx}{\sin(x-a).\sin(x-b)}$

Multiplying and dividing by $\sin(b-a)$ we get,

$$I = \frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x+b-x-a)}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin[(x-a)-(x-b)]}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a).\cos(x-b)}{\sin(x-a).\sin(x-b)} - \frac{\cos(x-a).\sin(x-b)}{\sin(x-a).\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int\left[\frac{\cos(x-b)}{\sin(x-b)} - \frac{\cos(x-a)}{\sin(x-a)}\right]dx$$

$$= \frac{1}{\sin(b-a)}\int[\cot(x-b)-\cot(x-a)]\,dx$$

$$= \frac{1}{\sin(b-a)}[\log\sin(x-b)-\log\sin(x-a)]+C$$

$$= \frac{1}{\sin(b-a)}.\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$$

$$I = \text{cosec}\,(b-a).\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$$

Hence, the correct option is (*c*).

Q50. $\int \tan^{-1}\sqrt{x}\,dx$ is equal to

(*a*) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$ (*b*) $x\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(*c*) $\sqrt{x}-x\tan^{-1}\sqrt{x}+C$ (*d*) $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+C$

Sol. Let $\quad I = \int \tan^{-1}\sqrt{x}\,dx$

Put $\sqrt{x} = \tan\theta \Rightarrow x = \tan^2\theta \Rightarrow dx = 2\tan\theta\sec^2\theta\, d\theta$

$\therefore \quad I = \int \tan^{-1}(\tan\theta).2\tan\theta\sec^2\theta\, d\theta = 2\int \theta.\tan\theta.\sec^2\theta\, d\theta$

$= 2\left[\theta.\int \tan\theta.\sec^2\theta\, d\theta - \int\left(D(\theta).\int \tan\theta\sec^2\theta\, d\theta\right) d\theta\right]$

Let us take

$I_1 = \int \tan\theta\sec^2\theta\, d\theta$

Put $\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt$

$\therefore \quad I_1 = \int t\, dt = \frac{1}{2}t^2 = \frac{1}{2}\tan^2\theta$

$\therefore \quad I = 2\left[\theta.\frac{1}{2}\tan^2\theta - \int\left(1.\frac{1}{2}\tan^2\theta\right) d\theta\right]$

$= \theta\tan^2\theta - \int \tan^2\theta\, d\theta = \theta\tan^2\theta - \int(\sec^2\theta - 1)\, d\theta$

$= \theta\tan^2\theta - (\tan\theta - \theta) + C = \theta\tan^2\theta - \tan\theta + \theta + C$

$\therefore \quad I = \tan^{-1}\sqrt{x}.x - \sqrt{x} + \tan^{-1}\sqrt{x} + C$

$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

Hence, the correct option is (*a*).

Q51. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is equal to

(*a*) $\frac{e^x}{1+x^2} + C$ (*b*) $\frac{-e^x}{1+x^2} + C$

(*c*) $\frac{e^x}{(1+x^2)^2} + C$ (*d*) $\frac{-e^x}{(1+x^2)^2} + C$

Sol. Let $I = \int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

$= \int e^x\left[\frac{1+x^2-2x}{(1+x^2)^2}\right] dx = \int e^x\left[\frac{(1+x^2)}{(1+x^2)^2} - \frac{2x}{(1+x^2)^2}\right] dx$

$= \int e^x\left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right] dx$

Here $f(x) = \frac{1}{1+x^2} \quad \therefore f'(x) = \frac{-2x}{(1+x^2)^2}$

Using $\int e^x[f(x) + f'(x)]\, dx = e^x.f(x) + C$

$\therefore \quad I = e^x.\frac{1}{(1+x^2)} + C = \frac{e^x}{1+x^2} + C$

Hence, the correct option is (*a*).

Q52. $\int \frac{x^9}{(4x^2+1)^6} dx$ is equal to

(a) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (b) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(c) $\frac{1}{10x}(1+4)^{-5}+C$ (d) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

Sol. Let $I = \int \frac{x^9}{(4x^2+1)^6} dx = \int \frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)^6} dx = \int \frac{1}{x^3\left(4+\frac{1}{x^2}\right)^6} dx$

Put $\left(4+\frac{1}{x^2}\right) = t \Rightarrow \frac{-2}{x^3} dx = dt \Rightarrow \frac{dx}{x^3} = -\frac{1}{2} dt$

$\therefore \quad I = -\frac{1}{2}\int \frac{dt}{t^6}$

$= -\frac{1}{2} \times -\frac{1}{5} t^{-5} + C = \frac{1}{10} t^{-5} + C = \frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5} + C$

Hence, the correct option is (*d*).

Q53. If $\int \frac{dx}{(x+2)(x^2+1)} = a \log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$ then

(a) $a = -\frac{1}{10}, b = \frac{-2}{5}$ (b) $a = \frac{1}{10}, b = \frac{-2}{5}$

(c) $a = -\frac{1}{10}, b = \frac{2}{5}$ (d) $a = \frac{1}{10}, b = \frac{2}{5}$

Sol. Let $\quad I = \int \frac{dx}{(x+2)(x^2+1)}$

Let us resolve the given integrand into partial fractions

Put $\quad \frac{1}{(x+2)(x^2+1)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2+1)}$

$1 = A(x^2+1) + (x+2)(Bx+C)$

$1 = Ax^2 + A + Bx^2 + Cx + 2Bx + 2C$

$1 = (A+B)x^2 + (C+2B)x + (A+2C)$

Comparing the like terms, we have

$A + B = 0$ ...(*i*)

$2B + C = 0$ ...(*ii*)

$A + 2C = 1$ ...(*iii*)

Subtracting (*i*) from (*iii*) we get

$2C - B = 1 \quad \therefore B = 2C - 1$

Putting the value of B in eqn. (*ii*) we have

$$2(2C-1)+C=0 \Rightarrow 4C-2+C=0$$

$$5C=2 \quad \therefore C=\frac{2}{5}$$

$$\therefore \qquad B=2\left(\frac{2}{5}\right)-1=-\frac{1}{5} \text{ and } A=\frac{1}{5}$$

$$\therefore \int \frac{1}{(x+2)(x^2+1)}dx=\int \frac{\frac{1}{5}}{(x+2)}dx+\int \frac{-\frac{1}{5}x+\frac{2}{5}}{(x^2+1)}dx$$

$$=\frac{1}{5}\int \frac{1}{(x+2)}dx-\frac{1}{5}\int \frac{x-2}{(x^2+1)}dx$$

$$=\frac{1}{5}\int \frac{1}{x+2}dx-\frac{1}{5}\int \frac{x}{x^2+1}dx+\frac{2}{5}\int \frac{1}{x^2+1}dx$$

$$=\frac{1}{5}\int \frac{1}{x+2}dx-\frac{1}{10}\int \frac{2x}{x^2+1}dx+\frac{2}{5}\int \frac{1}{x^2+1}dx$$

$$\therefore \qquad I=\frac{1}{5}\log|x+2|-\frac{1}{10}\log|x^2+1|+\frac{2}{5}\tan^{-1}x+C$$

Putting the given value of I

$$\therefore\ a\log|1+x^2|+b\tan^{-1}x+\frac{1}{5}\log|x+2|+C$$

$$=\frac{1}{5}\log|x+2|-\frac{1}{10}\log|x^2+1|+\frac{2}{5}\tan^{-1}x+C$$

$$\therefore a=-\frac{1}{10} \quad \text{and} \quad b=\frac{2}{5}$$

Hence, the correct option is (*c*).

Q54. $\int \frac{x^3}{x+1}dx$ is equal to

(*a*) $x+\frac{x^2}{2}+\frac{x^3}{3}-\log|1-x|+C$ (*b*) $x+\frac{x^2}{2}-\frac{x^3}{3}-\log|1-x|+C$

(*c*) $x-\frac{x^2}{2}-\frac{x^3}{3}-\log|1+x|+C$ (*d*) $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

Sol. Let $I=\int \frac{x^3}{x+1}dx$

$$\therefore \qquad I=\int\left(x^2-x+1-\frac{1}{x+1}\right)dx=\frac{x^3}{3}-\frac{x^2}{2}+x-\log|x+1|+C$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\log|x+1|+C$$

Hence, the correct option is (*d*).

Q55. $\int \frac{x+\sin x}{1+\cos x}\,dx$ is equal to

(*a*) $\log|1+\cos x|+C$ (*b*) $\log|x+\sin x|+C$

(*c*) $x-\tan\frac{x}{2}+C$ (*d*) $x.\tan\frac{x}{2}+C$

Sol. Let $I = \int \frac{x+\sin x}{1+\cos x}\,dx$

$$= \int \frac{x}{1+\cos x}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$$

$$= \int \frac{x}{2\cos^2\frac{x}{2}}\,dx + \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int x.\sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[x.\int \sec^2\frac{x}{2}\,dx - \int\left(D(x).\int \sec^2\frac{x}{2}dx\right)dx\right] + \int \tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[x.2\tan\frac{x}{2} - \int 2\tan\frac{x}{2}\,dx\right] + \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx + \int \tan\frac{x}{2}.dx + C$$

$\therefore$ $I = x\tan\frac{x}{2}+C$

Hence, the correct option is (*d*).

Q56. If $\int \frac{x^3}{\sqrt{1+x^2}}\,dx = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$, then

(*a*) $a=\frac{1}{3}, b=1$ (*b*) $a=-\frac{1}{3}, b=1$

(*c*) $a=-\frac{1}{3}, b=-1$ (*d*) $a=\frac{1}{3}, b=-1$

Sol. Let $I = \int \frac{x^3}{\sqrt{1+x^2}}\,dx$

Put $1+x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$\therefore$ $I = \frac{1}{2}\int \frac{(t-1)}{\sqrt{t}}\,dt$

$$= \frac{1}{2}\int \frac{t}{\sqrt{t}}\,dt - \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\int \sqrt{t}\,dt - \frac{1}{2}\int t^{-1/2}\,dt$$

$$= \frac{1}{2} \times \frac{2}{3}(t)^{3/2} - \frac{1}{2}.2\sqrt{t} + C = \frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C$$

But $I = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$

Comparing the like terms we get, $a = \frac{1}{3}$ and $b = -1$

Hence, the correct option is (*d*).

Q57. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ is equal to

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$

$$= \int_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx = \frac{1}{2}[\tan x]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2}\left[\tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right] = \frac{1}{2}[1+1] = \frac{1}{2} \times 2 = 1$$

Hence, the correct option is (*a*).

Q58. $\int_0^{\pi/2} \sqrt{1-\sin 2x}\, dx$ is equal to

(*a*) $2\sqrt{2}$ (*b*) $2(\sqrt{2}+1)$ (*c*) 2 (*d*) $2(\sqrt{2}-1)$

Sol. Let $I = \int_0^{\pi/2} \sqrt{1-\sin 2x}\, dx = \int_0^{\pi/2} \sqrt{(\sin^2 x + \cos^2 x - 2\sin x \cos x)}\, dx$

$$= \int_0^{\pi/2} \sqrt{(\sin x - \cos x)^2}\, dx = \int_0^{\pi/2} \pm(\sin x - \cos x)\, dx$$

$$= \int_0^{\pi/4} -(\sin x - \cos x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \left[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0)\right] - \left[\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)\right]$$

$$= \left[\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-(+1)\right]-\left[(0+1)-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right]$$

$$= \left(\frac{2}{\sqrt{2}}-1\right)-\left(1-\frac{2}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}}-1-1+\frac{2}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}-2 = 2\sqrt{2}-2 = 2(\sqrt{2}-1)$$

Hence, the correct option is (*d*).

Fill in the blanks in each of the following Exercises 59 to 63:

Q59. $\int_0^{\pi/2} \cos x . e^{\sin x} dx$ is equal to ________.

Sol. Let $\quad I = \int_0^{\pi/2} \cos x . e^{\sin x} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

When $x = 0$ then $t = \sin 0 = 0$; When $x = \frac{\pi}{2}$ then $t = \sin\frac{\pi}{2} = 1$

$\therefore \quad I = \int_0^1 e^t\, dt = [e^t]_0^1 = (e^1 - e^0) = e - 1$

Hence, $I = e - 1$.

Q60. $\int \frac{x+3}{(x+4)^2} . e^x\, dx =$ ________

Sol. Let $\quad I = \int \frac{x+3}{(x+4)^2} . e^x\, dx = \int \frac{x+4-1}{(x+4)^2} . e^x\, dx$

$$= \int \left[\frac{x+4}{(x+4)^2} - \frac{1}{(x+4)^2}\right] e^x\, dx = \int \left[\frac{1}{x+4} - \frac{1}{(x+4)^2}\right] e^x\, dx$$

Put $\quad \frac{1}{x+4} = t \Rightarrow -\frac{1}{(x+4)^2} dx = dt$

Let $\quad f(x) = \frac{1}{x+4} \quad \therefore f'(x) = -\frac{1}{(x+4)^2}$

Using $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$

$\therefore \quad I = e^x . \frac{1}{x+4} + C$

Hence, $I = \frac{e^x}{x+4} + C$.

Q61. If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then $a =$ ________.

Sol. Given that: $\int_0^a \frac{1}{1+4x^2}\,dx = \frac{\pi}{8}$

$$\Rightarrow \quad \frac{1}{4}\int_0^a \frac{1}{\left(\frac{1}{4}+x^2\right)}\,dx = \frac{\pi}{8} \quad \Rightarrow \int_0^a \frac{1}{\left[\left(\frac{1}{2}\right)^2+x^2\right]}\,dx = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{1}{1/2}\left[\tan^{-1}\frac{x}{1/2}\right]_0^a = \frac{\pi}{2} \Rightarrow 2\left[\tan^{-1}2a - \tan^{-1}0\right] = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}2a = \frac{\pi}{4} \Rightarrow 2a = \tan\frac{\pi}{4} \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

Hence, the value of $a = \frac{1}{2}$.

Q62. $\int \frac{\sin x}{3+4\cos^2 x}\,dx =$ ________

Sol. Let $\quad I = \int \frac{\sin x}{3+4\cos^2 x}\,dx$

Put $\quad \cos x = t$

$\therefore \quad -\sin x\,dx = dt \quad \Rightarrow \sin x\,dx = -dt$

$$\therefore\ I = -\int \frac{dt}{3+4t^2} = -\frac{1}{4}\int \frac{dt}{\frac{3}{4}+t^2} = -\frac{1}{4}\int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2+t^2}$$

$$= -\frac{1}{4}\times\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{t}{\sqrt{3}/2}\right)+C$$

$$= -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)+C = -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+C$$

Hence, $I = -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\cos x\right)+C.$

Q63. The value of $\int_{-\pi}^{\pi} \sin^3 x.\cos^2 x\,dx$ is ________

Sol. Let $\quad I = \int_{-\pi}^{\pi} \sin^3 x.\cos^2 x\,dx$

Let $\quad f(x) = \sin^3 x\cos^2 x$

$\quad f(-x) = \sin^3(-x).\cos^2(-x) = -\sin^3 x\cos^2 x = -f(x)$

$\therefore \int_{-\pi}^{\pi} \sin^3 x.\cos^2 x\,dx$ is an odd function

$\therefore \quad I = 0$

□□□