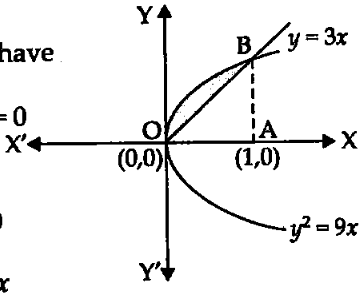
8 Application of Integrals

8.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$

Sol. We have, $y^2 = 9x$, $y = 3x$



Solving the two equations, we have

$$(3x)^2 = 9x$$

$\Rightarrow \quad 9x^2 - 9x = 0 \Rightarrow 9x(x-1) = 0$

$\therefore \quad x = 0, 1$

Area of the shaded region

= ar (region OAB) – ar (ΔOAB)

$$= -\int_0^1 y_1 . dx = \int_0^1 \sqrt{9x}\, dx - \int_0^1 3x\, dx$$

$$= 3\int_0^1 \sqrt{x}\, dx - 3\int_0^1 x\, dx = 3 \times \frac{2}{3}\left[x^{3/2}\right]_0^1 - 3\left[\frac{x^2}{2}\right]_0^1$$

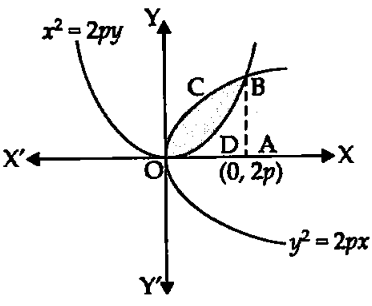
$$= 2\left[(1)^{3/2} - 0\right] - \frac{3}{2}[(1)^2 - 0] = 2(1) - \frac{3}{2}(1) = 2 - \frac{3}{2} = \frac{1}{2} \text{ sq. units}$$

Hence, the required area $= \frac{1}{2}$ sq. units.

Q2. Find the area of the region bounded by the parabola $y^2 = 2px$ and $x^2 = 2py$.

Sol. We are given that: $x^2 = 2py$ ...(*i*)

and $\quad y^2 = 2px \quad$ ...(*ii*)



From eqn. (*i*) we get $y = \frac{x^2}{2p}$

Putting the value of y in eqn. (*ii*) we have

$$\left(\frac{x^2}{2p}\right)^2 = 2px \Rightarrow \frac{x^4}{4p^2} = 2px$$

$$\Rightarrow x^4 = 8p^3x \quad \Rightarrow \quad x^4 - 8p^3x = 0$$

$$\Rightarrow x(x^3 - 8p^3) = 0 \quad \therefore x = 0, 2p$$

Required area = Area of the region (OCBA − ODBA)

$$= \int_0^{2p} \sqrt{2px}\, dx - \int_0^{2p} \frac{x^2}{2p} dx = \sqrt{2p} \int_0^{2p} \sqrt{x}\, dx - \frac{1}{2p} \int_0^{2p} x^2 dx$$

$$= \sqrt{2p} \cdot \frac{2}{3} [x^{3/2}]_0^{2p} - \frac{1}{2p} \cdot \frac{1}{3} [x^3]_0^{2p}$$

$$= \frac{2\sqrt{2}}{3} \sqrt{p}\, [(2p)^{3/2} - 0] - \frac{1}{6p} [(2p)^3 - 0]$$

$$= \frac{2\sqrt{2}}{3} \sqrt{p} \cdot 2\sqrt{2}\, p^{\frac{3}{2}} - \frac{1}{6p} \cdot 8p^3$$

$$= \frac{8}{3} \cdot p^2 - \frac{8}{6} p^2 = \frac{8}{6} p^2 = \frac{4}{3} p^2 \text{ sq. units}$$

Hence, the required area $= \frac{4}{3} p^2$ sq. units.

Q3. Find the area of the region bounded by the curve $y = x^3$, $y = x + 6$ and $x = 0$.

Sol. We are given that: $y = x^3$, $y = x + 6$ and $x = 0$

Solving $y = x^3$ and $y = x + 6$, we get

$$x + 6 = x^3$$

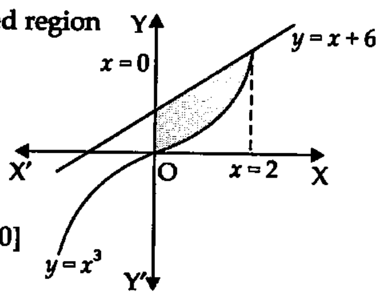
$$\Rightarrow \quad x^3 - x - 6 = 0$$

$$\Rightarrow \quad x^2 (x - 2) + 2x (x - 2) + 3 (x - 2) = 0$$

$$\Rightarrow \quad (x - 2)(x^2 + 2x + 3) = 0$$

$x^2 + 2x + 3 = 0$ has no real roots. $\therefore x = 2$

$\therefore$ Required area of the shaded region



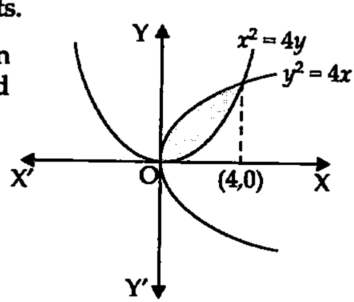
$$= \int_0^2 (x + 6)\, dx - \int_0^2 x^3 dx$$

$$= \left[\frac{x^2}{2} + 6x\right]_0^2 - \frac{1}{4}[x^4]_0^2$$

$$= \left(\frac{4}{2} + 12\right) - (0 + 0) - \frac{1}{4}[(2)^4 - 0]$$

$$= 14 - \frac{1}{4} \times 16 = 14 - 4 = 10 \text{ sq. units.}$$

Q4. Find the area of the region bounded by the curve $y^2 = 4x$ and $x^2 = 4y$.



Sol. We have $y^2 = 4x$ and $x^2 = 4y$.

$$y = \frac{x^2}{4}$$

$$\Rightarrow \quad \left(\frac{x^2}{4}\right)^2 = 4x$$

$\Rightarrow \quad \dfrac{x^4}{16} = 4x$

$\Rightarrow \quad x^4 = 64x \Rightarrow x^4 - 64x = 0$

$\Rightarrow \quad x(x^3 - 64) = 0$

$\therefore x = 0, x = 4$

Required area $= \int_0^4 \sqrt{4x}\, dx - \int_0^4 \dfrac{x^2}{4} dx = 2\int_0^4 \sqrt{x}\, dx - \dfrac{1}{4}\int_0^4 x^2\, dx$

$= 2 . \dfrac{2}{3}[x^{3/2}]_0^4 - \dfrac{1}{4} . \dfrac{1}{3}[x^3]_0^4$

$= \dfrac{4}{3}[(4)^{3/2} - 0] - \dfrac{1}{12}[(4)^3 - 0] = \dfrac{4}{3}[8] - \dfrac{1}{12}[64]$

$= \dfrac{32}{3} - \dfrac{16}{3} = \dfrac{16}{3}$ sq. units

Hence, the required area $= \dfrac{16}{3}$ sq. units.

Q5. Find the area of the region included between $y^2 = 9x$ and $y = x$.

Sol. Given that: $y^2 = 9x$ ...(*i*)

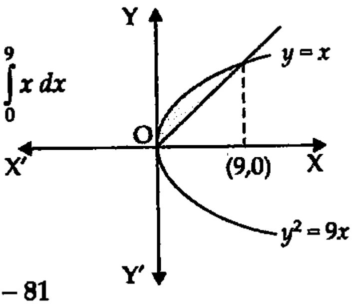
and $y = x$ ...(*ii*)

Solving eqns. (*i*) and (*ii*) we have

$x^2 = 9x \Rightarrow x^2 - 9x = 0$

$x(x-9) = 0 \quad \therefore \; x = 0, 9$

Required area



$= \int_0^9 \sqrt{9x}\, dx - \int_0^9 x\, dx = 3\int_0^9 \sqrt{x}\, dx - \int_0^9 x\, dx$

$= 3 . \dfrac{2}{3}[x^{3/2}]_0^9 - \dfrac{1}{2}[x^2]_0^9$

$= 2[(9)^{3/2} - 0] - \dfrac{1}{2}[(9)^2 - 0]$

$= 2(27) - \dfrac{1}{2}(81) = 54 - \dfrac{81}{2} = \dfrac{108 - 81}{2}$

$= \dfrac{27}{2}$ sq. units

Hence, the required area $= \dfrac{27}{2}$ sq. units.

Q6. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.

Sol. Here, $x^2 = y$ and $y = x + 2$

$\therefore \quad x^2 = x + 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad x^2 - 2x + x - 2 = 0$

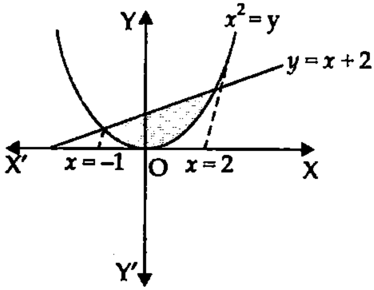
$\Rightarrow \quad x(x-2) + 1(x-2) = 0$

$\Rightarrow \quad (x-2)(x+1) = 0$

$\therefore x = -1, 2$

Graph of $y = x + 2$

x	0	–2
y	2	0



Area of the required region

$$= \int_{-1}^{2} (x+2)\,dx - \int_{-1}^{2} x^2\,dx = \left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{3}[x^3]_{-1}^{2}$$

$$= \left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] - \frac{1}{3}[8 - (-1)]$$

$$= \left(6 + \frac{3}{2}\right) - \frac{1}{3}(9) = \frac{15}{2} - 3 = \frac{9}{2} \text{ sq. units}$$

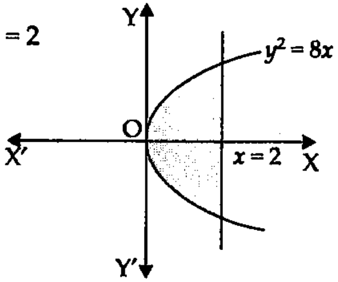
Hence, the required area $= \frac{9}{2}$ sq. units.

Q7. Find the area of the region bounded by the line $x = 2$ and parabola $y^2 = 8x$.

Sol. Here, $\quad y^2 = 8x$ and $x = 2$

$y^2 = 8(2) = 16$

$\therefore \quad y = \pm 4$



Required area

$$= 2\int_0^2 \sqrt{8x}\,dx = 2 \times 2\sqrt{2}\int_0^2 \sqrt{x}\,dx$$

$$= 4\sqrt{2} \times \frac{2}{3}[x^{3/2}]_0^2$$

$$= \frac{8\sqrt{2}}{3}[(2)^{3/2}] = \frac{8\sqrt{2}}{3} \times 2\sqrt{2} = \frac{32}{3} \text{ sq. units}$$

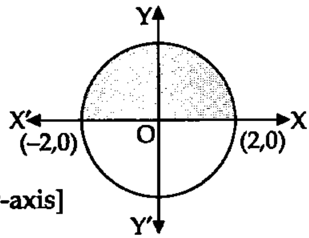
Hence, the area of the region $= \frac{32}{3}$ sq. units.

Q8. Sketch the region $\{(x, 0) : y = \sqrt{4 - x^2}\}$ and x-axis. Find the area of the region using integration.

Sol. Given that $\{(x, 0) : y = \sqrt{4 - x^2}\}$

$\Rightarrow \quad y^2 = 4 - x^2$

$\Rightarrow \quad x^2 + y^2 = 4$ which is a circle.



Required area

$$= 2.\int_0^2 \sqrt{4 - x^2}\, dx$$

[Since circle is symmetrical about y-axis]

$$= 2.\int_0^2 \sqrt{(2)^2 - x^2}\, dx$$

$$= 2.\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\left[\left(\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}(1)\right) - (0+0)\right]$$

$$= 2\left[2.\frac{\pi}{2}\right] = 2\pi \text{ sq. units}$$

Hence, the required area $= 2\pi$ sq. units.

Q9. Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.

Sol. Given the curves $y = 2\sqrt{x}$, $x = 0$ and $x = 1$.

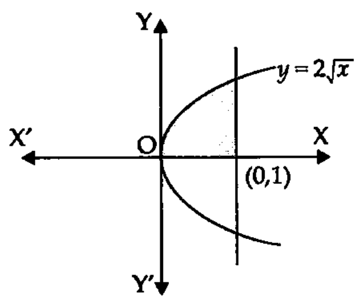
$$y = 2\sqrt{x} \Rightarrow y^2 = 4x \text{ (Parabola)}$$

Required area $= \int_0^1 (2\sqrt{x})\, dx$

$$= 2 \times \frac{2}{3}[x^{3/2}]_0^1$$

$$= \frac{4}{3}[(1)^{3/2} - 0]$$

$$= \frac{4}{3} \text{ sq. units}$$



Hence, required area $= \frac{4}{3}$ sq. units.

Q10. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.

Sol. Given that: $2y = 5x + 7$, x-axis, $x = 2$ and $x = 8$.

Let us draw the graph of $2y = 5x + 7 \Rightarrow y = \frac{5x+7}{2}$

x	1	–1
y	6	1

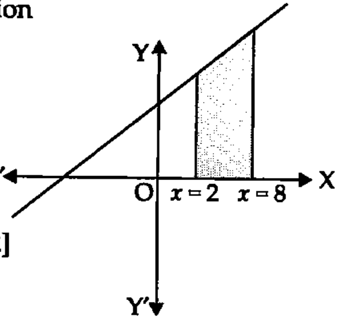
Area of the required shaded region

$$= \int_2^8 \left(\frac{5x+7}{2}\right) dx = \frac{1}{2}\left[\frac{5}{2}x^2 + 7x\right]_2^8$$

$$= \frac{1}{2}\left[\frac{5}{2}(64-4) + 7(8-2)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} \times 60 + 7 \times 6\right] = \frac{1}{2}[150 + 42]$$

$$= \frac{1}{2} \times 192 = 96 \text{ sq. units}$$



Hence, the required area = 96 sq. units.

Q11. Draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval [1, 5]. Find the area under the curve and between the lines $x = 1$ and $x = 5$.

Sol. Here, we have $y = \sqrt{x-1}$

$\Rightarrow \quad y^2 = x - 1$ (Parabola)

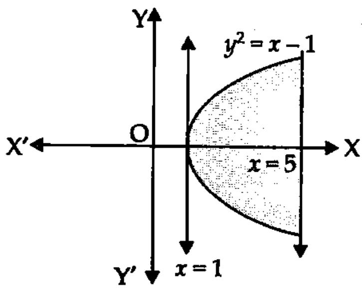
Area of the required region

$$= \int_1^5 \sqrt{x-1}\, dx$$

$$= \frac{2}{3}\left[(x-1)^{3/2}\right]_1^5$$

$$= \frac{2}{3}\left[(5-1)^{3/2} - 0\right] = \frac{2}{3} \times (4)^{3/2}$$

$$= \frac{2}{3} \times 8 = \frac{16}{3} \text{ sq. units}$$



Hence, the required area = $\frac{16}{3}$ sq. units.

Q12. Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.

Sol. Here, we are given $y = \sqrt{a^2 - x^2}$

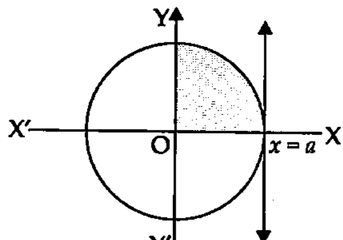
$\Rightarrow \quad y^2 = a^2 - x^2$

$\Rightarrow \quad x^2 + y^2 = a^2$

Area of the shaded region

$$= 2\left[(1)^{3/2} - 0\right] - \frac{3}{2}[(1)^2 - 0]$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$



$$= \left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} - 0 - 0\right]$$

$$= \frac{a^2}{2}\sin^{-1}(1) = \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}$$

Hence, the required area = $\frac{\pi a^2}{4}$ sq. units.

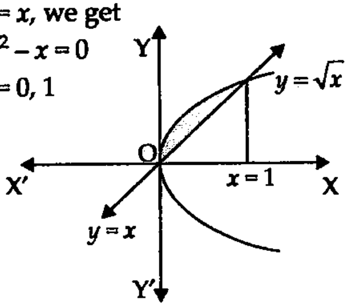
Q13. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

Sol. We are given the equations of curve $y = \sqrt{x}$ and line $y = x$.

Solving $y = \sqrt{x} \Rightarrow y^2 = x$ and $y = x$, we get

$$x^2 = x \Rightarrow x^2 - x = 0$$

$$\Rightarrow \quad x(x-1) = 0 \quad \therefore x = 0, 1$$



Required area of the shaded region

$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x\,dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^1 - \frac{1}{2}\left[x^2\right]_0^1$$

$$= \frac{2}{3}[(1)^{3/2} - 0] - \frac{1}{2}[(1)^2 - 0]$$

$$= \frac{2}{3} - \frac{1}{2} \Rightarrow \frac{4-3}{6} \Rightarrow \frac{1}{6} \text{ sq. units}$$

Hence, the required area = $\frac{1}{6}$ sq. units.

Q14. Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.

Sol. We are given that $y = -x^2$ or $x^2 = -y$

and the line $x + y + 2 = 0$

Solving the two equations, we get

$$x - x^2 + 2 = 0$$

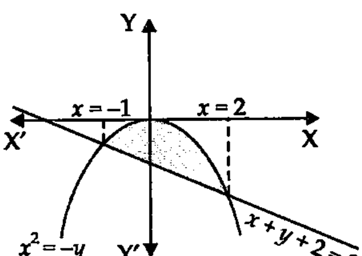
$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad x^2 - 2x + x - 2 = 0$$

$$\Rightarrow \quad x(x-2) + 1(x-2) = 0$$

$$\Rightarrow \quad (x-2)(x+1) = 0$$

$$\therefore \quad x = -1, 2$$



Area of the required shaded region

$$= \left| \int_{-1}^{2} (-x-2)\, dx - \int_{-1}^{2} -x^2\, dx \right|$$

$$\Rightarrow \quad \left| -\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} + \frac{1}{3}[x^3]_{-1}^{2} \right|$$

$$\Rightarrow \quad \left| -\left[\left(\frac{4}{2}+4\right) - \left(\frac{1}{2}-2\right)\right] + \frac{1}{3}(8+1) \right|$$

$$\Rightarrow \quad \left| -\left(6+\frac{3}{2}\right) + \frac{1}{3}(9) \right| \Rightarrow \left| -\frac{15}{2} + 3 \right|$$

$$\Rightarrow \quad \left| \frac{-15+6}{2} \right| = \left| \frac{-9}{2} \right| = \frac{9}{2} \text{ sq. units}$$

Q15. Find the area bounded by the curve $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis.

Sol. Given that: $y = \sqrt{x}$, $x = 2y + 3$, first quadrant and x-axis.

Solving $y = \sqrt{x}$ and $x = 2y + 3$, we get

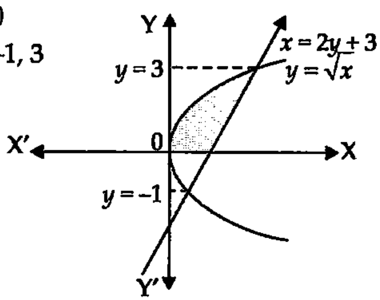
$$y = \sqrt{2y+3} \Rightarrow y^2 = 2y + 3$$

$$\Rightarrow \quad y^2 - 2y - 3 = 0 \Rightarrow y^2 - 3y + y - 3 = 0$$

$$\Rightarrow \quad y(y-3) + 1\,(y-3) = 0$$

$$\Rightarrow \quad (y+1)\,(y-3) = 0$$

$$\therefore \quad y = -1, 3$$



Area of shaded region

$$= \int_0^3 (2y+3)\, dy - \int_0^3 y^2 dy$$

$$= \left[2\frac{y^2}{2} + 3y\right]_0^3 - \frac{1}{3}\left[y^3\right]_0^3$$

$$= [(9+9) - (0+0)] - \frac{1}{3}[27 - 0]$$

$$= 18 - 9 = 9 \text{ sq. units}$$

Hence, the required area = 9 sq. units.

LONG ANSWER TYPE QUESTIONS

Q16. Find the area of the region bounded by the curve $y^2 = 2x$ and $x^2 + y^2 = 4x$.

Sol. Equations of the curves are given by

$$x^2 + y^2 = 4x \qquad \text{...(i)}$$

and

$$y^2 = 2x \qquad \text{...(ii)}$$

$\Rightarrow \quad x^2 - 4x + y^2 = 0$

$\Rightarrow \quad x^2 - 4x + 4 - 4 + y^2 = 0$

$\Rightarrow \quad (x-2)^2 + y^2 = 4$

Clearly it is the equation of a circle having its centre (2, 0) and radius 2.

Solving $x^2 + y^2 = 4x$ and $y^2 = 2x$

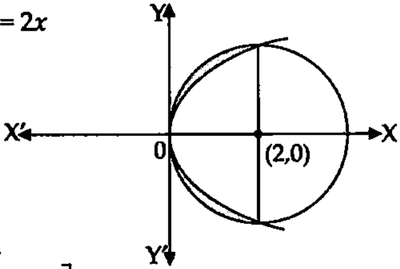
$x^2 + 2x = 4x$

$\Rightarrow \quad x^2 + 2x - 4x = 0$

$\Rightarrow \quad x^2 - 2x = 0$

$\Rightarrow \quad x(x-2) = 0$

$\therefore \quad x = 0, 2$



Area of the required region

$$= 2\left[\int_0^2 \sqrt{4-(x-2)^2}\,dx - \int_0^2 \sqrt{2x}\,dx\right]$$

[$\therefore$ Parabola and circle both are symmetrical about x-axis.]

$$= 2\left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\frac{x-2}{2}\right]_0^2 - 2.\sqrt{2}.\frac{2}{3}[x^{3/2}]_0^2$$

$$= 2\left[(0+0) - (0 + 2\sin^{-1}(-1)\right] - \frac{4\sqrt{2}}{3}[2^{3/2} - 0]$$

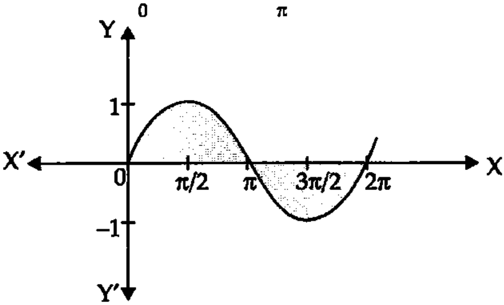
$$= -2 \times 2.\left(-\frac{\pi}{2}\right) - \frac{4\sqrt{2}}{3}.2\sqrt{2}$$

$$= 2\pi - \frac{16}{3} = 2\left(\pi - \frac{8}{3}\right) \text{ sq. units}$$

Hence, the required area $= 2\left(\pi - \frac{8}{3}\right)$ sq. units.

Q17. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

Sol. Required area $= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} |\sin x|\,dx$

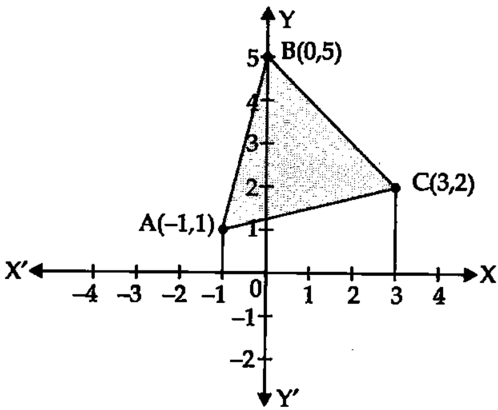


$$= -[\cos x]_0^{\pi} + |(-\cos x)|_{\pi}^{2\pi} = -[\cos \pi - \cos 0] + [\cos 2\pi - \cos \pi]$$

$$= -[-1-1] + [1+1] = 2 + 2 = 4 \text{ sq. units}$$

Q18. Find the area of the region bounded by the triangle whose vertices are (–1, 1), (0, 5) and (3, 2), using integration.

Sol. The coordinates of the vertices of ΔABC are given by A(–1, 1), B (0, 5) and C (3, 2).



Equation of AB is $y - 1 = \frac{5-1}{0+1}(x+1)$

$\Rightarrow$ $y - 1 = 4x + 4$

$\therefore$ $y = 4x + 4 + 1 \Rightarrow y = 4x + 5$ ...(*i*)

Equation of BC is $y - 5 = \frac{2-5}{3-0}(x-0)$

$\Rightarrow$ $y - 5 = -x$

$\therefore$ $y = 5 - x$ ...(*ii*)

Equation of CA is

$$y - 1 = \frac{2-1}{3+1}(x+1)$$

$\Rightarrow$ $y - 1 = \frac{1}{4}x + \frac{1}{4} \Rightarrow y = \frac{1}{4}x + \frac{1}{4} + 1$

$\therefore$ $y = \frac{1}{4}x + \frac{5}{4} = \frac{1}{4}(5+x)$

Area of ΔABC

$$= \int_{-1}^{0}(4x+5)\,dx + \int_{0}^{3}(5-x)\,dx - \int_{-1}^{3}\frac{1}{4}(5+x)\,dx$$

$$= \frac{4}{2}[x^2]_{-1}^{0} + 5[x]_{-1}^{0} + 5[x]_0^3 - \frac{1}{2}[x^2]_0^3 - \frac{1}{4}\left[5x + \frac{x^2}{2}\right]_{-1}^{3}$$

$$= 2(0-1)+5(0+1)+5(3-0)-\frac{1}{2}(9-0) - \frac{1}{4}\left[\left(15+\frac{9}{2}\right)-\left(-5+\frac{1}{2}\right)\right]$$

$$= -2+5+15-\frac{9}{2}-\frac{1}{4}\left(\frac{39}{2}+\frac{9}{2}\right)$$

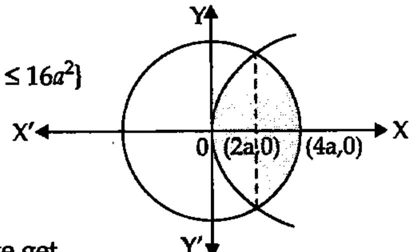
$$= 18-\frac{9}{2}-\frac{1}{4}\times\frac{48}{2} = 18-\frac{9}{2}-6 = 12-\frac{9}{2} = \frac{15}{2} \text{ sq. units}$$

Hence, the required area $= \frac{15}{2}$ sq. units.

Q19. Draw a rough sketch of the region $\{(x, y) : y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$. Also find the area of the region sketched using method of integration.

Sol. Given that:

$\{(x, y) : y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$



Equation of Parabola is

$y^2 = 6ax$ ...(*i*)

and equation of circle is

$x^2 + y^2 \le 16a^2$ ...(*ii*)

Solving eqns. (*i*) and (*ii*) we get

$$x^2 + 6ax = 16a^2$$

$$\Rightarrow \quad x^2 + 6ax - 16a^2 = 0$$

$$\Rightarrow \quad x^2 + 8ax - 2ax - 16a^2 = 0$$

$$\Rightarrow \quad x(x+8a) - 2a(x+8a) = 0$$

$$\Rightarrow \quad (x+8a)(x-2a) = 0$$

$\therefore$ $x = 2a$ and $x = -8a$. (Rejected as it is out of region)

Area of the required shaded region

$$= 2\left[\int_0^{2a} \sqrt{6ax}\, dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\, dx\right]$$

$$= 2\left[\sqrt{6a}\int_0^{2a} \sqrt{x}\, dx + \int_{2a}^{4a} \sqrt{(4a)^2 - x^2}\, dx\right]$$

$$= 2\sqrt{6a}\cdot\frac{2}{3}\cdot\left[x^{3/2}\right]_0^{2a} + 2\left[\frac{x}{2}\sqrt{(4a)^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= \frac{4\sqrt{6}}{3}\cdot\sqrt{a}\left[(2a)^{3/2} - 0\right] + \left[x\sqrt{(4a)^2 - x^2} + 16a^2\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= \frac{4\sqrt{6}}{3}\sqrt{a}\,.\,2\sqrt{2}\,.\,a^{3/2} + \left[0 + 16a^2\sin^{-1}\left(\frac{4a}{4a}\right) - 2a\sqrt{16a^2 - 4a^2} - 16a^2\sin^{-1}\frac{2a}{4a}\right]$$

$$= \frac{8\sqrt{12}}{3}a^2 + \left[16a^2.\sin^{-1}(1) - 2a\sqrt{12a^2} - 16a^2\sin^{-1}\frac{1}{2}\right]$$

$$= \frac{16\sqrt{3}}{3}a^2 + \left[16a^2.\frac{\pi}{2} - 2a\,.\,2\sqrt{3}a - 16a^2.\frac{\pi}{6}\right]$$

$$= \frac{16\sqrt{3}}{3}a^2 + 8\pi a^2 - 4\sqrt{3}a^2 - \frac{8}{3}\pi a^2$$

$$= \left(\frac{16\sqrt{3}}{3} - 4\sqrt{3}\right)a^2 + \frac{16}{3}\pi a^2 = \frac{4\sqrt{3}}{3}a^2 + \frac{16}{3}\pi a^2$$

$$= \frac{4}{3}\left(\sqrt{3} + 4\pi\right)a^2$$

Hence, required area $= \frac{4}{3}\left(\sqrt{3} + 4\pi\right)a^2$ sq. units.

Q20. Compute the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

Sol. Given that: $x + 2y = 2$ ...(*i*)

$y - x = 1$ ...(*ii*)

and $2x + y = 7$ ...(*iii*)

x	0	2
y	1	0

x	0	–1
y	1	0

x	0	7/2
y	7	0

Solving eqns. (*ii*) and (*iii*) we get

$$y = 1 + x$$

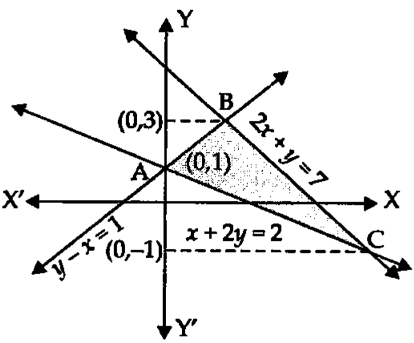
$\therefore$ $2x + 1 + x = 7$

$3x = 6$

$\Rightarrow$ $x = 2$

$\therefore$ $y = 1 + 2 = 3$

Coordinates of B = (2, 3)



Solving eqns. (*i*) and (*iii*) we get

$x + 2y = 2$

$\therefore$ $x = 2 - 2y$

$2x + y = 7$

$2(2 - 2y) + y = 7$

$\Rightarrow$ $4 - 4y + y = 7 \Rightarrow -3y = 3$

$\therefore$ $y = -1$ and $x = 4$

$\therefore$ Coordinates of $C = (4, -1)$ and coordinates of $A = (0, 1)$.

Taking the limits on y-axis, we get

$$\int_{-1}^{3} x_{BC}\, dy - \int_{-1}^{1} x_{AC}\, dy - \int_{1}^{3} x_{AB}\, dy$$

$$= \int_{-1}^{3} \frac{7-y}{2}\, dy - \int_{-1}^{1} (2-2y)\, dy - \int_{1}^{3} (y-1)\, dy$$

$$= \frac{1}{2}\left[7y - \frac{y^2}{2}\right]_{-1}^{3} - 2\left[y - \frac{y^2}{2}\right]_{-1}^{1} - \left[\frac{y^2}{2} - y\right]_{1}^{3}$$

$$= \frac{1}{2}\left[\left(21 - \frac{9}{2}\right) - \left(-7 - \frac{1}{2}\right)\right] - 2\left[\left(1 - \frac{1}{2}\right) - \left(-1 - \frac{1}{2}\right)\right] - \left[\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{33}{2} + \frac{15}{2}\right] - 2\left[\frac{1}{2} + \frac{3}{2}\right] - \left[\frac{3}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{2} \times 24 - 2 \times 2 - 2 \Rightarrow 12 - 4 - 2 = 6 \text{ sq.units}$$

Hence, the required area = 6 sq. units.

Q21. Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

Sol. Given that

$$y = 4x + 5 \quad \text{...(i)}$$

$$y = 5 - x \quad \text{...(ii)}$$

and $$4y = x + 5 \quad \text{...(iii)}$$

x	0	–5/4
y	5	0

x	0	5
y	5	0

x	0	–5
y	5/4	0

Solving eq. (*i*) and (*ii*) we get

$$4x + 5 = 5 - x$$

$$\Rightarrow \quad x = 0 \text{ and } y = 5$$

$\therefore$ Coordinates of A = (0, 5)

Solving eq. (ii) and (iii)

$$y = 5 - x$$

$$4y = x + 5$$

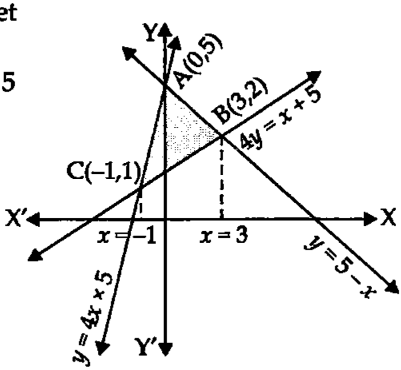
$$5y = 10$$

$\therefore y = 2$ and $x = 3$

$\therefore$ Coordinates of B = (3, 2)

Solving eq. (*i*) and (*iii*)

$$y = 4x + 5$$



$$4y = x + 5$$
$$\Rightarrow \quad 4(4x+5) = x + 5$$
$$\Rightarrow \quad 16x + 20 = x + 5 \Rightarrow 15x = -15$$
$$\therefore \quad x = -1 \text{ and } y = 1$$

$\therefore$ Coordinates of C = (–1, 1).

$\therefore$ Area of required regions

$$= \int_{-1}^{0} y_{AC}\, dx + \int_{0}^{3} y_{AB}\, dx - \int_{-1}^{3} y_{CB}\, dx$$

$$= \int_{-1}^{0} (4x+5)\, dx + \int_{0}^{3} (5-x)\, dx - \int_{-1}^{3} \frac{x+5}{4}\, dx$$

$$= \left[4\frac{x^2}{2} + 5x\right]_{-1}^{0} + \left[5x - \frac{x^2}{2}\right]_{0}^{3} - \frac{1}{4}\left[\frac{x^2}{2} + 5x\right]_{-1}^{3}$$

$$= [(0+0)-(2-5)] + \left[\left(15-\frac{9}{2}\right)-(0-0)\right] - \frac{1}{4}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]$$

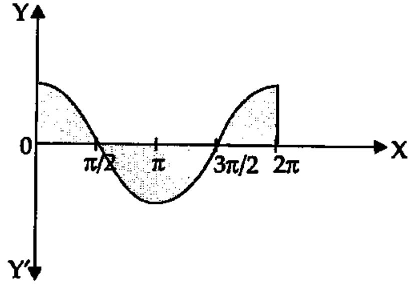
$$= 3 + \frac{21}{2} - \frac{1}{4}\left[\frac{39}{2} + \frac{9}{2}\right] = 3 + \frac{21}{2} - \frac{1}{4} \times 24 \Rightarrow 3 + \frac{21}{2} - 6$$

$$= \frac{15}{2} \text{ sq. units}$$

Hence, the required area = $\frac{15}{2}$ sq. units.

Q22. Find the area bounded by the curve $y = 2\cos x$ and the x-axis from $x = 0$ to $x = 2\pi$.

Sol. Given equation of the curve is $y = 2\cos x$



$\therefore$ Area of the shaded region

$$\int_{0}^{2\pi} 2\cos x\, dx = \int_{0}^{\pi/2} 2\cos x\, dx + \int_{\pi/2}^{3\pi/2} |2\cos x|\, dx + \int_{3\pi/2}^{2\pi} 2\cos x\, dx$$

$$= 2\left[\sin x\right]_0^{\pi/2} + \left|\left[2\sin x\right]_{\pi/2}^{3\pi/2}\right| + 2\left[\sin x\right]_{3\pi/2}^{2\pi}$$

$$= 2\left[\sin\frac{\pi}{2} - \sin 0\right] + \left|2\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right)\right| + 2\left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

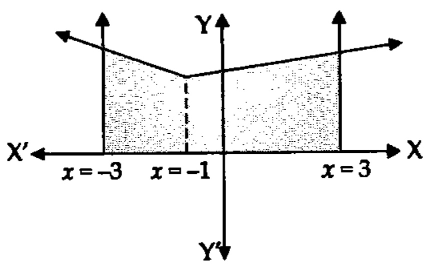
$$= 2(1) + |2(-1-1)| + 2(0+1) = 2+4+2 = 8 \text{ sq. units}$$

Q23. Draw a rough sketch of the given curve $y = 1 + |x+1|$, $x = -3$, $x = 3$, $y = 0$ and find the area of the region bounded by them, using integration.

Sol. Given equations are

$y = 1 + |x+1|, x = -3$

and $x = 3, y = 0$



Taking $y = 1 + |x+1|$

$\Rightarrow \quad y = 1 + x + 1$

$\Rightarrow \quad y = x + 2$

and $\quad y = 1 - x - 1 \Rightarrow y = -x$

On solving we get $x = -1$

Area of the required regions

$$= \int_{-3}^{-1} -x\,dx + \int_{-1}^{3} (x+2)\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3} = -\left[\frac{1}{2} - \frac{9}{2}\right] + \left[\left(\frac{9}{2}+6\right) - \left(\frac{1}{2}-2\right)\right]$$

$$= -(-4) + \left[\frac{21}{2} + \frac{3}{2}\right] = 4 + 12 = 16 \text{ sq. units}$$

Hence, the required area = 16 sq. units.

OBJECTIVE TYPE QUESTIONS

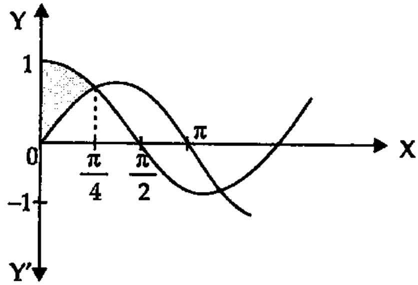
Choose the correct answer from the given four options in each of the Exercises 24 to 34.

Q24. The area of the region bounded by the y-axis, $y = \cos x$ and $y = \sin x$, where $0 \le x \le \frac{\pi}{2}$ is

(a) $\sqrt{2}$ sq. units (b) $(\sqrt{2}+1)$ sq. units

(c) $(\sqrt{2}-1)$ sq. units (d) $(2\sqrt{2}-1)$ sq. units

Sol. Given that y-axis, $y = \cos x$, $y = \sin x$, $0 \le x \le \frac{\pi}{2}$



$$\text{Required area} = \int_0^{\pi/4} \cos x\,dx - \int_0^{\pi/4} \sin x\,dx$$

$$= [\sin x]_0^{\pi/4} - [-\cos x]_0^{\pi/4}$$

$$= \left[\sin\frac{\pi}{4} - \sin 0\right] + \left[\cos\frac{\pi}{4} - \cos 0\right]$$

$$= \left[\frac{1}{\sqrt{2}} - 0 + \frac{1}{\sqrt{2}} - 1\right] = \frac{2}{\sqrt{2}} - 1$$

$$= (\sqrt{2} - 1) \text{ sq. units}$$

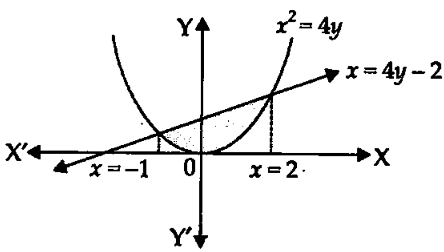
Hence, the correct option is (*c*).

Q25. The area of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

(*a*) $\frac{3}{8}$ sq. units (*b*) $\frac{5}{8}$ sq. units

(*c*) $\frac{7}{8}$ sq. units (*d*) $\frac{9}{8}$ sq. units

Sol. Given that: The equation of parabola is $x^2 = 4y$ ...(*i*)
and equation of straight line is $x = 4y - 2$ ...(*ii*)



Solving eqn. (*i*) and (*ii*) we get

$$y = \frac{x^2}{4}$$

$$x = 4\left(\frac{x^2}{4}\right) - 2$$

$$\Rightarrow \quad x = x^2 - 2$$

$$\Rightarrow \quad x^2 - x - 2 = 0 \Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow \quad x(x-2) + 1(x-2) = 0 \Rightarrow (x-2)(x+1) = 0 \therefore x = -1, x = 2$$

Required area $= \int_{-1}^{2} \frac{x+2}{4} dx - \int_{-1}^{2} \frac{x^2}{4} dx$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4} \cdot \frac{1}{3}[x^3]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] - \frac{1}{12}[8 + 1]$$

$$= \frac{1}{4}\left[6 + \frac{3}{2}\right] - \frac{1}{12}[9] = \frac{1}{4} \times \frac{15}{2} - \frac{3}{4}$$

$$= \frac{15}{8} - \frac{3}{4} = \frac{9}{8} \text{ sq. units}$$

Hence, the correct option is (*d*).

Q26. The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis is

(*a*) 8π sq. units (*b*) 20π sq. units

(*c*) 16π sq. units (*d*) 256π sq. units

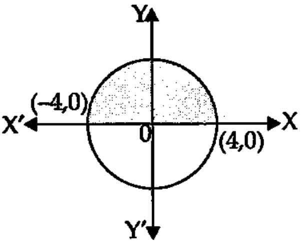
Sol. Here, equation of curve is $y = \sqrt{16 - x^2}$

Required area

$$= 2\left[\int_0^4 \sqrt{16 - x^2}\, dx\right]$$

$$= 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= 2\left[\left(0 + 8\sin^{-1}\frac{4}{4}\right) - (0 + 0)\right]$$



$$= 2[8\sin^{-1}(1)] = 16 \cdot \frac{\pi}{2} = 8\pi \text{ sq. units}$$

Hence, the correct option is (*a*).

Q27. Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is

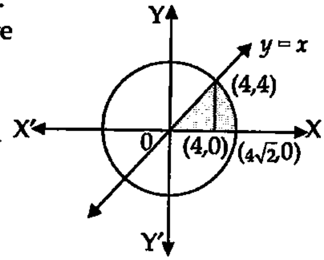
(*a*) 16π sq. units (*b*) 4π sq. units

(*c*) 32π sq. units (*d*) 24 sq. units

Sol. Given equation of circle is $x^2+y^2=32 \Rightarrow x^2+y^2=(4\sqrt{2})^2$ and the line is $y=x$ and the x-axis.

Solving the two equations we have

$$x^2+x^2=32$$
$$\Rightarrow \quad 2x^2=32$$
$$\Rightarrow \quad x^2=16$$
$$\therefore \quad x=\pm 4$$



Required area

$$=\int_0^4 x\,dx+\int_4^{4\sqrt{2}}\sqrt{(4\sqrt{2})^2-x^2}\,dx$$

$$=\frac{1}{2}[x^2]_0^4+\left[\frac{x}{2}\sqrt{(4\sqrt{2})^2-x^2}+\frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$=\frac{1}{2}[16-0]+\left[0+16\sin^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right)-2\sqrt{32-16}-16\sin^{-1}\frac{4}{4\sqrt{2}}\right]$$

$$=8+\left[16\sin^{-1}(1)-8-16\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$=8+16\cdot\frac{\pi}{2}-8-16\cdot\frac{\pi}{4}=8\pi-4\pi=4\pi \text{ sq. units}$$

Hence, the correct option is (*b*).

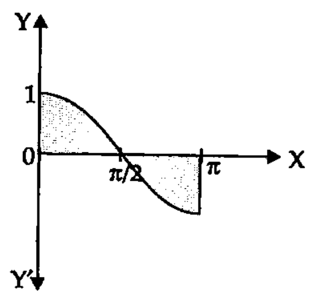
Q28. Area of the region bounded by the curve $y=\cos x$ between $x=0$ and $x=\pi$ is

(*a*) 2 sq. units (*b*) 4 sq. units

(*c*) 3 sq. units (*d*) 1 sq. units

Sol. Given that: $y=\cos x,\ x=0,\ x=\pi$

Required area



$$=\int_0^{\pi/2}\cos x\,dx+\left|\int_{\pi/2}^{\pi}\cos x\,dx\right|$$

$$=[\sin x]_0^{\pi/2}+\left|(\sin x)_{\pi/2}^{\pi}\right|$$

$$=\left[\sin\frac{\pi}{2}-\sin 0\right]+\left|\left[\sin\pi-\sin\frac{\pi}{2}\right]\right|$$

$$=(1-0)+|0-1|=1+1=2 \text{ sq. units}$$

Hence, the correct option is (*a*).

Q29. The area of the region bounded by parabola $y^2=x$ and the straight line $2y=x$ is (*a*) $\frac{4}{3}$ sq. units (*b*) 1 sq. unit

(*c*) $\frac{2}{3}$ sq. units (*d*) $\frac{1}{3}$ sq. units

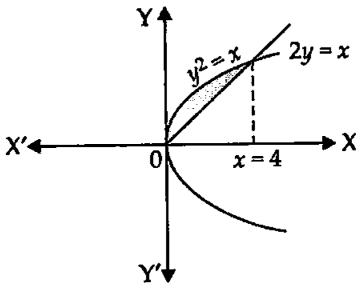
Sol. Given equation of parabola is $y^2 = x$ ...(*i*)

and equation of straight line is $2y = x$ ...(*ii*)

Solving eqns. (*i*) and (*ii*) we get

$$\left(\frac{x}{2}\right)^2 = x \Rightarrow \frac{x^2}{4} = x \Rightarrow x^2 = 4x$$

$$\Rightarrow x(x-4) = 0 \quad \therefore x = 0, 4$$



Required area

$$= \int_0^4 \sqrt{x}\, dx - \int_0^4 \frac{x}{2}\, dx$$

$$= \frac{2}{3}[x^{3/2}]_0^4 - \frac{1}{2} \cdot \frac{1}{2}[x^2]_0^4$$

$$= \frac{2}{3}\left[(4)^{3/2} - 0\right] - \frac{1}{4}\left[(4)^2 - 0\right] = \frac{2}{3} \times 8 - \frac{1}{4} \times 16$$

$$= \frac{16}{3} - 4 = \frac{4}{3} \text{ sq. units}$$

Hence, the correct answer is (*a*).

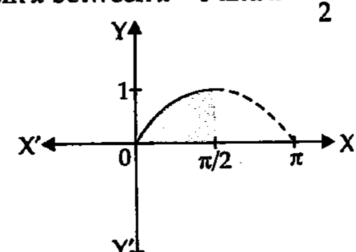
Q30. The area of the region bounded by the curve $y = \sin x$, between the ordinates $x = 0$ and $x = \frac{\pi}{2}$ and the x-axis is

(*a*) 2 sq. units (*b*) 4 sq. units

(*c*) 3 sq. units (*d*) 1 sq. units

Sol. Given equation of curve is $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$

Area of required region



$$= \int_0^{\pi/2} \sin x\, dx = -[\cos x]_0^{\pi/2}$$

$$= -\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -[0 - 1] = 1 \text{ sq. unit}$$

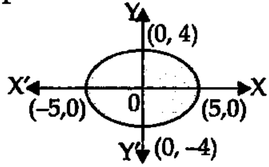
Hence, the correct answer is (*d*).

Q31. The area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

(*a*) 20π sq. units (*b*) $20\pi^2$ sq. units

(*c*) $16\pi^2$ sq. units (*d*) 25π sq. units

Sol. Given equation of ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$



$\Rightarrow \frac{y^2}{16}=1-\frac{x^2}{25} \Rightarrow y^2=\frac{16}{25}(25-x^2)$

$\therefore \quad y=\frac{4}{5}\sqrt{25-x^2}$

$\therefore$ Since the ellipse is symmetrical about the axes.

$\therefore$ Required area $=4\times\int_0^5 \frac{4}{5}\sqrt{25-x^2}\,dx=4\times\frac{4}{5}\int_0^5\sqrt{(5)^2-x^2}\,dx$

$=\frac{16}{5}\left[\frac{x}{2}\sqrt{(5)^2-x^2}+\frac{25}{2}\sin^{-1}\frac{x}{5}\right]_0^5$

$=\frac{16}{5}\left[0+\frac{25}{2}.\sin^{-1}\left(\frac{5}{5}\right)-0-0\right]=\frac{16}{5}\left[\frac{25}{2}.\sin^{-1}(1)\right]$

$=\frac{16}{5}\left[\frac{25}{2}.\frac{\pi}{2}\right]=20\pi$ sq. units

Hence, the correct answer is (a).

Q32. The area of the region bounded by the circle $x^2+y^2=1$ is

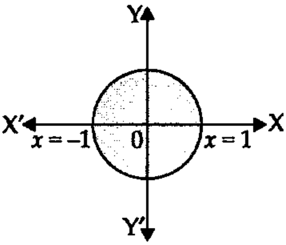
(*a*) 2π sq. units (*b*) π sq. units
(*c*) 3π sq. units (*d*) 4π sq. units

Sol. Given equation of circle is

$x^2+y^2=1 \quad \Rightarrow \quad y=\sqrt{1-x^2}$

Since the circle is symmetrical about the axes.

$\therefore$ Required area $=4\times\int_0^1\sqrt{1-x^2}\,dx$



$=4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1$

$=4\left[0+\frac{1}{2}\sin^{-1}(1)-0-0\right]$

$=4\times\frac{1}{2}\times\frac{\pi}{2}=\pi$ sq. units

Hence, the correct answer is (*b*).

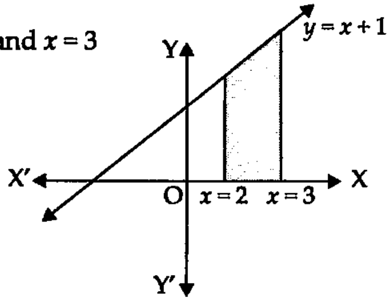
Q33. The area of the region bounded by the curve $y=x+1$ and the lines $x=2$ and $x=3$ is

(*a*) $\frac{7}{2}$ sq. units (*b*) $\frac{9}{2}$ sq. units

(*c*) $\frac{11}{2}$ sq. units (*d*) $\frac{13}{2}$ sq. units

Sol. Given equation of lines are

$$y = x + 1,\ x = 2 \text{ and } x = 3$$



Required area

$$= \int_2^3 (x+1)\,dx = \left[\frac{x^2}{2} + x\right]_2^3$$

$$= \left(\frac{9}{2} + 3\right) - \left(\frac{4}{2} + 2\right)$$

$$= \frac{15}{2} - 4 = \frac{7}{2} \text{ sq. units}$$

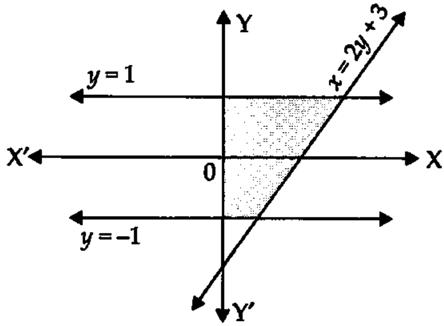
Hence, the correct option is (*a*).

Q34. The area of the region bounded by the curve $x = 2y + 3$ and the lines $y = 1$ and $y = -1$ is

(*a*) 4 sq. units (*b*) $\frac{3}{2}$ sq. units

(*c*) 6 sq. units (*d*) 8 sq. units

Sol. Given equations of lines are $x = 2y + 3$, $y = 1$ and $y = -1$



$$\text{Required area } = \int_{-1}^{1} (2y+3)\,dy$$

$$= 2 \cdot \frac{1}{2}\left[y^2\right]_{-1}^{1} + 3\,[y]_{-1}^{1}$$

$$= (1 - 1) + 3(1 + 1) = 6 \text{ sq. units}$$

Hence, the correct answer is (*c*).

❑❑❑