11 Three Dimensional Geometry

11.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the position vector of a point A in space such that $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY and $|\overrightarrow{OA}| = 10$ units.

Sol. Let $\alpha = 60°$, $\beta = 45°$ and the angle inclined to OZ axis be γ

We know that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2 60° + \cos^2 45° + \cos^2\gamma = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1 \quad \Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\gamma = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2\gamma = 1 \quad \Rightarrow \cos^2\gamma = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore \cos\gamma = \pm\frac{1}{2} \quad \Rightarrow \cos\gamma = \frac{1}{2}$$

(Rejecting $\cos\gamma = -\frac{1}{2}$, since $\gamma < 90°$)

$$\therefore \overrightarrow{OA} = |\overrightarrow{OA}|\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right) = 10\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$$

$$= 5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k}$$

Hence, the position vector of A is $(5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k})$.

Q2. Find the vector equation of the line which is parallel to the vector $3\hat{i} - 2\hat{j} + 6\hat{k}$ and which passes through the point $(1, -2, 3)$.

Sol. We know that the equation of line is

$$\vec{r} = \vec{a} + \vec{b}\lambda$$

Here, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore$ Equation of line is $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

Hence, the required equation is

$(x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

Q3. Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect. Also, find their point of intersection.

Sol. The given equations are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-4}{5}=\frac{y-1}{2}=z$$

Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$

$\therefore x=2\lambda+1,\ y=3\lambda+2$ and $z=4\lambda+3$

and $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}=\mu$

$\therefore x=5\mu+4,\ y=2\mu+1$ and $z=\mu$

If the two lines intersect each other at one point,

then $2\lambda+1=5\mu+4 \Rightarrow 2\lambda-5\mu=3$ ...(*i*)

$3\lambda+2=2\mu+1 \Rightarrow 3\lambda-2\mu=-1$ ...(*ii*)

and $4\lambda+3=\mu \Rightarrow 4\lambda-\mu=-3$ ...(*iii*)

Solving eqns. (*i*) and (*ii*) we get

$2\lambda-5\mu=3$ [multiply by 3]

$3\lambda-2\mu=-1$ [multiply by 2]

$\Rightarrow 6\lambda \quad -15\mu = 9$

$6\lambda \quad -4\mu = -2$

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$-11\mu = 11 \quad \therefore \mu=-1$

Putting the value of μ in eq. (*i*) we get,

$2\lambda-5(-1)=3$

$\Rightarrow 2\lambda+5=3$

$\Rightarrow 2\lambda=-2 \quad \therefore \lambda=-1$

Now putting the value of λ and μ in eq. (*iii*) then

$4(-1)-(-1)=-3$

$-4+1=-3$

$-3=-3$ (satisfied)

$\therefore$ Coordinates of the point of intersection are

$x=5(-1)+4=-5+4=-1$

$y=2(-1)+1=-2+1=-1$

$z=-1$

Hence, the given lines intersect each other at $(-1,-1,-1)$.

Alternately: If two lines intersect each other at a point, then the shortest distance between them is equal to 0.

For this we will use SD $=\frac{(\vec{a}_2-\vec{a}_1)(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}=0.$

Q4. Find the angle between the lines

$$\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and}$$
$$\vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k})$$

Sol. Here, $\vec{b}_1 = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b}_2 = 6\hat{i} + 3\hat{j} + 2\hat{k}$

$$\therefore \quad \cos\theta = \frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} = \frac{(2\hat{i} + \hat{j} + 2\hat{k}).(6\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{(2)^2 + (1)^2 + (2)^2}.\sqrt{(6)^2 + (3)^2 + (2)^2}}$$

$$= \frac{12 + 3 + 4}{\sqrt{4 + 1 + 4}.\sqrt{36 + 9 + 4}} = \frac{19}{\sqrt{9}.\sqrt{49}} = \frac{19}{3 \cdot 7} = \frac{19}{21}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

Hence, the required angle is $\cos^{-1}\left(\frac{19}{21}\right)$.

Q5. Prove that the line through A(0, – 1, – 1) and B(4, 5, 1) intersects the line through C(3, 9, 4) and D(– 4, 4, 4).

Sol. Given points are A(0, – 1, – 1) and B(4, 5, 1)
C(3, 9, 4) and D(– 4, 4, 4)

Cartesian form of equation AB is

$$\frac{x - 0}{4 - 0} = \frac{y + 1}{5 + 1} = \frac{z + 1}{1 + 1} \Rightarrow \frac{x}{4} = \frac{y + 1}{6} = \frac{z + 1}{2}$$

and its vector form is $\vec{r} = (-\hat{j} - \hat{k}) + \lambda(4\hat{i} + 6\hat{j} + 2\hat{k})$

Similarly, equation of CD is

$$\frac{x - 3}{-4 - 3} = \frac{y - 9}{4 - 9} = \frac{z - 4}{4 - 4} \Rightarrow \frac{x - 3}{-7} = \frac{y - 9}{-5} = \frac{z - 4}{0}$$

and its vector form is $\vec{r} = (3\hat{i} + 9\hat{j} + 4\hat{k}) + \mu(-7\hat{i} - 5\hat{j})$

Now, here $\vec{a}_1 = -\hat{j} - \hat{k}$, $\vec{b}_1 = 4\hat{i} + 6\hat{j} + 2\hat{k}$

$\vec{a}_2 = 3\hat{i} + 9\hat{j} + 4\hat{k}$, $\vec{b}_2 = -7\hat{i} - 5\hat{j}$

Shortest distance between AB and CD

$$\text{S.D.} = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$\vec{a}_2 - \vec{a}_1 = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (-\hat{j} - \hat{k}) = 3\hat{i} + 10\hat{j} + 5\hat{k}.$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}$$

$$= \hat{i}(0 + 10) - \hat{j}(0 + 14) + \hat{k}(-20 + 42)$$
$$= 10\hat{i} - 14\hat{j} + 22\hat{k}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(10)^2 + (-14)^2 + (22)^2}$$
$$= \sqrt{100 + 196 + 484} = \sqrt{780}$$
$$\therefore \quad \text{S.D} = \frac{(3\hat{i} + 10\hat{j} + 5\hat{k})\cdot(10\hat{i} - 14\hat{j} + 22\hat{k})}{\sqrt{780}}$$
$$= \frac{30 - 140 + 110}{\sqrt{780}} = 0$$

Hence, the two lines intersect each other.

Q6. Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular, if $pp' + rr' + 1 = 0$

Sol. Given that: $x = py + q \Rightarrow y = \dfrac{x-q}{p}$

and $z = ry + s \Rightarrow y = \dfrac{z-s}{r}$

$\therefore$ the equation becomes

$\dfrac{x-q}{p} = \dfrac{y}{1} = \dfrac{z-s}{r}$ in which d'ratios are $a_1 = p, b_1 = 1, c_1 = r$

Similarly $x = p'y + q' \Rightarrow y = \dfrac{x-q'}{p'}$

and $z = r'y + s' \Rightarrow y = \dfrac{z-s'}{r'}$

$\therefore$ the equation becomes

$\dfrac{x-q'}{p'} = \dfrac{y}{1} = \dfrac{z-s'}{r'}$ in which $a_2 = p', b_2 = 1, c_2 = r'$

If the lines are perpendicular to each other, then

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$
$$pp' + 1.1 + rr' = 0$$

Hence, $pp' + rr' + 1 = 0$ is the required condition.

Q7. Find the equation of a plane which bisects perpendicularly the line joining the points A(2, 3, 4), B(4, 5, 8) at right angles.

Sol. Given that A(2, 3, 4) and B(4, 5, 8)

Coordinates of mid-point C are $\left(\dfrac{2+4}{2}, \dfrac{3+5}{2}, \dfrac{4+8}{2}\right) = (3, 4, 6)$

Now direction ratios of the normal to the plane
= direction ratios of AB
$= 4 - 2, 5 - 3, 8 - 4 = (2, 2, 4)$

Equation of the plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$
$$\Rightarrow \quad 2(x - 3) + 2(y - 4) + 4(z - 6) = 0$$
$$\Rightarrow \quad 2x - 6 + 2y - 8 + 4z - 24 = 0$$
$$\Rightarrow \quad 2x + 2y + 4z = 38 \Rightarrow x + y + 2z = 19$$

Hence, the required equation of plane is

$x+y+2z=19$ or $\vec{r}(\hat{i}+\hat{j}+2\hat{k})=19.$

Q8. Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axis.

Sol. Since, the normal to the plane is equally inclined to the axes

$\therefore \cos\alpha=\cos\beta=\cos\gamma$

$\Rightarrow \cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$

$$\Rightarrow \quad 3\cos^2\alpha=1 \Rightarrow \cos\alpha=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \cos\alpha=\cos\beta=\cos\gamma=\frac{1}{\sqrt{3}}$$

So, the normal is $\quad \vec{N}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$

$\therefore$ Equation of the plane is $\vec{r}.\vec{N}=d$

$$\Rightarrow \quad \vec{r}.\frac{\vec{N}}{|\vec{N}|}=d$$

$$\Rightarrow \quad \frac{\vec{r}.\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)}{1}=3\sqrt{3}$$

$$\Rightarrow \quad \vec{r}.\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)=3\sqrt{3}$$

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k}).\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})=3\sqrt{3}$$

$$\Rightarrow \quad x+y+z=3\sqrt{3}.\sqrt{3} \Rightarrow x+y+z=9$$

Hence, the required equation of plane is $x+y+z=9$.

Q9. If the line drawn from the point $(-2,-1,-3)$ meets a plane at right angle at the point $(1,-3,3)$, find the equation of the plane.

Sol. Direction ratios of the normal to the plane are

$(1+2,-3+1,3+3) \Rightarrow (3,-2,6)$

Equation of plane passing through one point (x_1,y_1,z_1) is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

$$\Rightarrow \quad 3(x-1)-2(y+3)+6(z-3)=0$$

$$\Rightarrow \quad 3x-3-2y-6+6z-18=0$$

$$\Rightarrow \quad 3x-2y+6z-27=0 \Rightarrow 3x-2y+6z=27$$

Hence, the required equation is $3x-2y+6z=27$.

Q10. Find the equation of the plane passing through the points $(2,1,0)$, $(3,-2,-2)$ and $(3,1,7)$.

Sol. Since, the equation of the plane passing through the points (x_1,y_1,z_1), (x_2,y_2,z_2) and (x_3,y_3,z_3) is

$$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-1 & z-0 \\ 3-2 & -2-1 & -2-0 \\ 3-2 & 1-1 & 7-0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-1 & z \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)\begin{vmatrix} -3 & -2 \\ 0 & 7 \end{vmatrix} - (y-1)\begin{vmatrix} 1 & -2 \\ 1 & 7 \end{vmatrix} + z\begin{vmatrix} 1 & -3 \\ 1 & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-2)(-21) - (y-1)(7+2) + z(3) = 0$

$\Rightarrow \quad -21(x-2) - 9(y-1) + 3z = 0$

$\Rightarrow \quad -21x + 42 - 9y + 9 + 3z = 0$

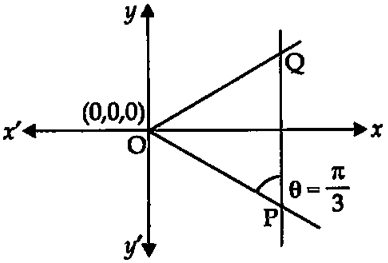
$\Rightarrow \quad -21x - 9y + 3z + 51 = 0 \Rightarrow 7x + 3y - z - 17 = 0$

Hence, the required equation is $7x + 3y - z - 17 = 0$.

Q11. Find the equations of two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

Sol. Any point on the given line is

$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} = \lambda$$



$\Rightarrow \quad x = 2\lambda + 3, y = \lambda + 3$

and $\quad z = \lambda$

Let it be the coordinates of P

$\therefore$ Direction ratios of OP are

$(2\lambda + 3 - 0), (\lambda + 3 - 0)$ and $(\lambda - 0) \Rightarrow 2\lambda + 3, \lambda + 3, \lambda$

But the direction ratios of the line PQ are 2, 1, 1

$$\therefore \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,.\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\frac{\pi}{3} = \frac{2(2\lambda+3) + 1(\lambda+3) + 1.\lambda}{\sqrt{(2)^2 + (1)^2 + (1)^2}\,.\sqrt{(2\lambda+3)^2 + (\lambda+3)^2 + \lambda^2}}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{4\lambda + 6 + \lambda + 3 + \lambda}{\sqrt{6}\,.\sqrt{4\lambda^2 + 9 + 12\lambda + \lambda^2 + 9 + 6\lambda + \lambda^2}}$$

$$\Rightarrow \quad \frac{\sqrt{6}}{2} = \frac{6\lambda + 9}{\sqrt{6\lambda^2 + 18\lambda + 18}} = \frac{6\lambda + 9}{\sqrt{6}\sqrt{\lambda^2 + 3\lambda + 3}}$$

$$\Rightarrow \quad \frac{6}{2} = \frac{3(2\lambda+3)}{\sqrt{\lambda^2+3\lambda+3}} \Rightarrow 3 = \frac{3(2\lambda+3)}{\sqrt{\lambda^2+3\lambda+3}}$$

$$\Rightarrow \quad 1 = \frac{2\lambda+3}{\sqrt{\lambda^2+3\lambda+3}} \Rightarrow \sqrt{\lambda^2+3\lambda+3} = 2\lambda+3$$

$\Rightarrow \quad \lambda^2+3\lambda+3 = 4\lambda^2+9+12\lambda$ (Squaring both sides)

$\Rightarrow \quad 3\lambda^2+9\lambda+6=0 \Rightarrow \lambda^2+3\lambda+2=0$

$\Rightarrow \quad (\lambda+1)(\lambda+2)=0$

$\therefore \quad \lambda=-1, \lambda=-2$

$\therefore$ Direction ratios are $[2(-1)+3, -1+3, -1]$ i.e., $1, 2, -1$ when $\lambda=-1$ and $[2(-2)+3, -2+3, -2]$ i.e., $-1, 1, -2$ when $\lambda=-2$.

Hence, the required equations are

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} \text{ and } \frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}.$$

Q12. Find the angle between the lines whose direction cosines are given by the equations $l+m+n=0$ and $l^2+m^2-n^2=0$

Sol. The given equations are

$$l+m+n=0 \quad ...(i)$$

$$l^2+m^2-n^2=0 \quad ...(ii)$$

From equation (*i*) $n=-(l+m)$

Putting the value of n in eq. (*ii*) we get

$l^2+m^2+[-(l+m)^2]=0$

$\Rightarrow \quad l^2+m^2-l^2-m^2-2lm=0$

$\Rightarrow \quad -2lm=0$

$\Rightarrow \quad lm=0 \Rightarrow (-m-n)m=0 \; [\because l=-m-n]$

$\Rightarrow \quad (m+n)m=0 \Rightarrow m=0 \text{ or } m=-n$

$\Rightarrow \quad l=0 \text{ or } l=-n$

$\therefore$ Direction cosines of the two lines are

$0, -n, n$ and $-n, 0, n \Rightarrow 0, -1, 1$ and $-1, 0, 1$

$$\therefore \quad \cos\theta = \frac{(0\hat{i}-\hat{j}+\hat{k}).(-\hat{i}+0\hat{j}+\hat{k})}{\sqrt{(-1)^2+(1)^2}\sqrt{(-1)^2+(1)^2}} = \frac{1}{\sqrt{2}.\sqrt{2}} = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Hence, the required angle is $\frac{\pi}{3}$.

Q13. If a variable line in two adjacent positions has direction cosines l, m, n and $l+\delta l, m+\delta m, n+\delta n$, show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

Sol. Given that l, m, n and $l+\delta l, m+\delta m, n+\delta n$, are the direction cosines of a variable line in two positions

$$\therefore \quad l^2+m^2+n^2=1 \quad ...(i)$$

and $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$ ...(ii)

$\Rightarrow l^2 + \delta l^2 + 2l.\delta l + m^2 + \delta m^2 + 2m.\delta m + n^2 + \delta n^2 + 2n.\delta n = 1$

$\Rightarrow (l^2 + m^2 + n^2) + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l.\delta l + m.\delta m + n.\delta n) = 1$

$\Rightarrow 1 + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l.\delta l + m.\delta m + n.\delta n) = 1$

$\Rightarrow l.\delta l + m.\delta m + n.\delta n = -\frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$

Let $\vec{a}$ and $\vec{b}$ be the unit vectors along a line with d'cosines l, m, n and $(l + \delta l), (m + \delta m), (n + \delta n)$.

$\therefore\ \vec{a} = l\hat{i} + m\hat{j} + n\hat{k}$ and $\vec{b} = (l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}$

$$\cos \delta\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos \delta\theta = \frac{(l\hat{i} + m\hat{j} + n\hat{k}).\left[(l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}\right]}{1.1}$$

$\left[\because |\vec{a}| = |\vec{b}| = 1\right]$

$\Rightarrow \cos \delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$

$\Rightarrow \cos \delta\theta = l^2 + l.\delta l + m^2 + m.\delta m + n^2 + n.\delta n$

$\Rightarrow \cos \delta\theta = (l^2 + m^2 + n^2) + (l.\delta l + m.\delta m + n.\delta n)$

$\Rightarrow \cos \delta\theta = 1 - \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$

$\Rightarrow 1 - \cos \delta\theta = \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$

$\Rightarrow 2\sin^2 \frac{\delta\theta}{2} = \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$

$\Rightarrow 4\sin^2 \frac{\delta\theta}{2} = \delta l^2 + \delta m^2 + \delta n^2$

$\left[\because \frac{\delta\theta}{2} \text{ is very small so, } \sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$

$\Rightarrow 4\left(\frac{\delta\theta}{2}\right)^2 = \delta l^2 + \delta m^2 + \delta n^2$

$\Rightarrow (\delta\theta)^2 = \delta l^2 + \delta m^2 + \delta n^2$ Hence proved.

Q14. O is the origin and A is (a, b, c). Find the direction cosines of the line OA and the equation of plane through A at right angle to OA.

Sol. We have $A(a, b, c)$ and $O(0, 0, 0)$

$\therefore$ direction ratios of OA $= a - 0, b - 0, c - 0$

$= a, b, c$

$\therefore$ direction cosines of line OA

$$= \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Now direction ratios of the normal to the plane are (a, b, c).

$\therefore$ Equation of the plane passing through the point $A(a, b, c)$ is

$a(x - a) + b(y - b) + c(z - c) = 0$

$\Rightarrow \quad ax - a^2 + by - b^2 + cz - c^2 = 0$

$\Rightarrow \quad ax + by + cz = a^2 + b^2 + c^2$

Hence, the required equation is $ax + by + cz = a^2 + b^2 + c^2$.

Q15. Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c' respectively from the origin, prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$.

Sol. Let OX, OY, OZ and ox, oy, oz be two rectangular systems

$\therefore$ Equations of two planes are

$\frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} = 1$ ...(i) and $\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1$ ...(ii)

Length of perpendicular from origin to plane (i) is

$$= \left| \frac{\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \right| = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

Length of perpendicular from origin to plane (ii)

$$= \left| \frac{\frac{0}{a'} + \frac{0}{b'} + \frac{0}{c'} - 1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}} \right| = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

As per the condition of the question

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

Hence, $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$

LONG ANSWER TYPE QUESTIONS

Q16. Find the foot of perpendicular from the point (2, 3, – 8) to the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.

Sol. Given that: $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$ is the equation of line

$\Rightarrow \quad \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = \lambda$

$\therefore$ Coordinates of any point Q on the line are

$x = -2\lambda + 4$, $y = 6\lambda$ and $z = -3\lambda + 1$

and the given point is P(2, 3, – 8)
Direction ratios of PQ are $-2\lambda + 4 - 2, 6\lambda - 3, -3\lambda + 1 + 8$
i.e., $-2\lambda + 2, 6\lambda - 3, -3\lambda + 9$
and the D'ratios of the given line are – 2, 6, – 3.
If PQ ⊥ line
then $-2(-2\lambda + 2) + 6(6\lambda - 3) - 3(-3\lambda + 9) = 0$
$\Rightarrow \quad 4\lambda - 4 + 36\lambda - 18 + 9\lambda - 27 = 0$
$\Rightarrow \quad 49\lambda - 49 = 0 \quad \Rightarrow \lambda = 1$
∴ The foot of the perpendicular is $-2(1) + 4, 6(1), -3(1) + 1$
i.e., 2, 6, – 2

$$\text{Now, distance } PQ = \sqrt{(2-2)^2 + (3-6)^2 + (-8+2)^2}$$
$$= \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

Hence, the required coordinates of the foot of perpendicular are 2, 6, – 2 and the required distance is $3\sqrt{5}$ units.

Q17. Find the distance of a point (2, 4, – 1) from the line

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}.$$

Sol. The given equation of line is

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = \lambda \text{ and any point } P(2, 4, -1)$$

Let Q be any point on the given line
∴ Coordinates of Q are $x = \lambda - 5, y = 4\lambda - 3, z = -9\lambda + 6$
D'ratios of PQ are $\lambda - 5 - 2, 4\lambda - 3 - 4, -9\lambda + 6 + 1$
i.e., $\lambda - 7, 4\lambda - 7, -9\lambda + 7$
and the d'ratios of the line are 1, 4, – 9
If PQ ⊥ line then
$$1(\lambda - 7) + 4(4\lambda - 7) - 9(-9\lambda + 7) = 0$$
$$\lambda - 7 + 16\lambda - 28 + 81\lambda - 63 = 0$$
$$\Rightarrow \quad 98\lambda - 98 = 0 \quad \therefore \lambda = 1$$
So, the coordinates of Q are $1 - 5, 4 \times 1 - 3, -9 \times 1 + 6$
i.e., – 4, 1, – 3

$$\therefore \quad PQ = \sqrt{(-4-2)^2 + (1-4)^2 + (-3+1)^2}$$
$$= \sqrt{(-6)^2 + (-3)^2 + (-2)^2} = \sqrt{36+9+4} = \sqrt{49} = 7$$

Hence, the required distance is 7 units.

Q18. Find the length and foot of perpendicular from the point $\left(1, \frac{3}{2}, 2\right)$ to the plane $2x - 2y + 4z + 5 = 0$.

Sol. Given plane is $2x - 2y + 4z + 5 = 0$ and given point is $\left(1, \frac{3}{2}, 2\right)$

D'ratios of the normal to the plane are 2, – 2, 4

So, the equation of the line passing through $\left(1, \frac{3}{2}, 2\right)$ and whose d'ratios are equal to the d'ratios of the normal to the plane i.e., 2, – 2, 4 is $\frac{x-1}{2} = \frac{y-\frac{3}{2}}{-2} = \frac{z-2}{4} = \lambda$

∴ Any point in the plane is $2\lambda + 1, \; -2\lambda + \frac{3}{2}, \; 4\lambda + 2$

Since, the point lies in the plane, then

$$2(2\lambda + 1) - 2\left(-2\lambda + \frac{3}{2}\right) + 4(4\lambda + 2) + 5 = 0$$

$$\Rightarrow 4\lambda + 2 + 4\lambda - 3 + 16\lambda + 8 + 5 = 0$$

$$\Rightarrow 24\lambda + 12 = 0 \quad \therefore \lambda = -\frac{1}{2}$$

So, the coordinates of the point in the plane are

$$2\left(-\frac{1}{2}\right) + 1, \; -2\left(-\frac{1}{2}\right) + \frac{3}{2}, \; 4\left(-\frac{1}{2}\right) + 2 \text{ i.e., } 0, \frac{5}{2}, 0$$

Hence, the foot of the perpendicular is $\left(0, \frac{5}{2}, 0\right)$ and the

required length $= \sqrt{(1-0)^2 + \left(\frac{3}{2} - \frac{5}{2}\right)^2 + (2-0)^2}$

$= \sqrt{1 + 1 + 4} = \sqrt{6}$ units

Q19. Find the equations of the line passing through the point (3, 0, 1) and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.

Sol. Given point is (3, 0, 1) and the equation of planes are

$$x + 2y = 0 \quad \text{...(i)}$$

and $$3y - z = 0 \quad \text{...(ii)}$$

Equation of any line l passing through (3, 0, 1) is

$$l : \frac{x-3}{a} = \frac{y-0}{b} = \frac{z-1}{c}$$

Direction ratios of the normal to plane (*i*) and plane (*ii*) are (1, 2, 0) and (0, 3, – 1)

Since the line is parallel to both the planes.

∴ $1.a + 2.b + 0.c = 0 \Rightarrow a + 2b + 0c = 0$

and $0.a + 3.b - 1.c = 0 \Rightarrow 0.a + 3b - c = 0$

So $$\frac{a}{-2-0} = \frac{-b}{-1-0} = \frac{c}{3-0} = \lambda$$

∴ $a = -2\lambda, b = \lambda, c = 3\lambda$

So, the equation of line is

$$\frac{x-3}{-2\lambda} = \frac{y}{\lambda} = \frac{z-1}{3\lambda}$$

Hence, the required equation is

$$\frac{x-3}{-2}=\frac{y}{1}=\frac{z-1}{3}$$

or in vector form is

$$(x-3)\hat{i}+y\hat{j}+(z-1)\hat{k} = \lambda(-2\hat{i}+\hat{j}+3\hat{k})$$

Q20. Find the equation of the plane through the points (2, 1, – 1) and (– 1, 3, 4), and perpendicular to the plane $x-2y+4z=10$.

Sol. Equation of the plane passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) with its normal's d'ratios is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \quad ...(i)$$

If the plane is passing through the given points (2, 1, – 1) and (– 1, 3, 4) then

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1) = 0$$

$$\Rightarrow \quad a(-1-2)+b(3-1)+c(4+1) = 0$$

$$\Rightarrow \quad -3a+2b+5c = 0 \quad ...(ii)$$

Since the required plane is perpendicular to the given plane $x-2y+4z=10$, then

$$1.a-2.b+4.c = 10 \quad ...(iii)$$

Solving (*ii*) and (*iii*) we get,

$$\frac{a}{8+10}=\frac{-b}{-12-5}=\frac{c}{6-2}=\lambda$$

$a=18\lambda$, $b=17\lambda$, $c=4\lambda$

Hence, the required plane is

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1) = 0$$

$$\Rightarrow \quad 18x-36+17y-17+4z+4 = 0$$

$$\Rightarrow \quad 18x+17y+4z-49 = 0$$

Q21. Find the shortest distance between the lines given by

$$\vec{r} = (8+3\lambda)\hat{i}-(9+16\lambda)\hat{j}+(10+7\lambda)\hat{k}$$

and $\quad \vec{r} = 15\hat{i}+29\hat{j}+5\hat{k}+\mu(3\hat{i}+8\hat{j}-5\hat{k})$.

Sol. Given equations of lines are

$$\vec{r} = (8+3\lambda)\hat{i}-(9+16\lambda)\hat{j}+(10+7\lambda)\hat{k} \quad ...(i)$$

and $\quad \vec{r} = 15\hat{i}+29\hat{j}+5\hat{k}+\mu(3\hat{i}+8\hat{j}-5\hat{k}) \quad ...(ii)$

Equation (*i*) can be re-written as

$$\vec{r} = 8\hat{i}-9\hat{j}+10\hat{k}+\lambda(3\hat{i}-16\hat{j}+7\hat{k}) \quad ...(iii)$$

Here, $\quad \vec{a}_1 = 8\hat{i}-9\hat{j}+10\hat{k}$ and $\vec{a}_2=15\hat{i}+29\hat{j}+5\hat{k}$

$\vec{b}_1 = 3\hat{i}-16\hat{j}+7\hat{k}$ and $\vec{b}_2=3\hat{i}+8\hat{j}-5\hat{k}$

$$\vec{a}_2-\vec{a}_1 = 7\hat{i}+38\hat{j}-5\hat{k}$$

$$\vec{b}_1\times\vec{b}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}$$

$$= \hat{i}(80-56) - \hat{j}(-15-21) + \hat{k}(24+48)$$
$$= 24\hat{i} + 36\hat{j} + 72\hat{k}$$

$\therefore$ Shortest distance, SD $= \left|\dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

$$= \left|\frac{(7\hat{i} + 38\hat{j} - 5\hat{k}).(24\hat{i} + 36\hat{j} + 72\hat{k})}{\sqrt{(24)^2 + (36)^2 + (72)^2}}\right|$$

$$= \left|\frac{168 + 1368 - 360}{\sqrt{576 + 1296 + 5184}}\right| = \left|\frac{168 + 1008}{\sqrt{7056}}\right| = \frac{1176}{84} = 14 \text{ units}$$

Hence, the required distance is 14 units.

Q22. Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$.

Sol. The given planes are

$P_1: \quad 5x + 3y + 6z + 8 = 0$

$P_2: \quad x + 2y + 3z - 4 = 0$

$P_3: \quad 2x + y - z + 5 = 0$

Equation of the plane passing through the line of intersection of P_2 and P_3 is

$$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$$

$\Rightarrow \quad (1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z - 4 + 5\lambda = 0 \quad$ ...(*i*)

Plane (*i*) is perpendicular to P_1, then

$$5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0$$

$\Rightarrow \quad 5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$

$\Rightarrow \quad 7\lambda + 29 = 0$

$\therefore \quad \lambda = \dfrac{-29}{7}$

Putting the value of λ in eq. (*i*), we get

$$\left[1+2\left(\frac{-29}{7}\right)\right]x + \left[2-\frac{29}{7}\right]y + \left[3+\frac{29}{7}\right]z - 4 + 5\left(\frac{-29}{7}\right) = 0$$

$\Rightarrow \dfrac{-15}{7}x - \dfrac{15}{7}y + \dfrac{50}{7}z - 4 - \dfrac{145}{7} = 0$

$\Rightarrow \quad -15x - 15y + 50z - 28 - 145 = 0$

$\Rightarrow \quad -15x - 15y + 50z - 173 = 0 \quad \Rightarrow \quad 51x + 15y - 50z + 173 = 0$

Q23. The plane $ax + by = 0$ is rotated about its line of intersection with plane $z = 0$ through an angle α. Prove that the equation of the plane in its new position is $ax + by \pm \left(\sqrt{a^2 + b^2} \tan \alpha\right)z = 0$.

Sol. Given planes are:

$$ax + by = 0 \quad \text{...(i)}$$
$$z = 0 \quad \text{...(ii)}$$

Equation of any plane passing through the line of intersection of plane (*i*) and (*ii*) is

$$(ax + by) + kz = 0 \quad \Rightarrow \quad ax + by + kz = 0 \quad \text{...(iii)}$$

Dividing both sides by $\sqrt{a^2 + b^2 + k^2}$, we get

$$\frac{a}{\sqrt{a^2+b^2+k^2}}x + \frac{b}{\sqrt{a^2+b^2+k^2}}y + \frac{k}{\sqrt{a^2+b^2+k^2}}z = 0$$

∴ Direction cosines of the normal to the plane are

$$\frac{a}{\sqrt{a^2+b^2+k^2}}, \frac{b}{\sqrt{a^2+b^2+k^2}}, \frac{k}{\sqrt{a^2+b^2+k^2}}$$

and the direction cosines of the plane (*i*) are

$$\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 0$$

Since, α is the angle between the planes (*i*) and (*iii*), we get

$$\cos\alpha = \frac{a.a + b.b + k.0}{\sqrt{a^2+b^2+k^2}.\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad \cos\alpha = \frac{a^2+b^2}{\sqrt{a^2+b^2+k^2}.\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad \cos\alpha = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+k^2}} \quad \Rightarrow \cos^2\alpha = \frac{a^2+b^2}{a^2+b^2+k^2}$$

$\Rightarrow \; (a^2+b^2+k^2)\cos^2\alpha = a^2+b^2$

$\Rightarrow a^2\cos^2\alpha + b^2\cos^2\alpha + k^2\cos^2\alpha = a^2+b^2$

$\Rightarrow \quad k^2\cos^2\alpha = a^2 - a^2\cos^2\alpha + b^2 - b^2\cos^2\alpha$

$\Rightarrow \quad k^2\cos^2\alpha = \alpha^2(1-\cos^2\alpha) + b^2(1-\cos^2\alpha)$

$\Rightarrow \quad k^2\cos^2\alpha = a^2\sin^2\alpha + b^2\sin^2\alpha$

$\Rightarrow \quad k^2\cos^2\alpha = (a^2+b^2)\sin^2\alpha$

$$\Rightarrow \quad k^2 = (a^2+b^2)\frac{\sin^2\alpha}{\cos^2\alpha} \quad \Rightarrow k = \pm\sqrt{a^2+b^2}.\tan\alpha$$

Putting the value of k in eq. (*iii*) we get

$ax + by \pm \left(\sqrt{a^2+b^2}.\tan\alpha\right)z = 0$ which is the required equation of plane.

Hence proved.

Q24. Find the equation of the plane through the intersection of the planes $\vec{r}.(\hat{i}+3\hat{j})-6=0$ and $\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})=0$, whose perpendicular distance from origin is unity.

Sol. Given planes are;

$$\vec{r}.(\hat{i}+3\hat{j})-6=0 \Rightarrow x+3y-6=0 \quad ...(i)$$

and $$\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})=0 \Rightarrow 3x-y-4z=0 \quad ...(ii)$$

Equation of the plane passing through the line of intersection of plane (*i*) and (*ii*) is

$$(x+3y-6)+k(3x-y-4z)=0 \quad ...(iii)$$

$$(1+3k)x+(3-k)y-4kz-6=0$$

Perpendicular distance from origin

$$=\left|\frac{-6}{\sqrt{(1+3k)^2+(3-k)^2+(-4k)^2}}\right|=1$$

$$\Rightarrow \frac{36}{1+9k^2+6k+9+k^2-6k+16k^2}=1 \text{ [Squaring both sides]}$$

$$\Rightarrow \frac{36}{26k^2+10}=1 \quad \Rightarrow 26k^2+10=36$$

$$\Rightarrow 26k^2=26 \quad \Rightarrow k^2=1 \quad \therefore k=\pm 1$$

Putting the value of k in eq. (*iii*) we get,

$$(x+3y-6)\pm(3x-y-4z)=0$$

$\Rightarrow x+3y-6+3x-y-4z=0$ and $x+3y-6-3x+y+4z=0$

$\Rightarrow 4x+2y-4z-6=0$ and $-2x+4y+4z-6=0$

Hence, the required equations are:

$4x+2y-4z-6=0$ and $-2x+4y+4z-6=0$.

Q25. Show that the points $(\hat{i}-\hat{j}+3\hat{k})$ and $3(\hat{i}+\hat{j}+\hat{k})$ are equidistant from the plane $\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=0$ and lies on opposite side of it.

Sol. Given points are $P(\hat{i}-\hat{j}+3\hat{k})$ and $Q(3\hat{i}+3\hat{j}+3\hat{k})$ and the plane $\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=0$

Perpendicular distance of $P(\hat{i}-\hat{j}+3\hat{k})$ from the plane

$$\vec{r}.(5\hat{i}+2\hat{j}-7\hat{k})+9=\left|\frac{(\hat{i}-\hat{j}+3\hat{k}).(5\hat{i}+2\hat{j}-7\hat{k})+9}{\sqrt{(5)^2+(2)^2+(-7)^2}}\right|$$

$$=\left|\frac{5-2-21+9}{\sqrt{25+4+49}}\right|=\left|\frac{-9}{\sqrt{78}}\right|$$

and perpendicular distance of $Q(3\hat{i}+3\hat{j}+3\hat{k})$ from the plane

$$=\left|\frac{(3\hat{i}+3\hat{j}+3\hat{k}).(5\hat{i}+2\hat{j}-7\hat{k})+9}{\sqrt{25+4+49}}\right|$$

$$=\left|\frac{15+6-21+9}{\sqrt{78}}\right|=\left|\frac{9}{\sqrt{78}}\right|$$

Hence, the two points are equidistant from the given plane. Opposite sign shows that they lie on either side of the plane.

Q26. $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

Sol. Position vector of A is $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$

So, equation of any line passing through A and parallel to $\overrightarrow{AB}$

$$\vec{r} = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k}) \quad ...(i)$$

Now any point P on $\overrightarrow{AB} = (6 + 3\lambda, 7 - \lambda, 4 + \lambda)$

Similarly, position vector of C is $-9\hat{j} + 2\hat{k}$

and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

So, equation of any line passing through C and parallel to $\overrightarrow{CD}$ is

$$\vec{r} = (-9\hat{j} + 2\hat{k}) + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) \quad ...(ii)$$

Any point Q on $\overrightarrow{CD} = (-3\mu, -9 + 2\mu, 2 + 4\mu)$

d'ratios of $\overrightarrow{PQ}$ are

$$(6 + 3\lambda + 3\mu, 7 - \lambda + 9 - 2\mu, 4 + \lambda - 2 - 4\mu)$$

$\Rightarrow$ $(6 + 3\lambda + 3\mu), (16 - \lambda - 2\mu), (2 + \lambda - 4\mu)$

Now $\overrightarrow{PQ}$ is $\perp$ to eq. (*i*), then

$3(6 + 3\lambda + 3\mu) - 1(16 - \lambda - 2\mu) + 1(2 + \lambda - 4\mu) = 0$

$\Rightarrow \quad 18 + 9\lambda + 9\mu - 16 + \lambda + 2\mu + 2 + \lambda - 4\mu = 0$

$\Rightarrow \quad 11\lambda + 7\mu + 4 = 0 \quad ...(iii)$

$\overrightarrow{PQ}$ is also $\perp$ to eq. (*ii*), then

$-3(6 + 3\lambda + 3\mu) + 2(16 - \lambda - 2\mu) + 4(2 + \lambda - 4\mu) = 0$

$\Rightarrow \quad -18 - 9\lambda - 9\mu + 32 - 2\lambda - 4\mu + 8 + 4\lambda - 16\mu = 0$

$\Rightarrow \quad -7\lambda - 29\mu + 22 = 0$

$\Rightarrow \quad 7\lambda + 29\mu - 22 = 0 \quad ...(iv)$

Solving eq. (*iii*) and (*iv*) we get

$$\begin{array}{rrrl} 77\lambda & + 49\mu & + 28 & = 0 \\ 77\lambda & + 319\mu & - 242 & = 0 \\ (-) & (-) & (+) & \\ \hline & -270\mu & + 270 & = 0 \end{array}$$

$\therefore \mu = 1$

Now using $\mu = 1$ in eq. (*iv*) we get

$$7\lambda + 29 - 22 = 0 \quad \Rightarrow \quad \lambda = -1$$

$\therefore$ Position vector of $P = [6 + 3(-1), 7 + 1, 4 - 1] = (3, 8, 3)$

and position vector of $Q = [-3(1), -9 + 2(1), 2 + 4(1)] = (-3, -7, 6)$

Hence, the position vectors of

$$P = 3\hat{i} + 8\hat{j} + 3\hat{k} \text{ and } Q = -3\hat{i} - 7\hat{j} + 6\hat{k}$$

Q27. Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.

Sol. Given that $2l + 2m - n = 0$ ...(*i*)

and $mn + nl + lm = 0$ ...(*ii*)

Eliminating m from eq. (*i*) and (*ii*) we get,

$$m = \frac{n - 2l}{2} \quad \text{[from (i)]}$$

$$\Rightarrow \left(\frac{n-2l}{2}\right)n + nl + l\left(\frac{n-2l}{2}\right) = 0$$

$$\Rightarrow \frac{n^2 - 2nl + 2nl + nl - 2l^2}{2} = 0$$

$$\Rightarrow n^2 + nl - 2l^2 = 0$$

$$\Rightarrow n^2 + 2nl - nl - 2l^2 = 0$$

$$\Rightarrow n(n + 2l) - l(n + 2l) = 0$$

$$\Rightarrow (n - l)(n + 2l) = 0$$

$$\Rightarrow n = -2l \quad \text{and} \quad n = l$$

$$\therefore \quad m = \frac{-2l - 2l}{2}, \quad m = \frac{l - 2l}{2}$$

$$\Rightarrow m = -2l, \quad m = \frac{-l}{2}$$

Therefore, the direction ratios are proportional to $l, -2l, -2l$ and $l, \frac{-l}{2}, l$.

$\Rightarrow$ 1, −2, −2 and 2, −1, 2

If the two lines are perpendicular to each other then

$$1(2) - 2(-1) - 2 \times 2 = 0$$

$$2 + 2 - 4 = 0$$

$$0 = 0$$

Hence, the two lines are perpendicular.

Q28. If l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$, makes equal angles with them.

Sol. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are such that

$$\vec{a} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$$

$$\vec{b} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$$
$$\vec{c} = l_3\hat{i} + m_3\hat{j} + n_3\hat{k}$$

and $\vec{d} = (l_1 + l_2 + l_3)\hat{i} + (m_1 + m_2 + m_3)\hat{j} + (n_1 + n_2 + n_3)\hat{k}$

Since the given d'cosines are mutually perpendicular then

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$
$$l_2l_3 + m_2m_3 + n_2n_3 = 0$$
$$l_1l_3 + m_1m_3 + n_1n_3 = 0$$

Let α, β and γ be the angles between $\vec{a}$ and $\vec{d}$, $\vec{b}$ and $\vec{d}$, $\vec{c}$ and $\vec{d}$ respectively.

$\therefore \cos\alpha = l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)$

$= l_1^2 + l_1l_2 + l_1l_3 + m_1^2 + m_1m_2 + m_1m_3 + n_1^2 + n_1n_2 + n_1n_3$

$= (l_1^2 + m_1^2 + n_1^2) + (l_1l_2 + m_1m_2 + n_1n_2) + (l_1l_3 + m_1m_3 + n_1n_3)$

$= 1 + 0 + 0 = 1$

$\therefore \cos\beta = l_2(l_1 + l_2 + l_3) + m_2(m_1 + m_2 + m_3) + n_2(n_1 + n_2 + n_3)$

$= l_1l_2 + l_2^2 + l_2l_3 + m_1m_2 + m_2^2 + m_2m_3 + n_1n_2 + n_2^2 + n_2n_3$

$= (l_2^2 + m_2^2 + n_2^2) + (l_1l_2 + m_1m_2 + n_1n_2) + (l_2l_3 + m_2m_3 + n_2n_3)$

$= 1 + 0 + 0 = 1$

Similarly,

$\therefore \cos\gamma = l_3(l_1 + l_2 + l_3) + m_3(m_1 + m_2 + m_3) + n_3(n_1 + n_2 + n_3)$

$= l_1l_3 + l_2l_3 + l_3^2 + m_1m_3 + m_2m_3 + m_3^2 + n_1n_3 + n_2n_3 + n_3^2$

$= (l_3^2 + m_3^2 + n_3^2) + (l_1l_3 + m_1m_3 + n_1n_3) + (l_2l_3 + m_2m_3 + n_2n_3)$

$= 1 + 0 + 0 = 1$

$\therefore \cos\alpha = \cos\beta = \cos\gamma = 1 \Rightarrow \alpha = \beta = \gamma$ which is the required result.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the Exercises from 29 to 36.

Q29. Distance of the point (α, β, γ) from *y*-axis is

(*a*) β (*b*) $|\beta|$ (*c*) $|\beta| + |\gamma|$ (*d*) $\sqrt{\alpha^2 + \gamma^2}$

Sol. The given point is (α, β, γ)

Any point on y-axis = (0, β, 0)

$\therefore$ Required distance $= \sqrt{(\alpha - 0)^2 + (\beta - \beta)^2 + (\gamma - 0)^2}$

$= \sqrt{\alpha^2 + \gamma^2}$

Hence, the correct option is (*d*).

Q30. If the direction cosines of a line are k, k, k, then

(a) $k > 0$ (b) $0 < k < 1$ (c) $k = 1$ (d) $k = \frac{1}{\sqrt{3}}$ or $\frac{-1}{\sqrt{3}}$

Sol. If l, m, n are the direction cosines of a line, then

$$l^2 + m^2 + n^2 = 1$$

So, $$k^2 + k^2 + k^2 = 1$$

$$\Rightarrow \quad 3k^2 = 1 \quad \Rightarrow \quad k = \pm\frac{1}{\sqrt{3}}$$

Hence, the correct option is (d).

Q31. The distance of the plane $\vec{r}.\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 1$ from the origin is

(a) 1 (b) 7 (c) $\frac{1}{7}$ (d) None of these

Sol. Given that: $\vec{r}.\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 1$

So, the distance of the given plane from the origin is

$$= \left|\frac{-1}{\sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2}}\right| = \left|\frac{-1}{\sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}}}\right| = \left|\frac{1}{1}\right| = 1$$

Hence, the correct option is (a).

Q32. The sine of the angle between the straight line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and the plane $2x - 2y + z = 5$ is

(a) $\frac{10}{6\sqrt{5}}$ (b) $\frac{4}{5\sqrt{2}}$ (c) $\frac{2\sqrt{3}}{5}$ (d) $\frac{\sqrt{2}}{10}$

Sol. Given that: $l: \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$

and $\quad P: 2x - 2y + z = 5$

d'ratios of the line are 3, 4, 5

and d'ratios of the normal to the plane are 2, −2, 1

$$\therefore \quad \sin\theta = \frac{3(2) + 4(-2) + 5(1)}{\sqrt{9+16+25}.\sqrt{4+4+1}}$$

$$\Rightarrow \quad \sin\theta = \frac{6-8+5}{\sqrt{50}.3} \quad \Rightarrow \quad \frac{3}{5\sqrt{2}.3} = \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}$$

Hence, the correct option is (d).

Q33. The reflection of the point (α, β, γ) in the xy-plane is

(a) $(\alpha, \beta, 0)$ (b) $(0, 0, \gamma)$ (c) $(-\alpha, -\beta, \gamma)$ (d) $(\alpha, \beta, -\gamma)$

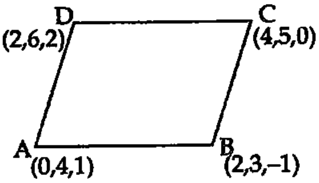
Sol. Reflection of point (α, β, γ) in xy-plane is $(\alpha, \beta, -\gamma)$.

Hence, the correct option is (d).

Q34. The area of the quadrilateral ABCD, where A(0,4,1), B(2,3,–1), C(4,5,0) and D(2,6,2) is equal to

(*a*) 9 sq. units (*b*) 18 sq. units
(*c*) 27 sq. units (*d*) 81 sq. units

Sol. Given points are
A(0, 4, 1), B(2,3,– 1), C(4, 5, 0) and D(2,6,2)
d'ratios of AB = 2,–1 –2
and d'ratios of DC = 2,–1,–2
$\therefore$ AB || DC
Similarly, d'ratios of AD = 2, 2, 1
and d'ratios of BC = 2, 2, 1
$\therefore$ AD || BC
So □ABCD is a parallelogram.



$$\overline{AB} = 2\hat{i} - \hat{j} - 2\hat{k}$$
$$\overline{AD} = 2\hat{i} + 2\hat{j} + \hat{k}$$

$\therefore$ Area of parallelogram ABCD $= |\overline{AB} \times \overline{AD}|$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = \hat{i}(-1+4) - \hat{j}(2+4) + \hat{k}(4+2) = 3\hat{i} - 6\hat{j} + 6\hat{k}$$

$$= \sqrt{(3)^2 + (-6)^2 + (6)^2} = \sqrt{9+36+36} = \sqrt{81} = 9 \text{ sq units}$$

Hence, the correct option is (*a*).

Q35. The locus represented by $xy + yz = 0$ is
(*a*) A pair of perpendicular lines
(*b*) A pair of parallel lines
(*c*) A pair of parallel planes
(*d*) A pair of perpendicular planes

Sol. Given that: $xy + yz = 0$
$$y.(x + z) = 0$$
$$y = 0 \text{ or } x + z = 0$$
Here $y = 0$ is one plane and $x + z = 0$ is another plane. So, it is a pair of perpendicular planes.
Hence, the correct option is (*d*).

Q36. The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

(*a*) $\frac{\sqrt{3}}{2}$ (*b*) $\frac{\sqrt{2}}{3}$ (*c*) $\frac{2}{7}$ (*d*) $\frac{3}{7}$

Sol. Direction ratios of the normal to the plane $2x - 3y + 6z - 11 = 0$ are $2, -3, 6$
Direction ratios of x-axis are 1, 0, 0

$\therefore$ angle between plane and line is

$$\sin\theta = \frac{2(1)-3(0)+6(0)}{\sqrt{(2)^2+(-3)^2+(6^2)}.\sqrt{(1)^2+(0)^2+(0)^2}}$$

$$= \frac{2}{\sqrt{4+9+36}} = \frac{2}{7}$$

Hence, the correct option is (*c*).

Fill in the blanks in each of the Exercises from 37 to 41.

Q37. A plane passes through the points (2, 0, 0), (0, 3, 0) and (0, 0, 4). The equation of plane is ....

Sol. Given points are (2, 0, 0), (0, 3, 0) and (0, 0, 4).
So, the intercepts cut by the plane on the axes are 2, 3, 4
Equation of the plane (intercept form) is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \Rightarrow \quad \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$$

Hence, the equation of plane is $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$.

Q38. The direction cosines of vector $(2\hat{i}+2\hat{j}-\hat{k})$ are ....

Sol. Let $\vec{a} = 2\hat{i}+2\hat{j}-\hat{k}$
direction ratios of $\vec{a}$ are 2, 2, – 1

So, the direction cosines are $\frac{2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{-1}{\sqrt{4+4+1}}$

$\Rightarrow \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$

Hence, the direction cosines of the given vector are $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

Q39. The vector equation of the line $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ is ....

Sol. The given equation is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$$

Here $\vec{a} = (5\hat{i}-4\hat{j}+6\hat{k})$ and $\vec{b} = (3\hat{i}+7\hat{j}+2\hat{k})$

Equation of the line is $\vec{r}=\vec{a}+\vec{b}\lambda$

Hence, the vector equation of the given line is

$$\vec{r} = (5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}+2\hat{k})$$

Q40. The vector equation of the line through the points (3, 4, – 7) and (1, – 1, 6) is ....

Sol. Given the points (3, 4, – 7) and (1, – 1, 6)

Here $\vec{a} = 3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 6\hat{k}$

Equation of the line is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

$\Rightarrow \vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda[(\hat{i} - \hat{j} + 6\hat{k}) - (3\hat{i} + 4\hat{j} - 7\hat{k})]$

$\Rightarrow \vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$

$\Rightarrow (x-3)\hat{i} + (y-4)\hat{j} + (z+7)\hat{k} = \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$

Hence, the vector equation of the line is

$(x-3)\hat{i} + (y-4)\hat{j} + (z+7)\hat{k} = \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$

Q41. The Cartesian equation of the plane $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 2$ is ....

Sol. Given equation is $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 2$

$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + \hat{j} - \hat{k}) = 2$

$\Rightarrow \quad x + y - z = 2$

Hence, the Cartesian equation of the plane is $x + y - z = 2$.

State True or False for the statements in each of the Exercises from 42 to 49.

Q42. The unit vector normal to the plane $x + 2y + 3z - 6 = 0$ is $\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$

Sol. Given plane is $x + 2y + 3z - 6 = 0$

Vector normal to the plane $\vec{n} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{(1)^2 + (2)^2 + (3)^2}} = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

Hence, the given statement is 'true'.

Q43. The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the coordinate axes are $-2, \frac{4}{3}, \frac{-4}{5}$.

Sol. Equation of the plane is $2x - 3y + 5z + 4 = 0$

$\Rightarrow \quad 2x - 3y + 5z = -4$

$\Rightarrow \quad \frac{2}{-4}x - \frac{3y}{-4} + \frac{5z}{-4} = 1$

$\Rightarrow \quad \frac{x}{-2} - \frac{y}{4/3} + \frac{z}{-4/5} = 1$

So, the required intercepts are $-2, \frac{4}{3}$ and $-\frac{4}{5}$

Hence, the given statement is 'true'.

Q44. The angle between the line $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r}.(3\hat{i}-4\hat{j}-\hat{k})+5=0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

Sol. Equation of line is $\vec{r}=(5\hat{i}-\hat{j}-4\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ and the equation of the plane is $\vec{r}.(3\hat{i}-4\hat{j}-\hat{k})+5=0$

Here, $\vec{b}_1=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{n}_2=3\hat{i}-4\hat{j}-\hat{k}$

$$\therefore \quad \sin\theta=\frac{\vec{b}_1\vec{n}_2}{|\vec{b}_1||\vec{n}_2|}$$

$$\Rightarrow \quad \sin\theta=\frac{(2\hat{i}-\hat{j}+\hat{k}).(3\hat{i}-4\hat{j}-\hat{k})}{\sqrt{4+1+1}.\sqrt{9+16+1}}=\frac{6+4-1}{\sqrt{6}.\sqrt{26}}=\frac{9}{\sqrt{6}.\sqrt{26}}$$

$$\Rightarrow \quad \sin\theta=\frac{9}{2\sqrt{39}} \text{ which is false.}$$

Hence, the given statement is 'false'.

Q45. The angle between the planes $\vec{r}.(2\hat{i}-3\hat{j}+\hat{k})=1$ and $\vec{r}.(\hat{i}-\hat{j})=4$ is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

Sol. The given planes are $\vec{r}.(2\hat{i}-3\hat{j}+\hat{k})=1$ and $\vec{r}.(\hat{i}-\hat{j})=4$

Here, $\vec{b}_1=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}_2=(\hat{i}-\hat{j})$

$$\text{So,} \quad \cos\theta=\frac{\vec{b}_1.\vec{n}_2}{|\vec{b}_1||\vec{n}_2|}$$

$$\Rightarrow \quad \cos\theta=\frac{(2i-3j+\hat{k}).(\hat{i}-\hat{j})}{\sqrt{4+9+1}.\sqrt{1+1}}=\frac{2+3}{\sqrt{14}.\sqrt{2}}=\frac{5}{\sqrt{28}}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{5}{\sqrt{28}}\right) \text{ which is false.}$$

Hence, the given statement is 'false'.

Q46. The line $\vec{r}=2\hat{i}-3\hat{j}-\hat{k}+\lambda(\hat{i}-\hat{j}+2\hat{k})$ lies in the plane $r\cdot(3\hat{i}+\hat{j}-\hat{k})+2=0$.

Sol. Direction ratios of the line $(\hat{i}-\hat{j}+2\hat{k})$

Direction ratios of the normal to the plane are $(3\hat{i}+\hat{j}-\hat{k})$

So $(\hat{i}-\hat{j}+2\hat{k}).(3\hat{i}+\hat{j}-\hat{k})=3-1-2=0$

Therefore, the line is parallel to the plane.

Now point through which the line is passing

$\vec{a}=2\hat{i}-3\hat{j}-\hat{k}$

If line lies in the plane then

$(2\hat{i} - 3\hat{j} - \hat{k}).(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$

$6 - 3 + 1 + 2 \neq 0$

So, the line does not lie in the plane.
Hence, the given statement is 'false'.

Q47. The vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ is $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.

Sol. The Cartesian form of the equation is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} = \lambda$$

$\therefore$ Here $x_1 = 5, y_1 = -4, z_1 = 6, a = 3, b = 7, c = 2$

So, the vector equation is $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

Hence, the given statement is 'true'.

Q48. The equation of a line, which is parallel to $2\hat{i} + \hat{j} + 3\hat{k}$ and which passes through the point (5, – 2, 4) is $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$.

Sol. Here, $x_1 = 5, y_1 = -2, z_1 = 4; a = 2, b = 1, c = 3$

We know that the equation of line is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

$$\Rightarrow \quad \frac{x-5}{2} = \frac{y+2}{1} = \frac{z-4}{3}$$

Hence, the given statement is 'false'.

Q49. If the foot of the perpendicular drawn from the origin to a plane is (5, –3, –2), then the equation of plane is $\vec{r}.(5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$.

Sol. The given equation of the plane is $\vec{r}.(5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$

If the foot of the perpendicular to this plane is

(5, – 3, – 2) i.e., $5\hat{i} - 3\hat{j} - 2\hat{k}$ then

$(5\hat{i} - 3\hat{j} - 2\hat{k}).(5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$

$\Rightarrow \quad 25 + 9 + 4 = 38$

$38 = 38$ (satisfied)

Hence, the given statement is 'true'.

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