1 Relations and Functions

Lesson at a Glance

1. *A Relation* R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$. *i.e.,* every subset of $A \times B$ is a relation from A to B. Thus, $(a, b) \in R \Rightarrow (a, b) \in A \times B$.

2. **Empty relation or void relation in a set.** A relation R in a set A is called empty relation or void relation if $R = \phi \subset A \times A$, *i.e.,* if no element of A is related to any element of A.

   Void relation ϕ is the **smallest** relation on the set A.

3. (a) **Identity relation in a Set A.** A relation R in a set A is called Identity relation if $(a, a) \in R$ for all $a \in A$ and it is denoted by I_A.

   (b) **Universal relation in a set.** A relation R in a set A is called universal relation if $R = A \times A$, *i.e.,* if each element of A is related to every element of A.

4. **Reflexive relation in a set.** A relation R in a set A is called **reflexive** if $(a, a) \in R$ for every $a \in A$, *i.e.,* every element of A is R-related to itself or aRa for all $a \in A$.

5. **Symmetric relation in a set.** A relation R in a set A is called symmetric if $(a, b) \in R$

   $\Rightarrow (b, a) \in R$ for all $a, b \in A$ *i.e.,* $aRb \Rightarrow bRa$ for all $a, b \in A$.

6. A relation R in a set A is symmetric iff $R = R^{-1}$.

7. **Transitive relation in a set.** A relation R in a set A is called transitive if $(a, b) \in R$ and $(b, c) \in R$

   $\Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

8. **Equivalence relation in a set.** A relation R in a set A is called an **equivalence relation** if R is reflexive, symmetric and transitive.

9. **Definition of function.** A relation f from a set A to a set B is a **function** if every element of A has **one and only** one image in the set B. Therefore every relation need not be a function, but every function is a relation.

10. A function f from set A to set B is denoted by $f : A \to B$. Set A is called **domain** of the function and set B is called its **co-domain.**

    If $B \subset R$, then f is called a *real valued* function. The set $f(A) = \{f(x) : x \in A\}$ is called **Range** of the function and range set $f(A)$ is a subset of co-domain set B.

Types of functions

(*i*) **One-one (or Injective) Function:** A function $f : X \to Y$ is said to be **one-one** function if distinct elements have distinct images.

i.e. for every pair $x_1, x_2 \in$ domain X s.t.

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

(*ii*) **Onto (or Surjective) Function:** A function $f : X \to Y$ is said to be an **onto** function if **Range set f (X) = Co-domain** Y.

i.e. if **every y ∈ co-domain Y, there exists.**

$x \in$ domain X s.t. $y = f(x)$.

(*iii*) **Bijective Function:** A function $f : X \to Y$ is said to be **bijective** if f is both one-one and onto.

(*iv*) **Identity Function:** A function $f : A \to A$ is said to be an **identity function** if $f(x) = x \ \forall \ x \in A$.

11. **Composition of Two Functions.** Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g (or composite of f and g) denoted by gof, is defined as the function $gof : A \to C$ given by

$$(gof)(x) = g(f(x)), \text{ for all } x \in A.$$

12. **Invertible function.** A function $f : X \to Y$ is said to be invertible if there exists a function $g : Y \to X$ such that **$gof : X \to X$ is identity fuction = I_X**

and **$fog : Y \to Y$ is identity fuction = I_Y**

The function g is called inverse of f and is denoted by f^{-1}

The function f is invertible (*i.e.* has an inverse) *iff* f **is one-one and onto.**

13. **Binary Operation on a set.** Let A be a non-empty set. A binary operation, usually denoted by $*$, on A is a function from A × A into A *i.e.*, $* : A \times A \to A$.

The unique element of the set A associated by $*$ with the ordered pair $(a, b) \in A \times A$ is denoted by $a * b$.

14. **Commutative Binary Operation on a set.** A binary operation $*$ on a set A is said to be **Commutative** if $a * b = b * a$ for all $a, b \in A$.

15. **Associative Binary Operation on a set.** A binary operation $*$ on a set A is said to be **associative** if $(a * b) * c = a * (b * c)$, for all $a, b, c \in A$.

16. **Identity element of a binary operation.** An element $e \in A$ is said to be the identity element for the binary operation $*$ on set A if $a * e = a = e * a$ for all $a \in A$.

17. **Inverse of an element w.r.t. a binary operation.** An element $b \in A$ is said to be the inverse of an element $a \in A$ w.r.t. a binary operation $*$ if $\boldsymbol{a * b = e = b * a}$.

TEXTBOOK QUESTIONS SOLVED

Exercise 1.1 *(Page No. 5-7)*

1. Determine whether each of the following relations are reflexive, symmetric and transitive:

(*i*) Relation R in the set A = {1, 2, 3, ..., 13, 14} defined as

$$R = \{(x, y) : 3x - y = 0\}$$

(*ii*) Relation R in the set N of natural numbers defined as

$$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$

(*iii*) Relation R in the set A = {1, 2, 3, 4, 5, 6} as

$$R = \{(x, y) : y \text{ is divisible by } x\}$$

(*iv*) Relation R in the set Z of all integers defined as

$$R = \{(x, y) : x - y \text{ is an integer}\}$$ **(*Important*)**

(*v*) Relation R in the set A of human beings in a town at a particular time given by

(*a*) R = {(x, y) : x and y work at the same place}

(*b*) R = {(x, y) : x and y live in the same locality}

(*c*) R = {(x, y) : x is exactly 7 cm taller than y}

(*d*) R = {(x, y) : x is wife of y}

(*e*) R = {(x, y) : x is father of y}

Sol. (*i*) **Given:** Set A = {1, 2, 3, ..., 13, 14}

Relation R in the set A is defined as

$$R = \{(x, y) : 3x - y = 0\}$$

i.e., $R = \{(x, y) : -y = -3x$ *i.e.*, $y = 3x\}$ ...(*i*)

Is R reflexive? Let $x \in A$. **Putting $y = x$** in (*i*), $x = 3x$

Dividing by $x \neq 0$ ($x \in$ A here is $\neq 0$), $1 = 3$ which is impossible. Therefore $(x, x) \notin$ R and hence R is not reflexive.

Is R symmetric? Let $(x, y) \in$ R, therefore by (*i*) $y = 3x$

Interchanging x and y in (*i*), $x = 3y$ which is not true

$(\because\ y = 3x \Rightarrow x = \frac{y}{3}$ and $\neq 3y)$

For example, $2 \in$ A, $6 \in$ A and $6 = 3(2)$ (*i.e.*, $y = 3x$)

but $2 \neq 3(6) = 18$.

$\therefore$ $(2, 6) \in$ R but $(6, 2) \notin$ R $\therefore$ R is not symmetric.

Is R transitive? Let $(x, y) \in$ R and $(y, z) \in$ R.

Therefore by (*i*), $y = 3x$ and $z = 3y$.

To eliminate y, Putting $y = 3x$ in $z = 3y$, $z = 3(3x) = 9x$

$\therefore$ By (*i*), $(x, z) \notin$ R.

$\therefore$ R is not transitive.

(*ii*) **Given:** Relation R in the set of natural numbers defined as

$$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$

i.e., $R = \{(x, x + 5) \text{ and } x = 1, 2, 3\}$

$\therefore$ In Roster form; $R = \{(1, 1 + 5), (2, 2 + 5), (3, 3 + 5)\}$

i.e., $R = \{(1, 6), (2, 7), (3, 8)\}$

Domain of R is the set of x co-ordinates of R. *i.e.,* the set $\{1, 2, 3\}$.

R is not reflexive because $(1, 1) \notin R$, $(2, 2) \notin R$, $(3, 3) \notin R$

R is not symmetric because $(1, 6) \in R$ but $(6, 1) \notin R$.

R is transitive because there are no two pairs of the type (x, y) and $(y, z) \in R = \{(1, 6), (2, 7), (3, 8)\}$;

so we should have no reason to expect $(x, z) \in R$.

Remark 1. Please note carefully and learn that relation R in the above question is transitive.

Remark 2. Whenever set A is a small finite set, it is always better to write R in roster form.

(*iii*) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ defined as $R = \{(x, y) : y \text{ is divisible by } x\}$.

$\therefore$ In roster form; $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

R is reflexive because $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)$ all $\in R$, *i.e.,* $(x, x) \in R$ for all $x \in A$.

R is not symmetric because $(2, 4) \in R$ but $(4, 2) \notin R$ as 2 is not divisible by 4.

Is R transitive? Let $x, y, z \in A$ such that $(x, y) \in R$ and $(y, z) \in R$.

$\therefore$ By definition of R in this question, y is divisible by x and z is divisible by y.

$\therefore$ There exist natural numbers m and n such that

$$y = mx \text{ and } z = ny.$$

To eliminate y: Putting $y = mx$ in $z = ny$, we have

$$z = n(mx) = (nm)x$$

$\therefore$ z is a multiple of, x, *i.e.,* z is divisible by x.

$\therefore$ $(x, z) \in R$. $\therefore$ R is transitive.

(*iv*) **Given:** Relation R in the set Z of all integers defined as

$R = \{(x, y) : (x - y) \text{ is an integer}\}$ ...(*i*)

Is R reflexive? Let $x \in Z$. Putting $y = x$ in (*i*),

$x - x = 0$ is an integer which is true.

$\therefore$ $(x, x) \in R$ for all $x \in Z$ $\therefore$ R is reflexive.

Is R symmetric? Let $x \in Z$, $y \in Z$, and $(x, y) \in R$.

$\therefore$ By (*i*) $(x - y)$ is an integer.

i.e., $-(y - x)$ is an integer and hence $y - x$ is an integer.

$\therefore$ $(y, x) \in R$. $\therefore$ R is symmetric.

Is R transitive? Let $x \in Z$, $y \in Z$ and $z \in Z$ such that $(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow$ By (*i*), $x - y$ is an integer and $y - z$ is also an integer.

Adding to eliminate y, $x - y + y - z$ = Integer + Integer

i.e., $x - z$ is an integer.

$\Rightarrow$ $(x, z) \in R$ by (*i*). $\therefore$ R is transitive.

(*v*) R is a relation in the set A of human beings of a town.

(*a*) **Given:** R = {$(x, y) : x$ and y work at the same place} ...(*i*)

Is R reflexive? Let $x \in$ A.

Putting $y = x$ in (*i*), x and x work at the same place which is true ($\because$ x and x is just one person x)

Is R symmetric? Let $x \in$ A, $y \in$ A and $(x, y) \in$ R.

$\therefore$ By (*i*), x and y work at the same place.

i.e., same thing as y and x work at the same place. Therefore by (*i*), $(y, x) \in$ R.

Is R transitive? Let $x \in$ A, $y \in$ A, $z \in$ A such that $(x, y) \in$ R and $(y, z) \in$ R.

$\therefore$ By (*i*), x and y work at the same place.

Also By (*i*), y and z work at the same place.

Therefore, we can say that x and z also work at the same place.

$\therefore$ By (*i*), $(x, z) \in$ R. Therefore, R is transitive.

(*b*) **Same solution as of part (*a*)**

(Replace the word "work" by "live" and "place" by "locality" in the solution of part (*a*)).

(*c*) R = {$(x, y) : x$ is exactly 7 cm taller than y) ...(*i*)

Is R reflexive? Let $x \in$ A. Putting $y = x$ in (*i*), we have x is exactly 7 cm taller than x, which is false.

($\because$ No body can be taller than oneself.)

Is R symmetric? Let $x \in$ A, $y \in$ A and $(x, y) \in$ R.

Therefore, by (*i*), x is exactly 7 cm taller than y.

$\therefore$ y is exactly 7 cm shorter than x.

$\therefore$ By (*i*), $(y, x) \notin$ R. $\therefore$ R is not symmetric.

Is R transitive? Let $x \in$ A, $y \in$ A, $z \in$ A such that $(x, y) \in$ R and $(y, z) \in$ R.

$\therefore$ By (*i*), x is exactly 7 cm taller than y and y is exactly 7 cm taller than z.

$\therefore$ x is exactly $(7 + 7) = 14$ cm (and not 7 cm) taller than z.

$\therefore$ By (*i*), $(x, z) \notin$ R. $\therefore$ R is not transitive.

(*d*) **Given:** R = {$(x, y) : x$ is wife of y} ...(*i*)

Is R reflexive? Let $x \in$ A. Putting $y = x$ in (*i*), we have x is wife of x which is false.

($\because$ No lady can be wife of herself.)

$\therefore$ R is not reflexive.

Is R symmetric? Let $x \in$ A, $y \in$ A and $(x, y) \in$ R.

$\therefore$ By (*i*), x is wife of y.

Hence y is husband (and not wife) of x.

$\therefore$ $(y, x) \notin$ R. $\therefore$ R is not symmetric.

Is R transitive?

There can't be any three $x, y, z \in$ A (Set of Human beings) such that both $(x, y) \in$ R and $(y, z) \in$ R.

($\because$ $(x, y) \in R \Rightarrow x$ is wife of y ($\Rightarrow$ Husband *i.e.,* Man) and hence (y, z) will never belong to R as no man y can be wife of any human being z).

Hence we no reason to expect that $(x, z) \in$ R. $\therefore$ R is transitive.

(*e*) **Given:** R = {(x, y) : x is father of y} ...(*i*)

Is R reflexive? Let $x \in$ A. Putting $y = x$ in (*i*), we have x is father of x which is false.

($\because$ No body can be father of oneself)

Is R symmetric? Let $x \in$ A, $y \in$ A such that $(x, y) \in$ R.

$\therefore$ By (*i*), x is father of y.

Hence y is son or daughter (and not father) of x.

$\therefore$ $(y, x) \notin$ R. $\therefore$ R is not symmetric.

Is R transitive? Let $x, y, z \in$ A such that $(x, y) \in$ R and $(y, z) \in$ R.

$\therefore$ By (*i*), x is father of y and y is father of z.

Hence x is grandfather (and not father) of z.

$\therefore$ $(x, z) \notin$ R. $\therefore$ R is not transitive.

2. Show that the relation R in the set R of real numbers, defined as R = {(a, b) : $a \le b^2$} is neither reflexive nor symmetric nor transitive.

Sol. Relation R = {(a, b) : a, b are real and $a \le b^2$} ...(*i*)

Is R reflexive? Let a be any real number.

Putting $b = a$ in (*i*), $a \le a^2$ which is not true for any positive real number less than 1.

For example, for $a = \frac{1}{2}$, $\frac{1}{2} \le \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ (*i.e.,* $a \le a^2$) is not true as $\frac{1}{2} > \frac{1}{4}$. $\therefore$ R is not reflexive.

Is R symmetric?

Let us take $a = 1$ and $b = 2$. Now $a = 1 \le 2^2 = 4$ (= b^2) is true.

$\therefore$ By (*i*), $(a, b) \in$ R.1414

But $b = 2 > 1^2 = 1$ (a^2) *i.e.,* b is not less than equal to a^2, therefore $(b, .a) \notin$ R. $\therefore$ R is not symmetric.

Is R transitive?

Let us take $a = 10$, $b = 4$, and $c = 2$ (All three are real numbers)

Now by (*i*), $(a, b) = (10, 4) \in$ R ($\because$ $a = 10 \le b^2$ (= 4^2) is true)

Again by (*i*), $(b, c) = (4, 2) \in$ R ($\because$ $b = 4 \le c^2$ (= 2^2) is true)

($4 \le 2^2 \Rightarrow 4 \le 4 \Rightarrow$ Either $4 < 4$ or $4 = 4$ But $4 = 4$ is true).

But $(a, c) = (10, 2) \notin$ R [$\because$ $a = 10 > 2^2$ (= b^2)]

$\therefore$ R is not transitive.

$\therefore$ R is neither reflexive, nor symmetric nor transitive.

Remark. It may be noted that $4 \le 4$ is true. Also $4 \ge 4$ is true. $8 \ge 5$ is true and $3 \le 6$ is true.

3. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {$(a, b) : b = a + 1$} is reflexive, symmetric or transitive.

Sol. Given: Set {1, 2, 3, 4, 5, 6} = set A (say)

Relation R = {$(a, b) : b = a + 1$} = {$(a, a + 1) : a \in$ A} ...(*i*)

Putting a = 1, 2, 3, 4, 5, 6 (given) in (*i*).

Roster form of relation R is {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)} ...(*ii*)

R is not reflexive. [$\because$ By (*ii*), $(1, 1) \notin$ R, $(2, 2) \notin$ R]

($\because$ We know that for relation R to be reflexive, $(a, a) \in$ R for all $a \in$ A)

R is not symmetric because by (*ii*), $(1, 2) \in$ R but $(2, 1) \notin$ R.

R is not transitive because $(x, y) = (1, 2) \in$ R, $(y, z) = (2, 3) \in$ R but $(x, z) = (1, 3) \notin$ R.

4. Show that the relation R in R defined as R = {$(a, b) : a \le b$}, is reflexive and transitive but not symmetric.

Sol. Given: Relation R in the set of real numbers is defined as

R = {$(a, b) : a \le b$} ...(*i*)

Is R reflexive? Putting $b = a$ in (*i*), we have $a \le a$ which is true.

[$\because a \le a \Rightarrow$ Either $a < a$ or $a = a$ and out of the two $a = a$ is true)

R is not **symmetric** because by (*i*), $(1, 2) \in$ R as $a = 1 \le b$ (= 2) but $(2, 1) \notin$ R because b (= 2) > 1(= a) (*i.e.*, $b \le a$ is not true).

Is R transitive? Let a, b, c be three real numbers such that $(a, b) \in$ R and $(b, c) \in$ R. $\therefore$ By (*i*), $a \le b$ and $b \le c$.

Therefore $a \le c$ and hence by (*i*), $(a, c) \in$ R. $\therefore$ R is transitive.

5. Check whether the relation R in R defined by R = {$(a, b) : a \le b^3$} is reflexive, symmetric or transitive.

Sol. Relation R = {(a, b) : a, b are real and $a \le b^3$} ...(*i*)

Is R reflexive? Let a be any real number.

Putting $b = a$ in (*i*), $a \le a^3$ which is not true for any positive real number less than 1.

For example, for $a = \frac{1}{2}$, $\frac{1}{2} \le \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ (*i.e.*, $a \le a^3$) is not true as $\frac{1}{2} > \frac{1}{8}$. $\therefore$ R is not reflexive.

Is R symmetric?

Let us take $a = 1$ and $b = 2$. Now $a = 1 \le 2^3 = 8$ (= b^3) is true. Therefore by (*i*), $(a, b) \le$ R.

Now, $b = 2 > 1^3$ (= 1) (a^3) *i.e.*, b is not less than or equal to a^3. Therefore, $(b, a) \notin$ R. $\therefore$ R is not symmetric.

Is R transitive?

Let us take $a = 10$, $b = 4$, $c = 2$ (All three are real numbers)

Now by (*i*), $(a, b) = (10, 4) \in$ R ($\because a = 10 \le b^3$ (= 4^3 = 64 is true)

Again by (*i*), $(b, c) = (4, 2) \in$ R ($\because$ $b = 4 \le c^3$ (= 2^3 = 8) is true.

But $(a, c) = (10, 2) \notin$ R ($\because$ $a = 10 > 2^3$ (= c^3 = 8)).

$\therefore$ R is not transitive.

$\therefore$ R is neither reflexive nor symmetric nor transitive.

6. Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

Sol. **Given:** Set is {1, 2, 3} = A (say)

Given relation R = {(1, 2), (2, 1)} ...(*i*)

R is symmetric because $(x, y) = (1, 2) \in$ R and also $(y, x) = (2, 1) \in$ R and these two are the only elements of R.

i.e., $(x, y) \in R \Rightarrow (y, x) \in R$ for all $(x, y) \in$ R.

R is not reflexive because $1 \in$ A but $(1, 1) \notin$ R.

$2 \in$ A but $(2, 2) \notin$ R, $3 \in$ A but $(3, 3) \notin$ R.

R is not transitive because $(x, y) = (1, 2) \in$ R and $(y, z) = (2, 1) \in$ R but $(x, z) = (1, 1) \notin$ R.

7. Show that the relation R in the set A of all the books in a library of a college given by R = {(*x*, *y*) : *x* and *y* have same number of pages} is an equivalence relation.

Sol. **Given:** Set A of all the books in a library of a college.

Given: Relation R : {(*x*, *y*) : *x* and *y* have same number of pages} ...(*i*)

Is R reflexive? Let (book) $x \in$ A

Putting $y = x$ in (*i*), we have (book) *x* and *x* have same number of pages which is clearly true, because we are talking of pages of one book only.

$\therefore$ $(x, x) \in$ R for all $x \in$ A. $\therefore$ R is reflexive.

Is R symmetric? Let $x \in$ A, $y \in$ A and $(x, y) \in$ R.

$\therefore$ By (*i*), books *x* and *y* have the same number of pages *i.e.,* books *y* and *x* have the same number of pages. Therefore $(y, x) \in$ R.

$\therefore$ R is symmetric.

Is R transitive? Let $x, y, z \in$ A such that $(x, y) \in$ R and $(y, z) \in$ R.

By (*i*), $(x, y) \in R \Rightarrow$ Books *x* and *y* have the same number of pages. ...(*ii*)

By (*i*), $(y, z) \in R \Rightarrow$ Books *y* and *z* have the same number of pages. ...(*iii*)

From (*ii*) and (*iii*) books *x* and *z* have the same number of pages. Therefore by (*i*), $(x, z) \in$ R. $\therefore$ R is transitive.

So have proved that the relation R given by (*i*) is reflexive, symmetric and transitive and hence is an equivalence relation.

8. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by R = {(*a*, *b*) : | *a* – *b* | is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

Sol. (*i*) For all $a \in$ A, $| a - a | = 0$ is even so that $(a, a) \in$ R. Therefore, R is reflexive.

(*ii*) Further $(a, b) \in R \Rightarrow |a - b|$ is even

$\Rightarrow |b - a|$ is even, since $|b - a| = |-(a - b)|$

$= |a - b|$ $(\because |-t| = |t|)$

$\Rightarrow (b, a) \in R$ $\therefore$ R is symmetric.

(*iii*) Also $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow |a - b|$ is even and $|b - c|$ is even

$\Rightarrow a - b$ is even and $b - c$ is even

Adding the two $\Rightarrow a - c =$ even

[$\because$ addition of even numbers is even]

$\Rightarrow |a - c|$ is even $\Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

Further all the elements of {1, 3, 5} are related to each other since all the elements of this subset of A are odd and difference of two odd numbers is even.

[$\because |1 - 1| = 0, |3 - 3| = 0, |5 - 5| = 0$ are also even].

Similarly, all the elements of {2, 4} are related to each other since all the elements of this subset of A are even and difference of two even numbers is even.

Also, no element of {1, 3, 5} is related to any element of {2, 4} since the difference of an odd number and an even number is never even.

[$\because |1 - 2| = 1, |1 - 4| = 3, |3 - 2| = 1, |3 - 4| = 1,$
$|5 - 2| = 3, |5 - 4| = 1$ are odd and not even.]

9. Show that each of the relation R in the set $A = \{x \in Z, 0 \le x \le 12\}$ given by

(*i*) $R = \{(a, b) : |a - b|$ is a multiple of 4}

(*ii*) $R = \{(a, b) : a = b\}$

is an equivalence relation. Find the set of all elements related to 1 in each case.

Sol. (*i*) **Given:** Set $A = \{x \in Z, 0 \le x \le 12\}$

and $R = \{(a, b) : |a - b|$ is a multiple of 4} ...(*i*)

Is R reflexive? Let $a \in A$.

Putting $b = a$ (*i*), $|a - a| = |0| = 0$ is a multiple of 4. which is true.

$\therefore (a, a) \in R$ for all $a \in A$. $\therefore$ R is reflexive.

Is R symmetric? Let $a \in A$, $b \in A$ and $(a, b) \in R$.

$\therefore$ By (*i*), $|a - b|$ is a multiple of 4.

i.e., $|-(b - a)| = |b - a|$

$(\because |-t| = |t|$ for every real $t)$ is a multiple of 4.

$\therefore$ By (*i*), $(b, a) \in R$. $\therefore$ R is symmetric.

Is R Transitive? Let $a, b, c \in A$ such that $(a, b) \in R$ and $(b, c) \in R$.

By (*i*), $(a, b) \in R \Rightarrow |a - b|$ is a multiple of 4.

$\therefore$ There exists an integer m such that $|a - b| = 4m$

$\therefore \quad a - b = \pm 4m$ ...(*ii*)

Similarly, there exists an integer n such that $b - c = \pm 4n$ ...(*iii*)

Adding (*ii*) and (*iii*) (to eliminate b)

$$a - c = \pm 4m \pm 4n = \pm 4(m + n)$$

$\therefore \quad | a - c | = 4(m + n)$

i.e., $| a - c |$ is a multiple of 4.

$\therefore$ By (*i*), $(a, c) \in$ R. $\therefore$ R is transitive.

Hence R is an equivalence relation.

To find the set of elements related to 1 (∈ A).

Let $a \in$ A be related to $1 \in$ A.

$\therefore \quad (a, 1) \in$ R.

$\therefore$ By (*i*), $| a - 1 |$ is a multiple of 4.

Testing a = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 ∈ A, only a = 1, 5, 9 satisfy the above relation, *i.e.*, $|1-1| = 0$, $| 5 - 1 | = 4$, $| 9 - 1 | = 8$ are multiples of 4.

$\therefore$ The set of elements of A related to 1 ∈ A is {1, 5, 9}.

(*ii*) R = $\{(a, b) : a = b\}$ ...(*i*)

Is R reflexive? Putting $b = a$ in (*i*), we have $a = a$ which is true.

$\therefore \quad (a, a) \in$ R for all $a \in$ A. $\therefore$ R is reflexive.

Is R symmetric? Let $a \in$ A, $b \in$ A and $(a, b) \in$ R.

$\therefore$ By (*i*), $a = b$. Hence $b = a$.

$\therefore$ By (*i*), $(b, a) \in$ R. $\therefore$ R is symmetric.

Is R transitive? Let $a, b, c \in$ A such that

$(a, b) \in$ R and $(b, c) \in$ R.

$\therefore$ By (*i*), $a = b$ and $b = c$.

Hence $a = c$. Therefore by (*i*), $(a, c) \in$ R. $\therefore$ R is transitive.

Hence R is an equivalence relation.

To find set of elements of A related to 1 ∈ A.

Let $a \in$ A be related to $1 \in$ A.

$\therefore \quad (a, 1) \in$ R. $\therefore$ By (*i*), $a = 1$.

$\therefore$ The set of elements of A related to 1 ∈ A is {1}.

10. Give an example of a relation, which is

(*i*) Symmetric but neither reflexive nor transitive.

(*ii*) Transitive but neither reflexive nor symmetric.

(*iii*) Reflexive and symmetric but not transitive.

(*iv*) Reflexive and transitive but not symmetric.

(*v*) Symmetric and transitive but not reflexive.

Sol. Let A = {1, 2, 3, 4}, then

A × A = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}

We know that a relation in A is a subset of A × A.

(*i*) The relation R = {(1, 2), (2, 1), (2, 4), (4, 2)} is symmetric but neither reflexive nor transitive, because

R^{-1} = {(2, 1), (1, 2), (4, 2), (2, 4)} = R

⇒ R is symmetric

1 ∈ A but (1, 1) ∉ R ⇒ R is not reflexive

(1, 2) ∈ R and (2, 4) ∈ R but (1, 4) ∉ R

⇒ R is not transitive.

(*ii*) The relation R = {(1, 2), (2, 3), (1, 3)} is transitive but neither reflexive nor symmetric, because

(1, 2) ∈ R, (2, 3) ∈ R and (1, 3) ∈ R

⇒ R is transitive

1 ∈ A but (1, 1) ∉ R ⇒ R is not reflexive

(1, 2) ∈ R but (2, 1) ∉ R ⇒ R is not symmetric.

(*iii*) The relation R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (1, 3), (3, 1)} is reflexive and symmetric but not transitive, because

(*a*, *a*) ∈ R for all *a* ∈ A ⇒ R is reflexive

(*a*, *b*) ∈ R ⇒ (*b*, *a*) ∈ R ⇒ R is symmetric

(2, 1) ∈ R and (1, 3) ∈ R but (2, 3) ∉ R

⇒ R is not transitive.

(*iv*) The relation R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3)} is reflexive and transitive but not symmetric, because

(*a*, *a*) ∈ R for all *a* ∈ A ⇒ R is reflexive

If (*a*, *b*) ∈ R and (*b*, *c*) ∈ R then (*a*, *c*) ∈ R

⇒ R is transitive

(1, 2) ∈ R but (2, 1) ∉ R ⇒ R is not symmetric

[Or R^{-1} ≠ R because R^{-1} = {(1, 1), (2, 2), (3, 3), (4, 4), (2, 1), (3, 2), (3, 1)} ≠ R.

⇒ R is not symmetric].

(*v*) The relation R = {(1, 1), (1, 2), (2, 1), (2, 2)} is symmetric and transitive but not reflexive, because

If (*a*, *b*) ∈ R then (*b*, *a*) ∈ R ⇒ R is symmetric

[Or R^{-1} = R ⇒ R is symmetric]

If (*a*, *b*) ∈ R and (*b*, *c*) ∈ R then (*a*, *c*) ∈ R

⇒ R is transitive

3 ∈ A but (3, 3) ∉ R ⇒ R is not reflexive.

11. Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

Sol. **Given:** Set A of points in a plane and relation R = {(P, Q) : distance of point P from origin is same as the distance of the point Q from the origin}.

i.e., R = {(P, Q) : OP = OQ where O is the origin} ...(*i*)

Is R reflexive? Let point P $\in$ A.

Putting Q = P in (*i*), we have OP = OP which is true.

$\therefore$ (P, P) $\in$ R for all points P $\in$ A. $\therefore$ R is reflexive.

Is R symmetric? Let points P, Q $\in$ A such that (P, Q) $\in$ R.

$\therefore$ By (*i*), OP = OQ *i.e.*, OQ = OP

$\therefore$ By (*i*), (Q, P) $\in$ R. $\therefore$ R is symmetric.

Is R transitive? Let points P, Q, S $\in$ A such that (P, Q) $\in$ R and (Q, S) $\in$ R.

$\therefore$ By (*i*), OP = OQ and OQ = OS. $\therefore$ OP = OS (= OQ)

$\therefore$ By (*i*), (P, S) $\in$ R $\therefore$ R is transitive.

Hence R is an equivalence relation.

Now given point P $\neq$ (0, 0).

Let Q be any point of set A related to point P.

$\therefore$ By (*i*), OQ = OP for all points Q $\in$ A, related to point P.

i.e., OQ = constant distance OP = *k* (say)

$\therefore$ By definition of circle, all points Q of set A related to a given point P of A lie on a circle with centre at the origin O.

12. Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2) : T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Sol. Given: Relation R defined in the set (say A) of all triangles as

R = {(T_1, T_2) : T_1 is similar to T_2} ...(*i*)

We know that two triangles are said to be similar if the ratio of their corresponding sides is same. ...(*ii*)

Is R reflexive? Let T_1 $\in$ A.

Putting T_2 = T_1 in (*i*), we have triangle T_1 is similar to triangle T_1 which is always true.

Therefore (T_1, T_1) $\in$ R for all T_1 $\in$ A. $\therefore$ R is reflexive.

Is R symmetric? Let T_1 $\in$ A, T_2 $\in$ A such that (T_1, T_2) $\in$ R.

$\therefore$ By (*i*), triangle T_1 is similar to triangle T_2.

$\therefore$ We can say that triangle T_2 is similar to triangle T_1.

$\therefore$ By (*i*), (T_2, T_1) $\in$ R. $\therefore$ R is symmetric.

Is R transitive? Let T_1 $\in$ A, T_2 $\in$ A, T_3 $\in$ A such that (T_1, T_2) $\in$ R and (T_2, T_3) $\in$ R.

By (*i*), (T_1, T_2) $\in$ R $\Rightarrow$ T_1 is similar to T_2

and (T_2, T_3) $\in$ R $\Rightarrow$ T_2 is similar to T_3.

$\therefore$ By definition of similar triangles given above (in (*ii*)),

Triangle T_1 is similar to triangle T_3.

$\therefore$ By (*i*), (T_1, T_3) $\in$ R.

$\therefore$ R is transitive.

Hence R is an equivalence relation.

Now given that triangle T_1 has sides 3, 4, 5. ...(*iii*)

Triangle T_2 has sides 5, 12, 13. ...(*iv*)

Triangle T_3 has sides 6, 8, 10. ...(*v*)

Triangle T_1 in (*iii*) is not similar to triangle T_2 in (*iv*)

($\because$Ratio of their corresponding sides is not same *i.e.*, $\frac{3}{5}, \frac{4}{12}, \frac{5}{13}$ are not equal.)

$\therefore$ By (*i*), T_1 is not related to T_2.

Triangle T_2 in (*iv*) is not similar to triangle T_3 in (*v*) because ratio of their corresponding sides is not same *i.e.*, $\frac{5}{6}, \frac{12}{8}$ and $\frac{13}{10}$ are not equal.

$\therefore$ By (*i*), T_2 is not related to T_3.

But triangle T_1 in (*iii*) is similar to triangle T_3 in (*v*).

[$\because$ Ratio of their corresponding sides is same

$$i.e., \frac{3}{6} = \frac{4}{8} = \frac{5}{10}\left(=\frac{1}{2}\right)\Bigg]$$

$\therefore$ By (*i*), triangle T_1 is related to triangle T_3.

13. Show that the relation R defined in the set A of all polygons as R = {(P_1, P_2) : P_1 and P_2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

Sol. Given: Relation R defined in the set A of all polygons as

R = {(P_1, P_2) : P_1 and P_2 have same number of sides} ...(*i*)

Is R reflexive? Let polygon $P_1 \in$ A.

Putting $P_2 = P_1$ in (*i*), we have P_1 and P_1 have same number of sides which is of course true.

($\because$ P_1 and P_1 is same polygon) $\therefore$ R is reflexive.

Is R symmetric? Let $P_1 \in$ A and $P_2 \in$ A such that $(P_1, P_2) \in$ R.

$\therefore$ By (*i*), polygons P_1 and P_2 have the same number of sides, *i.e.*, polygons P_2 and P_1 have the same number of sides.

$\therefore$ By (*i*), $(P_2, P_1) \in$ R. $\therefore$ R is symmetric.

Is R transitive? Let $P_1, P_2, P_3 \in$ A such that $(P_1, P_2) \in$ R and $(P_2, P_3) \in$ R.

$\therefore$ By (*i*), polygons P_1 and P_2 have same number of sides.

Also polygons P_2 and P_3 have same number of sides.

$\therefore$ Polygons P_1 and P_3 (also) have same number of sides.

$\therefore$ By (*i*), $(P_1, P_3) \in$ R. $\therefore$ R is transitive.

$\therefore$ R is an equivalence relation.

Now given a right angled triangle T with sides 3, 4, 5.

$\therefore$ T also $\in$ A ($\because$ A triangle is a polygon with three sides)

$\therefore$ By (*i*), the set of all triangles (polygons) of set A (and not only right angled triangles ∈ A) is the set of all elements related to this given right angled triangle T.

14. Let L be the set of all lines in XY plane and R be the relation in L defined as R = {(L_1, L_2) : L_1 is parallel to L_2}. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Sol. Given: L = set of all the lines in XY plane.

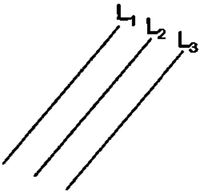
Relation R = {(L_1, L_2) : L_1 is parallel to L_2} ...(*i*)

Is R reflexive?

Let $L_1 \in A$. Putting $L_2 = L_1$ in (*i*), we have L_1 is parallel to L_1 which is true.

∴ (L_1 L_1) ∈ R. ∴ R is reflexive.

Is R symmetric?



Let $L_1 \in A$, $L_2 \in A$ such that (L_1, L_2) ∈ R.

∴ By (*i*), line L_1 is parallel to line L_2.

∴ Line L_2 is parallel to line L_1.

∴ By (*i*), (L_2, L_1) ∈ R.

∴ R is symmetric.

Is R transitive?

Let $L_1 \in A$, $L_2 \in A$, $L_3 \in A$ such that (L_1, L_2) ∈ R and (L_2, L_3) ∈ R.

∴ By (*i*), line L_1 is parallel to L_2 and L_2 is parallel to L_3.

Hence L_1 is parallel to L_3.

∴ By (*i*), (L_1, L_3) ∈ R. ∴ R is transitive.

Hence R is an equivalence relation.

Now to find set of all lines related to the line $y = 2x + 4$.

We know that equations of parallel lines differ only in constant term.

∴ By (i), (Equation), *i.e.*, set of all lines parallel to the line $y = 2x + 4$ is $y = 2x + c$ where c is an arbitrary constant.

15. Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation.

Sol. Given: Set is {1, 2, 3, 4} = A (say)

Relation R in A is {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)} ...(*i*)

R is reflexive because (1, 1), (2, 2), (3, 3), (4, 4) all belong to R *i.e.*, (a, a) ∈ R for all $a \in A$.

Now (1, 2) ∈ R but (2, 1) ∉ R. ∴ R is not symmetric.

Is R transitive?

Yes, because in relation (i) whenever (x, y) and $(y, z) \in$ R; then (x, z) also $\in$ R.

As for (1, 2) and (2, 2) $\in$ R; (1, 2) $\in$ R

For (1, 3) and (3, 2) $\in$ R; (1, 3) $\in$ R

$\therefore$ R is transitive. Therefore, option (B) is the correct option.

16. Let R be the relation in the set N given by R = $\{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer.

(A) (2, 4) $\in$ R (B) (3, 8) $\in$ R (C) (6, 8) $\in$ R (D) (8, 7) $\in$ R.

Sol. Relation R in N defined by R = $\{(a, b), a = b - 2, b > 6\}$ ...(i)

Out of the given options only option (C) satisfies (i) *i.e.*, (6, 8) $\in$ R.

$\because$ $a = 6, b = 8 > 6$ and $8 - 2 = 6$ *i.e.*, $a = b - 2$.

Exercise 1.2 *(Page No. 10-11)*

1. Show that the function $f : R_* \to R_*$, defined by $f(x) = \frac{1}{x}$ is one-one and onto, where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_*?

Sol. $f(x) = \frac{1}{x}$ (given)

$\therefore$ For $x_1, x_2 \in R_*$ with $f(x_1) = f(x_2)$ we have $\frac{1}{x_1} = \frac{1}{x_2}$ $\left(\because f(x) = \frac{1}{x}\right)$

$\Rightarrow$ $x_1 = x_2$ $\therefore$ f is one-one.

Again $f(x) = y = \frac{1}{x}$ $\Rightarrow$ $xy = 1$ $\Rightarrow$ $x = \frac{1}{y}$.

$\therefore$ Given any $y \in R_*$, the co-domain, we can choose $x = \frac{1}{y}$ such that

$f(x) = \frac{1}{x} = \frac{1}{\left(\frac{1}{y}\right)} = y$. Thus, every element in the co-domain has pre-image in the domain under f. Therefore, f is onto.

The function $f : N \to R_*$ again defined by $f(x) = \frac{1}{x}$ is also one-one,

since $x_1, x_2 \in N$ with $f(x_1) = f(x_2)$ $\Rightarrow$ $\frac{1}{x_1} = \frac{1}{x_2} \Rightarrow$ $x_1 = x_2$.

But f is not onto since $2 \in R_*$ and 2 is not the reciprocal of any natural number, *i.e.*, there exists no natural number x such that $f(x) = 2$ or $\frac{1}{x} = 2$ or $x = \frac{1}{2}$ because $x = \frac{1}{2} \notin$ N.

2. Check the injectivity and surjectivity of the following functions:

(*i*) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$
(*ii*) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$
(*iii*) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$
(*iv*) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$
(*v*) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^3$

Sol. Note. Checking **injectivity** ⇒ Testing for **one-one** function checking **surjectivity** ⇒ Testing for **onto** function.

(*i*) **Given:** $f : \underset{\text{domain}}{\underset{\downarrow}{\text{N}}} \to \underset{\text{co-domain}}{\underset{\downarrow}{\text{N}}}$ given by $f(x) = x^2$ ...(*i*)

Is f injective (one-one)? Let $x_1, x_2 \in$ Domain N such that $f(x_1) = f(x_2)$

Putting $x = x_1$ and $x = x_2$ in (*i*), $x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

Rejecting negative sign ($\because$ $x_1, x_2 \in$ N and hence are positive), $x_1 = x_2$

$\therefore$ f is injective *i.e.*, one-one.

Is f surjective (onto)?

Putting x = 1, 2, 3, ..., ∈ domain N, in (*i*),

Range set $= \{f(x) = x^2 ; x \in \text{N}\} = \{1^2, 2^2, 3^2, ...\} = \{1, 4, 9, ...\}$ ≠ co-domain N

[$\because$ 2, 3, 5, 8, ... ∈ co-domain N but don't belong range]

$\therefore$ f is not surjective.

$\therefore$ f is injective but not surjective.

(*ii*) **Given:** $f : \underset{\text{domain}}{\underset{\downarrow}{\text{Z}}} \to \underset{\text{co-domain}}{\underset{\downarrow}{\text{Z}}}$ given by $f(x) = x^2$ ...(*i*)

Is f injective (one-one)? Let $x_1, x_2 \in$ Domain Z such that $f(x_1) = f(x_2)$ ⇒ $x_1^2 = x_2^2$ (By (*i*))

⇒ $x_1 = \pm x_2$

Now negative sign can't be rejected because $x_1, x_2 \in$ Z (set of integers) can be negative also.

$\therefore$ $f(x_1) = f(x_2)$ ⇒ $x_1 = \pm x_2$.

Hence f is not one-one.

(Also for example 2 ∈ Z, – 2 ∈ Z, 2 ≠ – 2 but by (*i*), $f(2) = 2^2 = 4 = f(-2) = (= (-2)^2 = 4)$).

Is f onto?

Putting x = 0, ± 1, ± 2, ± 3 ..., ∈ domain Z, in (*i*),

Range set $= \{f(x) = x^2 ; x \in \text{Z}\} = \{0, 1, 4, 9, ...\}$

≠ co-domain Z

[$\because$ 2, – 2, etc. belong to co-domain Z but don't belong to range]

$\therefore$ f is not surjective. (*i.e.*, not onto)

$\therefore$ f is neither injective nor surjective.

(*iii*) **Given:** $f : \underset{\text{domain}}{\underset{\downarrow}{R}} \to \underset{\text{co-domain}}{\underset{\downarrow}{R}}$ given by $f(x) = x^2$ ...(*i*)

Is f injective? [Replace Z by R in the solution for part (*ii*) namely "Is f injective".]

Is f surjective?

Range set = $\{f(x) = x^2; x \in R\}$ is the set of all positive real numbers only including 0.

($\because$ $x^2 \geq 0$ for all $x \in R$) and hence $\neq$ co-domain R.

$\therefore$ f is not surjective.

$\therefore$ f is neither injective nor surjective.

(*iv*) **Given:** $f : \underset{\text{domain}}{\underset{\downarrow}{N}} \to \underset{\text{co-domain}}{\underset{\downarrow}{N}}$ given by $f(x) = x^3$ ...(*i*)

Is f injective? Let $x_1, x_2 \in$ domain N such that $f(x_1) = f(x_2)$

$\therefore$ By (*i*), ${x_1}^3 = {x_2}^3 \Rightarrow x_1 = x_2$ (only). $\therefore$ f is one-one (injective).

Is f surjective ?

Putting $x = 1, 2, 3, ... \in$ domain N, in (*i*),

Range set = $\{f(x) = x^3; x \in N\} = \{1^3, 2^3, 3^3, 4^3, ...\}$

$= \{1, 8, 27, 64, ...\} \neq$ co-domain N

($\because$ 2, 3, 5, 8 etc. belong to co-domain N but don't belong to range set)

$\therefore$ f is not surjective.

Hence f is injective but not surjective.

(*v*) **Given:** $f : \underset{\text{domain}}{\underset{\downarrow}{Z}} \to \underset{\text{co-domain}}{\underset{\downarrow}{Z}}$ given by $f(x) = x^3$ ...(*i*)

Is f injective? Let $x_1, x_2 \in$ domain Z such that $f(x_1) = f(x_2)$

$\therefore$ By (*i*), ${x_1}^3 = {x_2}^3 \Rightarrow x_1 = x_2$ (only).

$\therefore$ f is one-one (injective).

Is f surjective?

Putting $x = 0, 1, -1, 2, -2, ... \in$ domain Z, in (*i*),

Range set = $\{f(x) = x^3; x \in Z\} = \{0, 1, -1, 8, -8, ...\} \neq$ co-domain Z.

($\because$ 2, – 2, 3, – 3, ... belong to co-domain Z but don't belong to range set)

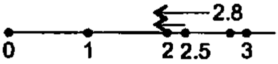
$\therefore$ f is not surjective. $\therefore$ f is injective but not surjective.

3. Prove that the Greatest Integer Function $f : R \to R$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Sol. Given: Greatest integer function

$f : R \to R$ given by $f(x) = [x]$ ...(*i*) where $[x]$ denotes the **greatest integer less than** or **equal to** x.

Is f one-one? Let us take $x_1 = 2.5$ and $x_2 = 2.8 \in$ domain R.



From (i), $f(x_1) = f(2.5) = [2.5] = 2$

and $f(x_2) = f(2.8) = [2.8] = 2$

$\therefore$ $f(x_1) = f(x_2)$ $(= 2)$ but x_1 $(= 2.5) \neq x_2$ $(= 2.8)$

$\therefore$ f is not one-one.

Is f onto? By (i), range set $= \{f(x) = [x], x \in \text{R}\}$ is the set Z of all integers ($\because$ value of $[x]$ is always an integer)

But $\neq$ co-domain R.

$\therefore$ f is not onto.

$\therefore$ f is neither one-one nor onto.

4. Show that the Modulus Function $f : \text{R} \to \text{R}$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Sol. Given: Modulus function $f : \text{R} \to \text{R}$ given $f(x) = |x|$ ...(i)

Is f one-one? Let us take $x_1 = -1$ and $x_2 = 1 \in$ domain R

From (i), $f(x_1) = f(-1) = |-1| = 1$

and $f(x_2) = f(1) = |1| = 1$

$\therefore$ $f(x_1) = f(x_2)$ $(= 1)$ but $x_1(= -1) \neq x_2$ $(= 1)$

$\therefore$ f is not one-one.

Is f onto?

By (i), Range set $= \{f(x) = |x|, x \in \text{R}\}$ is the set of positive real numbers (only) including 0. ($\because$ $|x| \geq 0$ for all $x \in \text{R}$)

but $\neq$ co-domain R.

$\therefore$ f is not onto. $\therefore$ f is neither one-one nor onto.

5. Show that the Signum Function $f : \text{R} \to \text{R}$, given by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

Sol. Is f one-one?

Let $x_1 = 2$, $x_2 = 3$ $\therefore$ $f(x_1) = f(2) = 1$ and $f(x_2) = f(3) = 1$

[$\because$ $2 > 0$, $3 > 0$ and it is given that $f(x) = 1$ for $x > 0$]

$\therefore$ $f(x_1) = f(x_2) = 1$ but x_1 $(= 2) \neq x_2$ $(= 3)$ $\therefore$ f is not one-one.

or Let $x_1 = -2$, $x_2 = -3$

$\therefore$ $f(x_1) = f(-2) = -1$ and $f(x_2) = f(-3) = -1$

[$\because$ $-2 < 0, -3 < 0$ and it is given that $f(x) = -1$ for $x < 0$]

$\therefore$ $f(x_1) = f(x_2) = -1$ but x_1 $(= -2) \neq x_2$ $(= -3)$

$\therefore$ f is not one-one.

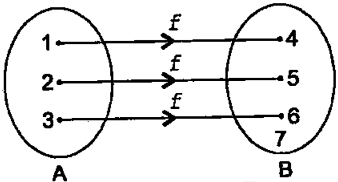
Is f onto?

According to given $f(x) = 1$ if $x > 0$, $= 0$ if $x = 0$ and $= -1$ if $x < 0$.

$\therefore$ Range set $= \{f(x) : x \in \text{R}\} = \{1, 0, -1\} \neq$ co-domain R.

$\therefore$ f is not onto. $\therefore$ f is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.



Sol. Given: set A = {1, 2, 3}, B = {4, 5, 6, 7} and

f = {(1, 4), (2, 5), (3, 6)} is a function from A to B.

⇒ $f(1) = 4, f(2) = 5, f(3) = 6$.

⇒ Distinct elements 1, 2 and 3 of domain A have distinct f-images 4, 5, 6 in co-domain B.

Therefore f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$.

(ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$.

Sol. (*i*) **Given:** $f : \underset{\text{domain}}{\underset{\downarrow}{\mathrm{R}}} \to \underset{\text{co-domain}}{\underset{\downarrow}{\mathrm{R}}}$ defined by $f(x) = 3 - 4x$ ...(*i*)

Is f one-one? Let $x_1, x_2 \in$ domain R such that

$f(x_1) = f(x_2)$ ⇒ $3 - 4x_1 = 3 - 4x_2$

[Putting $x = x_1$ and $x = x_2$ in (*i*), $f(x_1) = 3 - 4x_1$

and $f(x_2) = 3 - 4x_2$] ⇒ $-4x_1 = -4x_2$

Dividing by -4, $x_1 = x_2$ ∴ f is one-one.

Is f onto? Let y be any element of co-domain R.

Let us suppose that $y = f(x)$

∴ $y = 3 - 4x$ (By (*i*)) ...(*ii*)

Let us find x in terms of y.

From (*ii*), $4x = 3 - y$ ⇒ $x = \dfrac{3-y}{4}$

Now $x = \dfrac{3-y}{4} \in$ domain R for each $y \in$ co-domain.

$$\therefore \text{ By } (i),\ f(x) = 3 - 4x = 3 - 4\left(\frac{3-y}{4}\right) = 3 - (3 - y)$$

$$= 3 - 3 + y = y$$

Hence our supposition that $y = f(x)$ is correct.

Hence co-domain = Range

∴ f is onto. ∴ f is both one-one and onto, *i.e.*, f is bijective.

(*ii*) **Given:** f : R → R defined by $f(x) = 1 + x^2$ ...(*i*)

Is f one-one? Let $x_1, x_2 \in$ domain R such that $f(x_1) = f(x_2)$.

∴ From (*i*), $1 + x_1^2 = 1 + x_2^2$ ⇒ $x_1^2 = x_2^2$ ⇒ $x_1 = \pm x_2$

Negative sign can't be rejected because $x_1, x_2 \in$ R can be negative.

∴ $f(x_1) = f(x_2)$ ⇒ $x_1 = \pm x_2$ and not $x_1 = x_2$.

Hence f is not one-one.

[For example, from (*i*), $f(1) = 1 + 1^2 = 1 + 1 = 2$

and $f(-1) = 1 + (-1)^2 = 1 + 1 = 2.$

Now $1 \neq -1$ but $f(1) = f(-1)]$

Is f onto?

Range set $= \{ f(x) = 1 + x^2 : x \in \text{R}\}$

$= \{ f(x) \geq 1 \ (\because \ x^2 \geq 0$ always$)\}$

$= [1, \infty) \neq$ co-domain R $(= (-\infty, \infty))$

$\therefore$ f is not onto. $\therefore$ f is neither one-one nor onto.

8. Let A and B be sets. Show that f : A × B → B × A such that $f(a, b) = (b, a)$ is bijective function.

Sol. Let $(a, b), (c, d) \in \text{A} \times \text{B}$ with $f(a, b) = f(c, d)$, then $(b, a) = (d, c)$

[$\because$ The given rule is $f(a, b) = (b, a)$]

Equating corresponding entries $b = d$ and $a = c$

$\Rightarrow (a, b) = (c, d)$ $\therefore$ f is injective (one-one) function. ...(*i*)

Also, for all $(b, a) \in$ co-domain B × A, there exists $(a, b) \in$ domain A × B such that by the given rule $f(a, b) = (b, a)$. Thus every $(b, a) \in \text{B} \times \text{A}$ is the image of $(a, b) \in \text{A} \times \text{B}$.

$\therefore$ f is surjective (onto) function. ...(*ii*)

From (*i*) and (*ii*), f is a one-one onto *i.e.*, bijective function.

9. Let f : N → N be defined by

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in \text{N}.$$

State whether the function f is onto, one-one or bijective. Justify your answer.

Sol. $$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

To examine whether f is one-one.

Now $3 \in \text{N}$ and $4 \in \text{N}$ and $3 \neq 4$

$3 \in \text{N}$ is odd, $\therefore$ from (*i*), $f(3) = \dfrac{3+1}{2} = 2$

$4 \in \text{N}$ is even, $\therefore$ from (*ii*), $f(4) = \dfrac{4}{2} = 2$

$\therefore f(3) = f(4)$ but $3 \neq 4$ $\therefore$ f is not one-one.

To examine whether f is onto.

Let us find range of f.

Putting $n = 1, 3, 5, \ldots 2n - 1, \ldots$ (all odd) in (*i*),

$$f(1) = \frac{1+1}{2} = 1, f(3) = \frac{3+1}{2} = 2, f(5) = \frac{5+1}{2} = 3, \ldots$$

$$f(2n-1) = \frac{2n-1+1}{2} = n, \ldots$$

Putting $n = 2, 4, 6, \dots 2n, \dots$ (all even) in (*ii*),

$$f(2) = \frac{2}{2} = 1, f(4) = \frac{4}{2} = 2, f(6) = \frac{6}{2} = 3,$$

$$f(2n) = \frac{2n}{2} = n, \dots\dots$$

$\therefore$ Range of $f = \{1, 2, 3, 4, \dots, n, \dots\}$ = co-domain N

$\therefore$ f is onto. $\therefore$ f is not one-one but onto

Hence, f is not bijective.

Note. While discussing **onto,** whenever for $y \in$ co-domain, there are two or more than two $x \in$ domain, for which $f(x) = y$. Then f is never one-one.

Here in the above example for $y = 1 \in$ co-domain N, there are two values of x (*i.e.*, two pre-images) $x = 1, x = 2 \in$ domain N for which $f(x) = y$

$\therefore$ f is not one-one.

10. Let A = R – {3} and B = R – {1}. Consider the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x-3}$. Is f one-one and onto? Justify your answer.

Sol. For $x_1, x_2 \in A$ with $f(x_1) = f(x_2) \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$\left(\text{On putting } x = x_1 \text{ and } x = x_2 \text{ in } f(x) = \frac{x-2}{x-3}\right)$

Cross-multiplying, $(x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow -3x_1 - 2x_2 = -2x_1 - 3x_2 \Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\therefore$ f is one-one.

Now let $y = f(x) = \frac{x-2}{x-3}$, $\Rightarrow (x - 3)y = x - 2$

$\Rightarrow xy - 3y = x - 2$ $\Rightarrow xy - x = 3y - 2$

$\Rightarrow x(y - 1) = 3y - 2$ $\Rightarrow x = \frac{3y-2}{y-1}$

$\therefore$ For every $y \neq 1$, *i.e.*, when $y \in$ co-domain B, there exists $x = \frac{3y-2}{y-1} \in A = R - \{3\}$ such that

$$f(x) = \frac{x-2}{x-3} = \frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3} = \frac{3y - 2 - 2y + 2}{3y - 2 - 3y + 3} = y$$

$\therefore$ f is onto.

11. Let $f : R \to R$ be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto **(B) f is many-one onto**

(C) f is one-one but not onto

(D) f is neither one-one nor onto.

Sol. $f : R \to R$ is defined as $f(x) = x^4$ ...(*i*)

Is f one-one? No

Because 2 and – 2 both belong to domain R, but by (*i*),

$$f(2) = 2^4 = 16 \text{ and } f(-2) = (-2)^4 = 16$$

$\therefore \quad f(2) = f(-2)$ but $2 \neq -2$

(*i.e.*, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$)

Is f onto? No

Because range set $= \{f(x) = x^4, x \in R\}$

$= \{f(x) \geq 0\}$ $\quad (\because \; x^4 \geq 0$ for all real $x)$

$= [0, \infty) \neq$ co-domain R $(= (-\infty, \infty))$

$\therefore$ f is not onto. $\qquad \therefore$ f is neither one-one nor onto.

$\therefore$ Option (D) is the correct answer.

12. Let $f : R \to R$ be defined as $f(x) = 3x$. Choose the correct answer.

(A) f is one-one onto **(B) f is many-one onto**

(C) f is one-one but not onto

(D) f is neither one-one nor onto.

Sol. $f : R \to R$ is defined as $f(x) = 3x$ ...(*i*)

Is f one-one?

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$. $\quad \therefore$ By (*i*), $3x_1 = 3x_2$

Dividing by 3, $x_1 = x_2$ $\qquad \therefore$ f is one-one.

Is f onto?

Let $y \in$ co-domain R.

Let $y = f(x)$ $\quad \therefore \quad y = 3x$ (By (*i*))

$\therefore \quad x = \dfrac{y}{3} \in$ domain R for every $y \in$ co-domain R

$\therefore$ By (*i*), $f(x) = 3x = 3\left(\dfrac{y}{3}\right) = y$ $\qquad$ *i.e.*, Range = co-domain.

$\therefore$ f is onto. $\qquad \therefore$ f is both one-one and onto.

$\therefore$ Option (A) is the correct answer.

Exercise 1.3 *(Page No. 18-19)*

1. Let $f : \{1, 3, 4\} \to \{1, 2, 5\}$ and $g : \{1, 2, 5\} \to \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof.

Sol. Let A = {1, 3, 4}, B = {1, 2, 5} and C = {1, 3}.

Given: $f : A \to B$ and $g : B \to C \Rightarrow gof : A \to C$

i.e., $gof : \{1, 3, 4\} \to \{1, 3\}$.

Now $f = \{(1, 2), (3, 5), (4, 1)\} \Rightarrow f(1) = 2, f(3) = 5, f(4) = 1$

Also $g = \{(1, 3), (2, 3), (5, 1)\} \Rightarrow g(1) = 3, g(2) = 3, g(5) = 1$

$\therefore \quad (gof)(1) = g(f(1)) = g(2) = 3,$

$(gof)(3) = g(f(3)) = g(5) = 1$ and $(gof)(4) = g(f(4)) = g(1) = 3$

Hence, $gof = \{(1, 3), (3, 1), (4, 3)\}$.

2. Let f, g and h be functions from R to R. Show that

(i) $(f + g)oh = foh + goh$ (ii) $(fg)oh = (foh) . (goh)$

Sol. Definition of Equal functions

Two functions f and g are said to be equal if they have the same domain D and $f(x) = g(x)$ for all $x \in$ D.

(i) $[(f + g)oh](x) = (f + g)(h(x))$

(By Def. of composite function)

$= f(h(x)) + g(h(x))$

(By Def. of sum function)

$= (foh)(x) + (goh)(x)$

(By Def. of composite function)

$= (foh + goh)x$ (By Def. of sum function) for all $x \in$ R, domain.

$\Rightarrow (f + g)oh = foh + goh$ (By Def. of equal functions)

(ii) $[(fg)oh](x) = (fg)(h(x))$ (By Def. of composite function)

$= f(h(x)) . g(h(x))$ (By Def. of product function)

$= (foh)(x) . (goh)(x)$(By Def. of composite function)

$= ((foh) . (goh))x$ (By Def. of product function)

$\Rightarrow (fg)oh = (foh) . (goh)$. (By Def. of equal functions)

3. Find gof and fog if

(i) $f(x) = |x|$ and $g(x) = |5x - 2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{1/3}$.

Sol. (i) **Given:** $f(x) = |x|$ ...(i)

and $g(x) = |5x - 2|$ ...(ii)

We know that $(gof)x = g(f(x)) = g(|x|)$ [By (i)]

Changing $x \to |x|$ in (ii), $= |5|x| - 2|$

Again $(fog)x = f(g(x)) = f(|5x - 2|)$ [By (ii)]

Changing x to $|5x - 2|$ in (i) $= ||5x - 2|| = |5x - 2|$

[$\because$ $t \in$ R $\Rightarrow$ $|t| \ge 0$ and hence $||t||$ is number $|t|$ itself]

$\therefore$ $(gof)x = |5|x| - 2|$ and $(fog)x = |5x - 2|$.

(ii) **Given:** $f(x) = 8x^3$ ...(i)

and $g(x) = x^{1/3}$ ...(ii)

We know that $(gof)x = g(f(x)) = g(8x^3)$ (By (i))

Changing x to $8x^3$ in (ii), $= (8x^3)^{1/3} = ((2x)^3)^{1/3} = 2x$

Again $(fog)x = f(g(x)) = f(x^{1/3})$ (By (ii))

Changing x to $x^{1/3}$ in (i), $= 8(x^{1/3})^3 = 8x$

$\therefore$ $(gof)x = 2x$ and $(fog)x = 8x$.

4. If $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$, for all $x \neq \dfrac{2}{3}$. What is the inverse of f?

Sol. Here $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$ ...(i)

$$\therefore \quad fof(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right) \qquad \text{[by } (i)]$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} \qquad \left[\text{Replacing } x \text{ by } \frac{4x+3}{6x-4} \text{ in } (i)\right]$$

Multiplying by L.C.M. = $(6x - 4)$

$$(fof)x = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x = I_A(x) \text{ for all}$$

$$x \neq \frac{2}{3}$$

$$\Rightarrow \quad fof = I_A \text{ where } A = R - \left\{\frac{2}{3}\right\} = \text{domain of } f$$

$$\Rightarrow \quad f^{-1} = f.$$

5. State with reason whether following functions have inverse. Find the inverse, if it exists.

(*i*) $f : \{1, 2, 3, 4\} \rightarrow \{10\}$
with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(*ii*) $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$
with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(*iii*) $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$
with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

Sol. (*i*) Since $f(1) = f(2) = f(3) = f(4) = 10$, f is many-one and not one-one, so that f does not have inverse.

(*ii*) Since $g(5) = g(7) = 4$, g is many-one and not one-one, so that g does not have inverse.

(*iii*) Here distinct elements of the domain have distinct images under h, so that h is one-one. Moreover range of $h = \{7, 9, 11, 13\}$ = co-domain, so that h is onto.

Since h is one-one and onto, therefore, h has inverse given by $h^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$.

6. Show that $f : [-1, 1] \rightarrow R$, given by $f(x) = \frac{x}{x+2}$ is one-one. Find the inverse of the function $f : [-1, 1] \rightarrow$ Range f.

Sol. For $x_1, x_2 \in [-1, 1]$.

Putting $x = x_1$ and $x = x_2$ in $f(x) = \frac{x}{x+2}$,

$$f(x_1) = f(x_2) \quad \Rightarrow \quad \frac{x_1}{x_1+2} = \frac{x_2}{x_2+2}$$

Cross-multiplying,

$$x_1(x_2 + 2) = x_2(x_1 + 2)$$

$\Rightarrow \quad x_1x_2 + 2x_1 = x_1x_2 + 2x_2$

$\Rightarrow \quad 2x_1 = 2x_2 \quad \Rightarrow x_1 = x_2 \quad \therefore f$ is one-one.

Now we have to find the inverse of the function $f : X \to Y$ (say) $=$ Range of f (given)

where $X = [-1, 1]$

and $\quad Y = \left\{y \in R : y = \dfrac{x}{x+2} \text{ for some } x \in X\right\}$ = range of f.

Therefore, f is onto.

[$\because$ Co-domain Y = Range of f (given)]

Thus, f is one-one and onto and therefore, f^{-1} exists.

$\therefore \quad y = f(x) \quad \Rightarrow \quad x = f^{-1}(y)$ ...(*i*)

To find f^{-1}, let us find x in terms of y.

Given: $y = \dfrac{x}{x+2}$

Cross-multiplying,

$xy + 2y = x$ or $2y = x(1-y)$ or $x = \dfrac{2y}{1-y}$, $y \neq 1$ ...(*ii*)

From (*i*) and (*ii*),

$\therefore f^{-1} : Y \to X$ is defined by $f^{-1}(y) (= x) = \dfrac{2y}{1-y}$, $y \neq 1$.

7. Consider $f : R \to R$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

Sol. Given: $f : R \to R$ given by $f(x) = 4x + 3$ ...(*i*)

We know that a function f is invertible iff f is one-one and onto.

Is f one-one?

Let $x_1, x_2 \in$ domain R such that $f(x_1) = f(x_2)$

$\therefore$ By (*i*), $4x_1 + 3 = 4x_2 + 3 \Rightarrow 4x_1 = 4x_2$

Dividing by 4, $x_1 = x_2$. $\quad \therefore f$ is one-one.

Is f onto? Let $y \in$ co-domain R.

Let $y = f(x) = 4x + 3$. (By (*i*))

Let us find x in terms of y.

$y = 4x + 3 \Rightarrow y - 3 = 4x$

$\Rightarrow \quad x = \dfrac{y-3}{4}$ ...(*ii*)

$\therefore$ By (*i*), $f(x) = 4x + 3 = 4\left(\dfrac{y-3}{4}\right) + 3$

$= y - 3 + 3 = y$

i.e., Every y is $f(x)$ *i.e.*, co-domain = Range $\quad \therefore f$ is onto also.

$\therefore \quad f$ is both one-one and onto and hence f^{-1} exists.

$\therefore \quad f(x) = y \Rightarrow f^{-1}(y) = x \quad$ *i.e.*, $\quad f^{-1}(y) = \dfrac{y-3}{4}$. (By (*ii*))

8. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with inverse f^{-1} given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Sol. A function $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. ...(*i*)

We know that a function f is invertible iff f is one-one and onto.

Is f one-one? Let $x_1, x_2 \in$ domain R_+ such that $f(x_1) = f(x_2)$.

$\therefore$ By (*i*), $x_1^2 + 4 = x_2^2 + 4$ or $x_1^2 = x_2^2$

$\therefore$ $x_1 = \pm x_2$

Rejecting negative sign ($\because$ $x_1, x_2 \in$ domain R_+ and hence can't be negative).

$\therefore$ $x_1 = x_2$.

$\therefore$ f is one-one.

Is f onto? Let $y \in$ co-domain $[4, \infty)$.

Let $y = f(x) = x^2 + 4$

Let us find x in terms of y. $y = x^2 + 4 \Rightarrow y - 4 = x^2$

$\Rightarrow$ $x^2 = y - 4 \Rightarrow x = \pm \sqrt{y-4}$

Rejecting negative sign ($\because$ $x \in R_+$),

$$x = \sqrt{y-4} \qquad ...(ii)$$

and $x = \sqrt{y-4} \in R_+$ because for all $y \in$ co-domain $[4, \infty)$,

$y \ge 4$ *i.e.*, $y - 4 \ge 0$ and hence $\sqrt{y-4} \in R_+$

Putting $x = \sqrt{y-4}$ in $f(x) = x^2 + 4$, we have

$f(x) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$. Every $y = f(x)$

i.e., co-domain = Range *i.e.*, f is onto (also). $\therefore$ f^{-1} exists.

and $f(x) = y \Rightarrow f^{-1}(y) = x$ $\therefore$ $f^{-1}(y) = \sqrt{y-4}$ (By (*ii*))

9. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$.

Show that f is invertible with $f^{-1}(y) = \left(\frac{(\sqrt{y+6})-1}{3}\right)$.

Sol. Given: $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$ ...(*i*)

We know that f is invertible iff f is one-one and onto.

Is f one-one? Let $x_1, x_2 \in R_+$ such that $f(x_1) = f(x_2)$.

$\therefore$ By (*i*), $9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow$ $9x_1^2 - 9x_2^2 + 6x_1 - 6x_1 = 0$

$\Rightarrow$ $9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow$ $9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$

$\Rightarrow$ $(x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$\Rightarrow$ $(x_1 - x_2)(9x_1 + 9x_2 + 6) = 0$

But $9x_1 + 9x_2 + 6 > 0$ and hence $\ne 0$

($\because$ $x_1, x_2 \in R_+$ $\Rightarrow$ $x_1 \ge 0$ and $x_2 \ge 0$)

$\therefore$ $x_1 - x_2 = 0$ *i.e.*, $x_1 = x_2$ $\therefore$ f is one-one.

Is f onto? Let $y \in$ co-domain $[-5, \infty)$

Let $y = f(x) = 9x^2 + 6x - 5$ (By (*i*))

Let us find x in terms of y by completing squares on R.H.S.

$$y = 9x^2 + 6x - 5 = 9\left[x^2 + \frac{6x}{9} - \frac{5}{9}\right] = 9\left[x^2 + \frac{2x}{3} - \frac{5}{9}\right]$$

Adding and subtracting $\left(\frac{1}{2}\text{coeff. of } x\right)^2 = \left(\frac{1}{2}\times\frac{2}{3}\right)^2 = \left(\frac{1}{3}\right)^2$

$$y = 9\left[x^2 + \frac{2x}{3} + \left(\frac{1}{3}\right)^2 - \frac{1}{9} - \frac{5}{9}\right] = 9\left[\left(x + \frac{1}{3}\right)^2 - \frac{6}{9}\right]$$

or $y = 9\left(x + \frac{1}{3}\right)^2 - 6 \Rightarrow y + 6 = 9\left(x + \frac{1}{3}\right)^2$

$$\Rightarrow \left(x + \frac{1}{3}\right)^2 = \frac{y+6}{9} \Rightarrow x + \frac{1}{3} = \pm\sqrt{\frac{y+6}{9}}$$

$$\Rightarrow x = \frac{-1}{3} \pm \frac{\sqrt{y+6}}{3} = \frac{-1 \pm \sqrt{y+6}}{3}$$

Rejecting negative sign

$$\left(\because \text{ then } x = \frac{-1-\sqrt{y+6}}{3} = -\left(\frac{1+\sqrt{y+6}}{3}\right) \notin R_+\right)$$

$\therefore x = \frac{-1+\sqrt{y+6}}{3} \in R_+$ for every $y \in$ co-domain $[-5, \infty)$. ...(*ii*)

$\therefore$ By (*i*), $f(x) = 9x^2 + 6x - 5$

$$= 9\left(\frac{-1+\sqrt{y+6}}{3}\right)^2 + 6\left(\frac{-1+\sqrt{y+6}}{3}\right) - 5$$

$$= (-1 + \sqrt{y+6})^2 + 2(-1 + \sqrt{y+6}) - 5$$

$$= 1 + y + 6 - 2\sqrt{y+6} - 2 + 2\sqrt{y+6} - 5$$

or $f(x) = y$ *i.e.,* every $y = f(x)$

i.e., co-domain = range. $\therefore$ f is onto (also).

$\therefore$ f^{-1} exists. $\therefore$ $y = f(x) \Rightarrow f^{-1}(y) = x$

i.e., $f^{-1}(y) = \frac{-1+\sqrt{y+6}}{3}$ *i.e.,* $\frac{\sqrt{y+6}-1}{3}$.

10. Let $f : X \to Y$ is an invertible function, then f has unique inverse.

Sol. $\because$ $f : X \to Y$ is an invertible function, therefore f has an inverse.

Again because $f : X \to Y$ is invertible, therefore, f is one-one and onto.

i.e., for every $y \in Y$, $\exists$ a unique $x \in X$ such that $f(x) = y$ ...(*i*)

If possible, let $g_1 : Y \to X$ and $g_2 : Y \to X$ be two inverses of f so that

$g_1(y) = x$ and $g_2(y) = x$.

Also $g_1 of = I_X, fog_1 = I_Y$ ...(*ii*)

and also $g_2 of = I_X, fog_2 = I_Y$ ...(*iii*)

For all $y \in Y$, $g_1(y) = g_1(f(x))$ [By (*i*)]

$= (g_1 of)(x) = I_X(x)$ [By (*ii*)] $= (g_2 of)(x)$ [By (*iii*)]

$= g_2(f(x)) = g_2(y)$ [By (*i*)]

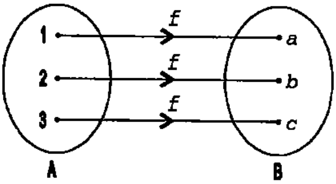
$\therefore$ $g_1(y) = g_2(y)$ for all $y \in Y$

$\Rightarrow$ $g_1 = g_2$ (By definition of equal functions).

11. Consider $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Sol. Given: A function $f : \{1, 2, 3\}$ (= set A say) $\rightarrow \{a, b, c\}$ (= set B (say)) such that $f(1) = a$,

$f(2) = b, f(3) = c$ ...(*i*)



f is one-one because distinct elements of A have distinct images in B.

f is onto.

($\because$ Range set = $\{a, b, c\}$ = co-domain B)

$\therefore$ f^{-1} B $\rightarrow$ A exists and hence from (*i*),

$f^{-1}(a) = 1, f^{-1}(b) = 2$ and $f^{-1}(c) = 3$. ...(*ii*)

Now again by definition of Inverse of a function.

We have from (*ii*), $f^{-1} : A \rightarrow B$ and $(f^{-1})^{-1}(1) = a$,

$(f^{-1})^{-1}(2) = b$ and $(f^{-1})^{-1}(3) = c$. ...(*iii*)

By definition of equal functions, from (*iii*) and (*i*), we have $(f^{-1})^{-1} = f$.

Note. Two functions f and g are said to be **equal** if they have (*i*) same domain and (*ii*), $f(x) = g(x)$ for all x in their common domain.

12. Let $f : X \rightarrow Y$ is an invertible function, then the inverse of f^{-1} is f, *i.e.*, $(f^{-1})^{-1} = f$.

Sol. Since $f : X \rightarrow Y$ is invertible, therefore, there exists a function

$g = f^{-1} : Y \rightarrow X$ defined by $g(y) = x$ where $f(x) = y$.

Also $gof = I_X$ and $fog = I_Y$ ...(*i*)

Now f is invertible

$\Rightarrow f$ is one-one and onto

$\Rightarrow g$, the inverse of f is also one-one and onto.

$\Rightarrow g$ is invertible

$\Rightarrow g^{-1}$ exists and $g^{-1} : X \rightarrow Y$.

From (*i*), we also have, f is inverse of g

$\Rightarrow f = g^{-1}$

Putting $g = f^{-1}$, we have $f = (f^{-1})^{-1}$

Hence $(f^{-1})^{-1} = f$.

13. If $f : R \rightarrow R$ be given by $f(x) = (3 - x^3)^{1/3}$ then $(fof)x$ is

(A) $x^{1/3}$ **(B) x^3** **(C) x** **(D) $(3 - x^3)$.**

Sol. $f : R \to R$ is given by $f(x) = (3 - x^3)^{1/3}$ ...(*i*)

We know by definition that

$$(fof)x = f(f(x)) = f((3 - x^3)^{1/3}) \quad \text{(By } (i))$$
$$= f(t) \text{ where } t = (3 - x^3)^{1/3}$$
$$= (3 - t^3)^{1/3} \quad \text{(By } (i))$$

Putting $t = (3 - x^3)^{1/3}$; $= (3 - ((3 - x^3)^{1/3})^3)^{1/3}$

$$= (3 - (3 - x^3))^{1/3}. \quad \left(\because 3 \times \frac{1}{3} = 1\right)$$
$$= (3 - 3 + x^3)^{1/3} = (x^3)^{1/3} = x^{3 \times 1/3} = x^1 = x$$

∴ Option (C) is the correct answer.

14. Let $f : R - \left\{-\frac{4}{3}\right\} \to R$ be a function defined as $f(x) = \frac{4x}{3x+4}$.

The inverse of f is the map g : Range $f \to R - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$ **(B) $g(y) = \frac{4y}{4-3y}$**

(C) $g(y) = \frac{4y}{3-4y}$ **(D) $g(y) = \frac{3y}{4-3y}$.**

Sol. **Given:** $f(x) = \frac{4x}{3x+4}$ ...(*i*)

Given: Inverse of f is g. Therefore f is one-one and onto.

$$\therefore \quad y = f(x) = \frac{4x}{3x+4}. \quad \text{(By } (i))$$

Let us find x in terms of y.

Cross-multiplying, $y(3x + 4) = 4x \Rightarrow 3xy + 4y = 4x$

$\Rightarrow$ $- 4x + 3xy = - 4y \Rightarrow - x(4 - 3y) = - 4y$

$\Rightarrow$ $x = \frac{4y}{4-3y} \Rightarrow f^{-1}(y) = \frac{4y}{4-3y}$

$[\because \ y = f(x) \Rightarrow x = f^{-1}(y)]$

$\Rightarrow g(y) = \frac{4y}{4-3y}$ $[\because \ g = f^{-1}$ (given)]

∴ Option (B) is correct option.

Exercise 1.4 *(Page No. 24-26)*

1. **Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.**
   - **(*i*) On Z^+, define $*$ by $a * b = a - b$**
   - **(*ii*) On Z^+, define $*$ by $a * b = ab$**
   - **(*iii*) On R, define $*$ by $a * b = ab^2$**
   - **(*iv*) On Z^+, define $*$ by $a * b = |a - b|$**
   - **(*v*) On Z^+, define $*$ by $a * b = a$.**

Sol. Definition: Binary operation $*$ on a set A is a function from $A \times A \to A$.

i.e., an operation $*$ on a set A is called binary operation if $a \in A$ and $b \in A \Rightarrow a * b \in A$.

We also know that Z^+ denotes the set of positive integers

i.e., $\mathbf{Z^+ = N}$.

(*i*) **Given:** On Z^+ define $*$ by $a * b = a - b$. ...(*i*)

Now $a = 2 \in Z^+$ and $b = 3 \in Z^+$, but $a * b = 2 * 3$

$= 2 - 3$ (By (*i*)) $= -1 \notin Z^+$

$\because$ Operation $*$ in (*i*) is **not** a binary operation and hence does not satisfy **closure law.**

Remark. It may be noted for the above (*i*) part that if $a \in Z^+$, $b \in Z^+$ and $a > b$, then $a * b = a - b$ is a positive integer and hence belongs to Z^+.

But let us know that truth is always truth. Hence a result which is sometimes true and sometimes not true is said to be not true.

(*ii*) **Given:** On Z^+, define $*$ by $a * b = ab$ ...(*i*)

Let $a \in Z^+ = N$ and $b \in Z^+ = N$; then

$a * b = ab$ is the product of two natural numbers and hence $\in N = Z^+$.

$\therefore$ This operation $*$ is a binary operation, *i.e.*, satisfies **closure law.**

(*iii*) **Given:** On R_+ define $*$ by $a * b = ab^2$ ...(*i*)

We know that R_+ denotes the set of positive real numbers including 0.

Let $a \in R_+$ and $b \in R_+ \Rightarrow a \geq 0$ and $b \geq 0$.

$\therefore$ By (*i*), $a * b = ab^2 \geq 0$ *i.e.*, $a * b \in R_+$

$\therefore$ This operation $*$ is a binary operation *i.e.*, satisfies closure law.

(*iv*) **Given:** On Z^+ define $*$ by $a * b = |a - b|$ ...(*i*)

We know that $a = 2 \in Z^+$ and $b = 2 \in Z^+$

But by (*i*), $a * b = |a - b| = |2 - 2| = 0$ is not a positive integer and hence $\notin Z^+$.

$\therefore$ This operation $*$ is not a binary operation *i.e.*, does not satisfy closure law.

Remark. In the above question if we replace the set Z^+ by Z_+ (the set of positive integers including 0); then by (*i*) $a * b = |a - b|$ is either zero or positive integer and hence $\in Z_+$ for all $a, b \in Z_+$ and hence is a binary operation.

(*v*) **Given:** On Z^+, define $*$ by $a * b = a$ ...(*i*)

Let $a \in Z^+$ and $b \in Z^+$. Therefore a and b are positive integers. Therefore by (*i*), $a * b = a$ is also a positive integer and hence $\in Z^+$.

$\therefore$ Operation $*$ defined in (*i*) is a binary operation *i.e.*, satisfies closure law.

2. For each binary operation $*$ defined below, determine whether $*$ is commutative or associative.

(i) On Z, define $a * b = a - b$

(ii) On Q, define $a * b = ab + 1$

(iii) On Q, define $a * b = \frac{ab}{2}$

(iv) On Z⁺, define $a * b = 2^{ab}$

(v) On Z⁺, define $a * b = a^b$

(vi) On R – {– 1}, define $a * b = \frac{a}{b+1}$

Sol. Definition: A binary operation $*$ on a set A is said to be

I **commutative** if $a * b = b * a$ for all $a, b \in$ A

II **Associative** if $(a * b) * c = a * (b * c)$ for all $a, b, c \in$ A.

(i) **Given:** Binary operation on Z, defined as

$a * b = a - b$ ...(i)

where $a \in$ Z and $b \in$ Z.

Is $*$ commutative? Interchanging a and b in (i),

$$b * a = b - a = -(a - b)$$

From (i) and (ii), $a * b \neq b * a$ (unless $a = b$)

For example, from (i), $2 * 3 = 2 - 3 = -1$

and $3 * 2 = 3 - 2 = 1$ $\therefore$ $2 * 3 \neq 3 * 2$

$\therefore$ Binary operation $*$ given by (i) is not commutative.

Is $*$ associative? Let $a \in$ Z, $b \in$ Z, $c \in$ Z.

$\therefore$ By (i), $(a * b) * c = (a - b) * c = (a - b) - c$

$= a - b - c$ ...(iii)

Again by (i),

$a * (b * c) = a * (b - c) = a - (b - c) = a - b + c$ ...(iv)

From (iii) and (iv), $(a * b) * c \neq a * (b * c)$

($\because$ Right hand sides are not equal)

For example, $(2 * 3) * 4 = (2 - 3) * 4 = -1 * 4$

$= -1 - 4 = -5$

Again $2 * (3 * 4) = 2 * (3 - 4) = 2 * (-1) = 2 - (-1) = 3$

$\therefore$ $(2 * 3) * 4 \neq 2 * (3 * 4)$

$\therefore$ Binary operation $*$ given by (i) is not associative.

Here $*$ is neither commutative nor associative.

(ii) **Given:** Binary operation $*$ on Q defined by

$a * b = ab + 1$ ...(i)

for all $a, b \in$ Q

Is $*$ commutative? Interchanging a and b in (i),

$b * a = ba + 1 = ab + 1$ ...(ii)

[$\because$ We know that $ab = ba$ for all $a, b \in$ Q]

From (i) and (ii), $a * b = b * a$ for all $a, b \in$ Q

$\therefore$ Binary operation $*$ given by (*i*) is commutative.

Is $*$ associative? Let $a \in Q$, $b \in Q$, $c \in Q$.

By (*i*), $(a * b) * c = (ab + 1) * c = (ab + 1)\, c + 1$

$= abc + c + 1$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * (bc + 1) = a(bc + 1) + 1$

$= abc + a + 1$ ...(*iv*)

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

[$\because$ Right hand sides of (*iii*) and (*iv*) are not equal]

$\therefore$ $*$ is not associative.

Here $*$ is commutative but not associative.

(*iii*) **Given:** Binary operation $*$ on Q defined as

$$a * b = \frac{ab}{2} \quad \text{...}(i)$$

for all $a, b \in Q$.

Is $*$ commutative? Interchanging a and b in (*i*),

We have $b * a = \dfrac{ba}{2} = \dfrac{ab}{2}$ ...(*ii*)

[$\because$ Ordinary multiplication in Q is commutative]

From (*i*) and (*ii*), we have

$a * b = b * a$ for all $a, b \in Q$

$\therefore$ Binary operation $*$ given by (*i*) is commutative.

Is $*$ associative? Let $a, b, c \in Q$

$$\text{By } (i),\ (a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4} \quad \text{...}(iii)$$

$$\text{Again by } (i),\ a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4} \quad \text{...}(iv)$$

From (*iii*) and (*iv*), $(a * b) * c = a * (b * c)$ for all $a, b, c \in Q$

($\because$ Right Hand sides of (*iii*) and (*iv*) are equal)

$\therefore$ $*$ is associative.

Here $*$ is both commutative and associative.

(*iv*) **Given:** Binary operation $*$ on Z^+ defined as

$a * b = 2^{ab}$ ...(*i*)

for all $a, b \in Z^+$

Is $*$ commutative?

Interchanging a and b in (*i*), we have

$b * a = 2^{ba} = 2^{ab}$ ...(*ii*)

From (*i*) and (*ii*), we have $a * b = b * a$ for all $a, b \in Z^+$

$\therefore$ $*$ given by (*i*) is commutative.

Is $*$ associative? Let $a, b, c \in Z^+$.

By (*i*), $(a * b) * c = 2^{ab} * c = 2^{(2^{ab}).c}$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * 2^{bc} = 2^{a.(2^{bc})}$ ...(*iv*)

From (*iii*) and (*iv*) $(a * b) * c \neq a * (b * c)$

$\therefore$ $*$ given by (*i*) is associative.

$\therefore$ $*$ given by (*i*), is commutative but not associative.

(*v*) **Given:** Binary operation $*$ on Z^+ defined as

$a * b = a^b$ ...(*i*)

for all $a, b \in Z^+$

Is $*$ commutative? Interchanging a and b in (*i*),

We have $b * a = b^a$ ...(*ii*)

From (*i*) and (*ii*), $a * b \neq b * a$.

$\therefore$ $*$ given by (*i*) is not commutative.

Is $*$ associative? Let $a, b, c \in Z^+$

By (*i*), $(a * b) * c = (a^b) * c = (a^b)^c = a^{bc}$ ...(*iii*)

Again by (2), $a * (b * c) = a * (b^c) = a^{(b^c)}$ ...(*iv*)

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

$\therefore$ $*$ is not associative.

$\therefore$ $*$ given by (*i*) is neither commutative nor associative.

(*vi*) Since $a * b = \dfrac{a}{b+1}$ (given) ...(*i*)

Interchanging a and b in (*i*), $b * a = \dfrac{b}{a+1}$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*), $a * b \neq b * a$

$$\left[e.g.,\ 2 * 3 = \frac{2}{3+1} = \frac{2}{4} = \frac{1}{2} \text{ and } 3 * 2 = \frac{3}{2+1} = \frac{3}{3} = 1\right]$$

$\Rightarrow$ The binary operation '$*$' is not commutative.

$$\text{Also} \quad (a * b) * c = \frac{a}{b+1} * c = \frac{\frac{a}{b+1}}{c+1} \qquad \text{[By } (i)]$$

$$= \frac{a}{(b+1)(c+1)}$$

$$\text{and} \quad a * (b * c) = a * \frac{b}{c+1}$$

$$= \frac{a}{\frac{b}{c+1}+1} = \frac{a(c+1)}{b+c+1} \qquad \text{[By } (i)]$$

so that $(a * b) * c \neq a * (b * c)$

$$\left[e.g.,\ (2 * 3) * 4 = \frac{2}{3+1} * 4 = \frac{1}{2} * 4 = \frac{1/2}{4+1} = \frac{1}{10}\right.$$

$$\left.\text{and} \quad 2 * (3 * 4) = 2 * \frac{3}{4+1} = 2 * \frac{3}{5} = \frac{2}{3/5+1} = \frac{10}{8} = \frac{5}{4}\right]$$

$\Rightarrow$ The binary operation '$*$' is not associative.

3. Consider the binary operation $\wedge$ on the set {1, 2, 3, 4, 5} defined as $a \wedge b = \min \{a, b\}$. Write the operation table of the operation $\wedge$.

Sol. Given: Set is {1, 2, 3, 4, 5} = A (say)

Given: Binary operation $\wedge$ on the set A is defined as

$$a \wedge b = \min. \{a, b\} \quad ...(i)$$

Operation (or composition) table of binary operation $\wedge$ given by (*i*) is being formed below:

$\wedge$	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓
1 →	$1 \wedge 1 = 1$	$1 \wedge 2 = 1$	$1 \wedge 3 = 1$	$1 \wedge 4 = 1$	$1 \wedge 5 = 1$
2 →	$2 \wedge 1 = 1$	$2 \wedge 2 = 2$	$2 \wedge 3 = 2$	$2 \wedge 4 = 2$	$2 \wedge 5 = 2$
3 →	$3 \wedge 1 = 1$	$3 \wedge 2 = 2$	$3 \wedge 3 = 3$	$3 \wedge 4 = 3$	$3 \wedge 5 = 3$
4 →	$4 \wedge 1 = 1$	$4 \wedge 2 = 2$	$4 \wedge 3 = 3$	$4 \wedge 4 = 4$	$4 \wedge 5 = 4$
5 →	$5 \wedge 1 = 1$	$5 \wedge 2 = 2$	$5 \wedge 3 = 3$	$5 \wedge 4 = 4$	$5 \wedge 5 = 5$

(For example) by (*i*), $4 \wedge 3 = \min. \{4, 3\} = 3$ and by (*i*),

$2 \wedge 5 = \min. \{2, 5\} = 2$)

Remark. (e =) 5 ∈ A is the identity element for this binary operation $\wedge$ because the top row headed by the binary operation $\wedge$ coincides with the row headed by 5 in the composition table.

(By definition of identity element $e \in A$ is said to be identity element of binary operation if $a \wedge e = a$ for all $a \in A$).

4. Consider a binary operation $*$ on the set {1, 2, 3, 4, 5} given by the following multiplication table

(*i*) Compute $(2 * 3) * 4$ and $2 * (3 * 4)$

(*ii*) Is $*$ commutative?

(*iii*) Compute $(2 * 3) * (4 * 5)$.

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Sol. (*i*) To compute $(2 * 3) * 4$

Now we know that $2 * 3$ is the entry at the intersection of the row headed by 2 and column headed by 3 in the table and this entry is 1.

∴ $(2 * 3) * 4 = 1 * 4 = 1$ as explained above.

Again $2 * (3 * 4) = 2 * 1 = 1$

∴ $(2 * 3) * 4 = 1$ and $2 * (3 * 4) = 1$

(*ii*) **Is $*$ commutative ?**

Let us look at each diagonal entry of the main diagonal starting with $*$. The corresponding entries of the column below any such entry and the adjacent row to the same diagonal entry are same.

$\therefore$ We can say that there is symmetry about this main diagonal.

Better method: The matrix formed by the given table is a **symmetric matrix** and hence $*$ is commutative.

$\therefore$ $*$ is commutative.

(*iii*) To compute $(2 * 3) * (4 * 5)$

Now $2 * 3$ = Entry in the table at the intersection of row headed by 2 and column headed by 3 = 1.

Similarly, $4 * 5 = 1$ $\therefore$ $(2 * 3) * (4 * 5) = 1 * 1 = 1$.

5. Let $*'$ be the binary operation on the set {1, 2, 3, 4, 5} defined by $a *' b$ = H.C.F. of a and b. Is the operation $*'$ same as the operation $*$ defined in Exercise 4 above? Justify your answer.

Sol. Given: Binary operation $*'$ on the set {1, 2, 3, 4, 5} is defined as $a *' b$ = H.C.F. of a and b *i.e.*, highest common factor of a and b ...(*i*)

$\therefore$ $2 *' 4$ = H.C.F. of 2 and 4

= (Highest common factor of 2 and 4) = 2

$3 *' 5$ = H.C.F. of 3 and 5

= (Highest common factor of 3 and 5) = 1 etc.

The composition (operation) table for this binary operation $*'$ namely H.C.F. a and b is being given below:

$*'$	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓
1 →	1	1	1	1	1
2 →	$2 *' 1 = 1$	$2 *' 2 = 2$	$2 *' 3 = 1$	$2 *' 4 = 2$	$2 *' 5 = 1$
3 →	1	1	$3 *' 3 = 3$	$3 *' 4 = 1$	1
4 →	1	$4 *' 2 = 2$	1	$4 *' 4 = 4$	1
5 →	1	1	1	1	$5 *' 5 = 5$

Now the corresponding entries in this composition table for this binary operation $*'$ and the composition table given in exercise 4 are same.

$\therefore$ Operation $*'$ of this Exercise 5 and operation $*$ of Exercise 4 defined on the same set {1, 2, 3, 4, 5} are same.

6. Let $*$ be the binary operation on N given by $a * b$ = L.C.M. of a and b. Find

(*i*) $5 * 7$, $20 * 16$ **(*ii*) Is $*$ commutative?**

(*iii*) Is $*$ associative? **(*iv*) Find the identity of $*$ in N**

(*v*) Which elements of N are invertible for the operation $*$?

Sol. Given: $*$ is a binary operation on N given by

$a * b$ = L.C.M. of a and b ...(*i*)

i.e., least common multiple of a and b.

(*i*) $\therefore$ $5 * 7$ = L.C.M. of 5 and 7 = $5 \times 7 = 35$

and $20 * 16$ = L.C.M. of 20 and 16

= Least common multiple of 20 and 16. = 80

($\because$ All common multiples of 20 and 16 are 80, 160, 240 etc.)

(*ii*) **Is $*$ commutative?**

Let $a \in$ N and $b \in$ N

$\therefore$ by (*i*), $a * b$ = L.C.M. of a and b

= L.C.M. of b and $a = b * a$.

$\therefore$ $*$ is commutative.

(*iii*) **Is $*$ associative?** Let $a \in$ N, $b \in$ N, $c \in$ N

$\therefore$ $(a * b) * c$ = (L.C.M. of a and b) $* c$ [By (*i*)]

= L.C.M. ((L.C.M. of a and b) and c) [By (*i*)]

or $(a * b) * c$ = L.C.M. of a, b, c ...(*ii*)

Again $a * (b * c) = a *$ (L.C.M. of b and c) [By (*i*)]

= L.C.M. (a and (L.C.M. of b and c)) [By (*i*)]

= L.C.M. of a, b and c ...(*iii*)

From (*ii*) and (*iii*), we have $(a * b) * c = a * (b * c)$

$\therefore$ $*$ is associative.

(*iv*) **To find the identity of $*$ on N.**

Let $a \in$ N. Now $1 \in$ N

By (*i*), $a * 1$ = L.C.M. of a and $1 = a$ for all $a \in$ N

(*i.e.,* $a * e = a$ with $e = 1$)

$\therefore$ $1 \in$ N is the identity element for $*$ on N.

(*v*) **Which elements of N are invertible for the operation $*$?**

Let $a \in$ N. Let $b \in$ N be inverse of a.

$\therefore$ **By definition of inverse, $a * b = e$**

$\therefore$ By (*i*), L.C.M. of a and b is $e = 1$.

Now this is true only when $a = 1$ and $b = 1$.

($\because$ Only L.C.M. of 1 and 1 is 1)

$\therefore$ $1 \in$N is the only element of N which is invertible.

7. Is $*$ defined on the set {1, 2, 3, 4, 5} by $a * b$ = L.C.M. of a and b a binary operation? Justify your answer.

Sol. Given: Set {1, 2, 3, 4, 5} = A (say)

and operation $*$ is defined on A as $a * b$ = L.C.M. of a and b ...(*i*)

Now $2 \in$ A and $3 \in$ A

But by (*i*), $2 * 3$ = L.C.M. of 2 and 3 is $2 \times 3 = 6 \notin$ A

$3 * 4$ = L.C.M. of 3 and 4 is $3 \times 4 = 12 \notin A$

$\therefore$ Operation $*$ on the set A is not a binary opeartion on the set A. (*i.e.*, operation $*$ given by (*i*) does not satisfy **closure law**).

Remark. For still better understanding of the concept, the reader is suggested to construct the operation (*i.e.*, composition) table for the operation given by (*i*) and then observe that entries 6, 10, 12, 15, 20 of the composition table don't belong to A and hence conclude that operation $*$ is not a binary operation.

8. Let $*$ be the binary operation on N defined by $a * b$ = H.C.F. of a and b. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N?

Sol. $*$ is a binary operation on N defined as:

$a * b$ = H.C.F. of a and b ...(*i*)

Is $*$ commutative?

Reproduce Exercise 6(*ii*) replacing the phrase "L.C.M." by "H.C.F."

Is $*$ associative?

Reproduce Exercise 6(*iii*) replacing the phrase "L.C.M." by "H.C.F."

Does there exists identity for this binary operation on N?

We know that there does not exist any natural number e such that H.C.F. of a and e is a for all $a \in$ N. It may be noted that $e \neq 1$ even because $a * 1$ = H.C.F. of a and 1 is 1 and $\neq a$. (For identity $a * e = a$ and $\neq e$).

9. Let $*$ be a binary operation on the set Q of rational numbers as follows:

(*i*) $a * b = a - b$ **(*ii*) $a * b = a^2 + b^2$**

(*iii*) $a * b = a + ab$ **(*iv*) $a * b = (a - b)^2$**

(*v*) $a * b = \frac{ab}{4}$ **(*vi*) $a * b = ab^2$.**

Find which of the binary operations are commutative and which are associative.

Sol. (*i*) Reproduce the solution of Exercise 2(*i*) replacing Z by Q.

(*ii*) The given binary operation $*$ is $a * b = a^2 + b^2$ ...(*i*)

for all $a, b \in$ Q.

Is $*$ commutative?

Interchanging a and b in (*i*), we have

$$b * a = b^2 + a^2 = a^2 + b^2 \quad ...(ii)$$

[$\because$ Addition is commutative in Q]

From (*i*) and (*ii*), we have $a * b = b * a$ for all $a, b \in$ Q.

$\therefore$ $*$ is commutative on Q.

Is $*$ associative? Let $a, b, c \in$ Q.

By (*i*), $(a * b) * c = (a^2 + b^2) * c = (a^2 + b^2)^2 + c^2$

$= a^4 + b^4 + 2a^2b^2 + c^2$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * (b^2 + c^2) = a^2 + (b^2 + c^2)^2$

$= a^2 + b^4 + c^4 + 2b^2c^2$ ...(*iv*)

From (*iii*) and (*iv*) $(a * b) * c \neq a * (b * c)$

∴ Binary operation * is commutative but not associative.

(*iii*) The given binary operation * on Q is $a * b = a + ab$ ...(*i*)

for all $a, b \in$ Q

Is * commutative?

Interchanging a and b in (*i*)

We have $b * a = b + ba = b + ab$ ...(*ii*)

From (*i*) and (*ii*), $a * b \neq b * a$

∴ * is not commutative.

Is * associative?

Let $a, b, c \in$ Q

From (*i*), $(a * b) * c = (a + ab) * c = a + ab + (a + ab)\,c$

$= a + ab + ac + abc$ ...(*iii*)

Again from (*i*), $a * (b * c) = a * (b + bc)$

$= a + a(b + bc) = a + ab + abc$ ...(*iv*)

From (*iii*) and (*iv*) $(a * b) * c \neq a * (b * c)$

∴ * is not associative.

∴ Binary operation * is neither commutative nor associative.

(*iv*) The given binary operation * on Q is

$a * b = (a - b)^2$ ...(*i*)

for all $a, b \in$ Q

Interchanging a and b in (*i*), we have

$b * a = (b - a)^2 = (-(a - b))^2 = (a - b)^2 = a * b$ [By (*i*)]

∴ Binary operation * is commutative.

Is * associative?

Let $a, b, c \in$ Q

By (*i*), $(a * b) * c = (a - b)^2 * c = ((a - b)^2 - c)^2$

$= (a^2 + b^2 - 2ab - c)^2$ ...(*ii*)

Again by (*i*),

$a * (b * c) = a * (b - c)^2 = (a - (b - c)^2)^2$

$= (a - (b^2 + c^2 - 2bc))^2 = (a - b^2 - c^2 + 2bc)^2$

$= (-(b^2 + c^2 - 2bc - a))^2$

$= (b^2 + c^2 - 2bc - a)^2$ ...(*iii*)

From (*ii*) and (*iii*), $(a * b) * c \neq a * (b * c)$

∴ Binary operation * is not associative.

∴ Binary operation * is neither commutative nor associative.

(*v*) **Given:** Binary operation $*$ on Q defined as $a * b = \frac{ab}{4}$

for all $a, b \in$ Q ...(*i*)

Is $*$ commutative?

Interchanging a and b in (*i*), $b * a = \frac{ba}{4} = \frac{ab}{4}$ ...(*ii*)

(Multiplication is commutative in Q)

From (*i*) and (*ii*), we have $a * b = b * a$ for all $a, b \in$ Q.

$\therefore$ $*$ is commutative on Q.

Is $*$ associative on Q?

Let $a \in$ Q, $b \in$ Q, $c \in$ Q.

By (*i*), $(a * b) * c = \left(\frac{ab}{4}\right) * c = \frac{\left(\frac{ab}{4}\right)c}{4} = \frac{abc}{16}$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * \frac{bc}{4} = \frac{a\left(\frac{bc}{4}\right)}{4} = \frac{abc}{16}$ ...(*iv*)

From (*iii*) and (*iv*), we have $(a * b) * c = a * (b * c)$ for all $a, b, c \in$ Q.

$\therefore$ $*$ is associative. $\therefore$ is commutative as well as associative.

(*vi*) **Given:** Binary operation $*$ on Q defined as

$a * b = ab^2$ ...(*i*)

for all, $a, b \in$ Q

Is $*$ commutative?

Interchanging a and b in (*i*), we have

$b * a = ba^2$ ...(*ii*)

From (*i*) and (*ii*), we have $a * b \neq b * a$

$\therefore$ $*$ is not commutative.

Is $*$ associative? Let $a, b, c \in$ Q.

By (*i*), $(a * b) * c = (ab^2) * c\ (ab^2)\ c^2 = ab^2c^2$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * (bc^2) = a\ (bc^2)^2$

$= ab^2c^4$ ...(*iv*)

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

$\therefore$ $*$ is not associative.

$\therefore$ $*$ is neither commutative nor associative.

10. Find which of the operations given above has identity.

Sol. (*i*) **Existence of identity for $*$.**

The binary operation on the set Q of rational numbers is defined as $a * b = a - b$. ...(*i*)

If possible, let $e \in Q$ be the identity for $*$.

$\therefore$ $a * e = e * a$ (= a) for all $a \in Q$.

$\therefore$ By (*i*), $a - e = e - a \Rightarrow -2e = -2a \Rightarrow e = a \in Q$

$\therefore$ e has infinite values. But this is impossible [$\because$ Identity e for a binary operation is unique]

$\therefore$ Identity for this $*$ does not exist.

Even $0 \in Q$ is not the identity for binary operation $*$ given by (*i*) as $a * 0$ (= $a - 0 = a \neq 0 * a$ (= $0 - a = -a$) for all $a \in Q$

(*ii*) **Existence of identity for *.**

Identity element $e \in Q$ does not exist for $*$ given by (*i*), as

$a * e = a^2 + e^2$ can never be equal to a.

(*iii*) **Existence of Identity for ***

If possible, let $e \in Q$ be the identity of binary operation $*$ given by (*i*).

$\therefore$ $a * e = e * a$ (= a) $\therefore$ By (*i*), $a + ae = e + ea$

$\Rightarrow a = e \Rightarrow e = a \in Q$

$\Rightarrow$ e has infinite values $a \in Q$

But this is impossible because we know that e is unique.

$\therefore$ e does not exist for this binary operation $*$.

(*iv*) **Existence of Identity for ***

If possible let $e \in Q$ be the identity for $*$ given by (*i*).

Therefore, $a * e = a$ for all $a \in Q$.

$\therefore$ By (*i*), $(a - e)^2 = a$ which is impossible to hold true for any $e \in Q$.

It is not true even for $e = 0$ because for $e = 0$, the above equation becomes $a^2 = a$ which is not true for every $a \in Q$.

$\therefore$ Identity element does not exist for this binary operation $*$.

(*v*) **Does there exist identity element for this binary operation *?**

If possible, let $e \in Q$, be the identity element for $*$.

$\therefore$ $a * e = a \Rightarrow$ By (*i*), $\frac{ae}{4} = a$

Cross-multiplying, $ae = 4a$

Dividing by a (if $a \neq 0$ *i.e.*, if Q is replaced by Q – {0}); then $e = 4 \in Q - \{0\}$ is the identity element of this $*$ on Q – {0}.

$\therefore$ For this binary operation $*$, e does not exist for Q but e exists for Q – {0} and $e = 4$.

(*vi*) **Does there exist identity element for this binary operation *?**

If possible, let $e \in Q$ be the identity element for this binary operation $*$.

$\therefore$ $a * e = a$ $\Rightarrow$ by (*i*), $ae^2 = a$

Dividing by a ($\neq 0$ if $a \in$ Q – {0}), $e^2 = 1$

$\therefore$ $e = \pm 1 \in$ Q – {0}

Even now e is not unique (because e has two values 1 and – 1)

$\therefore$ e does not exist for binary operation $*$ on Q nor exists for binary operation $*$ if Q is replaced by Q – {0}.

11. Let A = N × N and * be the binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$.

Show that * is commutative and associative. Find the identity element for * on A, if any.

Sol. For all $(a, b) \in$ A = N × N, we are given that

$$(a, b) * (c, d) = (a + c, b + d) \quad ...(i)$$
$$= (c + a, d + b)$$
[$\because$ Addition is commutative on N]
$$= (c, d) * (a, b) \quad [\text{By } (i)]$$

$\Rightarrow$ $*$ is commutative.

For all $(a, b), (c, d), (e, f) \in$ A = N × N, we have

$$[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f) \quad [\text{By } (i)]$$
$$= ((a + c) + e, (b + d) + f) \quad [\text{By } (i)]$$
$$= (a + (c + e), b + (d + f))$$
[$\because$ Addition is associative on N]
$$= (a, b) * (c + e, d + f) \quad [\text{By } (i)]$$
$$= (a, b) * [(c, d) * (e, f)] \quad (\text{By } (i))$$

$\Rightarrow$ $*$ is associative.

Now to find the identity element for * on A, if any

Now suppose (x, y) is the identity element in A = N × N: Then

$$(a, b) * (x, y) = (a, b), \forall (a, b) \in A \ (a * e = a)$$

$\Rightarrow$ $(a + x, b + y) = (a, b)$ [By (*i*)]

$\Rightarrow$ $a + x = a$ and $b + y = b$

$\Rightarrow$ $x = 0$ and $y = 0$

But $0 \notin$ N, therefore $(0, 0) \notin$ A = N × N

Hence $*$ has no identity element.

12. State whether the following statements are true or false. Justify.

(*i*) For any arbitrary binary operation * on a set N, $a * a = a \ \forall \ a \in$ N

(*ii*) If * is commutative binary operation on N, then $a * (b * c) = (c * b) * a$.

Sol. (*i*) **False.** Here given $a * a = a \ \forall \ a \in$ N ...(*i*)

Because we know that a binary operation $*$ on a set N is a function from N × N → N and by definition of function, Image of every element (a, b) of domain N × N must be possible.

But by given rule (*i*),
Image of (a, b) $(b \neq a)$ is undefined.

(*ii*) **True.** Because L.H.S. $= a * (b * c) = (b * c) * a$
[$\because$ $*$ is given to be commutative binary operation on N and $b * c \in$ N, $a \in$ N]
$= (c * b) * a$
[$\because$ $*$ is given to be commutative binary operation]
$=$ R.H.S.

13. Consider a binary operation $*$ on N defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is $*$ both associative and commutative?
(B) Is $*$ commutative but not associative?
(C) Is $*$ associative but not commutative?
(D) Is $*$ neither commutative nor associative?

Sol. Binary operation $*$ on N is defined as $a * b = a^3 + b^3$ ...(*i*)
for all $a, b \in$ N

Is $*$ commutative?

Interchanging a and b in (*i*),
$$b * a = b^3 + a^3 = a^3 + b^3 \quad ...(ii)$$
($\because$ Usual addition is commutative in N)

From (*i*) and (*ii*), $a * b = b * a$ $\quad\therefore$ $*$ is commutative.

Is $*$ associative?

Let $a, b, c \in$ N

By (*i*), $(a * b) * c = (a^3 + b^3) * c = (a^3 + b^3)^3 + c^3$ ...(*iii*)

Again $a * (b * c) = a * (b^3 + c^3) = a^3 + (b^3 + c^3)^3$ ...(*iv*)

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

$\therefore$ Binary operation $*$ is not associative.

$\therefore$ Binary operation $*$ is commutative but not associative.

$\therefore$ Option (B) is the correct answer.

MISCELLANEOUS EXERCISE *(Page No.: 29-31)*

1. Let f : R $\to$ R be defined by $f(x) = 10x + 7$. Find the function g : R $\to$ R such that $gof = fog = I_R$.

Sol. Given: f : R $\to$ R defined by $f(x) = 10x + 7$ ...(*i*)

We are to find a function g : R $\to$ R such that $gof = fog = I_R$

$\because$ It is given that $gof = fog = I_R$, therefore by definition of inverse of a function, f^{-1} exists and $g = f^{-1}$ ...(*ii*)

$\therefore$ To find function g $\Rightarrow$ To find f^{-1}.

Because f^{-1} exists, therefore f is one-one onto function.

$\therefore$ $\quad y(= f(x)) = 10x + 7$ ...(*iii*)

Let us find x in terms of y.

From (*iii*), $y - 7 = 10x$

$$\Rightarrow \quad x = \frac{y-7}{10} \quad \Rightarrow \quad f^{-1}(y) = \frac{y-7}{10} \quad \left| \because \; y = f(x) \Rightarrow x = f^{-1}(y) \right.$$

$\Rightarrow \quad g(y) = \dfrac{y-7}{10}.$ $\qquad$ ($\because$ By (*ii*), $f^{-1} = g$)

2. Let $f : W \to W$ be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Sol. We know that W, the set of all whole numbers = N $\cup$ {0} where N is the set of all natural numbers.

For one-one function: Let $n_1, n_2 \in W$ with $f(n_1) = f(n_2)$

If n_1 is odd and n_2 is even, then by the given rule of the function f, we will have $n_1 - 1 = n_2 + 1$, *i.e.*, $n_1 - n_2 = 2$ which is impossible since the difference of an odd and an even number is always odd. Similarly, n_1 is even and n_2 is odd is ruled out. Therefore, both n_1 and n_2 must be either odd or even.

Suppose n_1 and n_2 both are odd, then by the given rule of the function f, $f(n_1) = f(n_2) \;\Rightarrow\; n_1 - 1 = n_2 - 1$

$\Rightarrow \; n_1 = n_2$. Similarly, if n_1 and n_2 both are even, then

$f(n_1) = f(n_2)$

$\Rightarrow \; n_1 + 1 = n_2 + 1 \qquad \Rightarrow \qquad n_1 = n_2.$ $\qquad \therefore$ f is one-one.

Now let us prove that f is onto

According to given,

$f(n) = n - 1$ if n is odd $\Rightarrow f(n)$ is even ($\because$ Odd $-$ 1 = Even) ...(*i*)

and $\; f(n) = n + 1$ if n is even

$\Rightarrow \; f(n)$ is odd $\qquad$ ($\because$ Even + 1 = Odd) ...(*ii*)

Let $y \in$ Co-domain W

Case I. y is Even

$\therefore$ From (*i*), $y = f(n) = n - 1 \quad \therefore n = y + 1$ is odd ...(*iii*)

Case II. y is odd

$\therefore$ From (*ii*), $y = f(n) = n + 1 \; \therefore n = y - 1$ is even ...(*iv*)

$\therefore$ From (*iii*), for every even $y \in$ co-domain W,

there exsts odd $n = y + 1$ such that $f(n) = y$.

and from (*iv*), for every odd $y \in$ co-domain W,

there exists even $n = y - 1$ such that $f(n) = y$. $\qquad \therefore$ f is onto.

Thus, f is one-one and onto and therefore invertible.

$\therefore \; f(n) = y \qquad \Rightarrow \; f^{-1}(y) = n$

$\therefore$ From (*iv*) and (*iii*), we have

$$f^{-1}(y) \;(= n) = \begin{cases} y - 1 & \text{if } \; y \text{ is odd} \quad ...(v) \\ y + 1 & \text{if } \; y \text{ is even} \quad ...(vi) \end{cases}$$

From (*i*), (*ii*), (*v*) and (*vi*); $f = f^{-1}$

Note. $0 \in$ W is even.

3. Let $f : R \to R$ be defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Sol. $f : R \to R$ is defined by $f(x) = x^2 - 3x + 2$ ...(*i*)

$\therefore \qquad f(f(x)) = f(x^2 - 3x + 2)$ ...[By (*i*)]

Changing x to $x^2 - 3x + 2$ in (i)

$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$

($\because$ $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$)

$= x^4 - 6x^3 + 10x^2 - 3x.$

4. Show that the function $f : \mathrm{R} \to \{x \in \mathrm{R} : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in \mathrm{R}$ is one-one and onto function.

Sol. Given: $f : \mathrm{R} \to \{x \in \mathrm{R} : -1 < x < 1\}$

given by $f(x) = \dfrac{x}{1+|x|}$ ...(i)

To prove: f is one-one

Case I. $x \geq 0$

$\therefore$ $|x| = x$ and therefore from (i), $f(x) = \dfrac{x}{1+x}$ ...(ii)

Let $x_1 \in \mathrm{R}$, $x_2 \in \mathrm{R}$ (such that $x_1 \geq 0$, $x_2 \geq 0$) and $f(x_1) = f(x_2)$

$\therefore$ From (ii), $\dfrac{x_1}{1+x_1} = \dfrac{x_2}{1+x_2}$

Cross-multiplying, $x_1(1 + x_2) = x_2(1 + x_1)$

$\Rightarrow$ $x_1 + x_1x_2 = x_2 + x_1x_2 \Rightarrow x_1 = x_2.$

Case II. $x < 0$

$\therefore$ $|x| = -x$ and therefore from (i), $f(x) = \dfrac{x}{1-x}$ ...(iii)

Let $x_1 \in \mathrm{R}, x_2 \in \mathrm{R}$ (such that $x_1 < 0, x_2 < 0$) and $f(x_1) = f(x_2)$

$\therefore$ From (iii), $\dfrac{x_1}{1-x_1} = \dfrac{x_2}{1-x_2}$

Cross-multiplying, $x_1(1 - x_2) = x_2(1 - x_1)$

or $x_1 - x_1x_2 = x_2 - x_1x_2$ or $x_1 = x_2$

$\therefore$ In both the cases I and II,

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$ $\therefore$ f is one-one.

To prove: f is onto.

According to given,

Co-domain $= \{x : -1 < x < 1\}$

i.e., $= \{y : -1 < y < 1\}$ = open interval $(-1, 1)$

($\because$ Elements of co-domain are generally denoted by y)

Let us find range $f(x)$

From eqn. (ii), for $x \geq 0$, $f(x) = \dfrac{x}{1+x}$

We know that for $x \geq 0$, $x < 1 + x$

Dividing by $1 + x$, $\dfrac{x}{1+x} < 1$ *i.e.*, $f(x) < 1$ ...(iv)

From eqn. (*iii*) for $x < 0$, $f(x) = \frac{x}{1-x}$

We know that for $x < 0$, $x > x - 1$

($\because$ $x - 1$ is more negative than x)

$\Rightarrow$ $x > -1\ (1 - x)$

Dividing both sides by $(1 - x)$ (> 0), $\frac{x}{1-x} > -1$

i.e., $f(x) > -1$ *i.e.*, $-1 < f(x)$ ...(*v*)

From (*v*) and (*iv*) $-1 < f(x) < 1$

i.e., Range set = open interval $(-1, 1)$ = co domain (given)

$\therefore$ f is onto. $\therefore$ f is one-one and onto.

5. Show that the function $f : R \to R$ given by $f(x) = x^3$ is injective.

Sol. Function $f : R \to R$ given by $f(x) = x^3$ ...(*i*)

Let $x_1, x_2 \in$ domain R such that $f(x_1) = f(x_2)$

$\therefore$ By (*i*), $x_1^3 = x_2^3$ $\Rightarrow$ $x_1 = x_2$

$\therefore$ f is injective (function) *i.e.*, one-one function.

6. Give examples of two functions $f : N \to Z$ and $g : Z \to Z$ such that *gof* is injective but g is not injective.

Sol. Let us define $f : N \to Z$ as $f(x) = x$ ...(*i*)

and $g : Z \to Z$ as $g(x) = |x|$ ...(*ii*)

To prove: g is not injective *i.e.*, g is not one-one.

Now $x_1 = -1 \in Z$, $x_2 = 1 \in Z$

From (*ii*), $g(x_1) = g(-1) = |-1| = 1$

and $g(x_2) = g(1) = |1| = 1$

Now $g(x_1) = g(x_2)$ (= 1) but $x_1(= -1) \neq x_2(= 1)$

$\therefore$ g is not injective.

Let us find the function *gof*

$\because$ $f : N \to Z$ and $g : Z \to Z$, therefore $gof : N \to Z$

($\because$ Domain of *gof* by def. is always same as domain of *f*)

and $(gof)(x) = g(f(x)) = g(x)$ [By (*i*)]

$= |x|$ [By (*ii*)]

i.e., $(gof)x = |x|$ for all $x \in$ domain N

$= x$ ...(*iii*)

($\because$ $x \in N \Rightarrow x \geq 1 \Rightarrow x > 0$ and hence $|x| = x$)

To prove: *gof* is one-one.

Let $x_1, x_2 \in N$ (domain of *gof*) such that

$(gof)x_1 = (gof)\ x_2$ $\Rightarrow$ By (*iii*), $x_1 = x_2$

$\therefore$ *gof* is one-one (*i.e.*, injective)

$\therefore$ For the above example, *gof* is injective but g is not injective.

Remark. A second example for the above question is:

take $f(x) = 2x$ and $g(x) = x^2$.

7. Give examples of two functions $f : N \to N$ and $g : N \to N$ such that *gof* is onto but f is not onto.

Sol. Let $f : N \to N$ be defined by $f(x) = 2x$ ...(*i*)

$\therefore$ Range set $= \{f(x) = 2x : x \in N\}$

$= \{2 \times 1, 2 \times 2, 2 \times 3, ...\} = \{2, 4, 6, 8, 10, ...\}$

$=$ Set of even natural numbers $\neq$ co-domain N.

$\therefore$ f is not onto.

Let $g : N \to N$ be defined by $g(x) = \frac{1}{2}x$ ...(*ii*)

$\therefore$ gof is a function from $N \to N$ and is defined as

$$(gof)x = g(f(x)) = g(2x) \quad = \frac{1}{2}(2x) \qquad [\text{By } (i)]$$

$$= x$$

$\therefore$ $(gof) : N \to N$ and $(gof)x = x$

$\therefore$ $(gof)(1) = 1$, $(gof)(2) = 2$ etc.

i.e., Range of gof is N and equals to co-domain N.

$\therefore$ gof is onto but f is not onto (proved above).

8. Given a non-empty set X; consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if A $\subset$ B. Is R an equivalence relation on P(X)? Justify your answer.

Sol. Given: The set P(X) which is the set of all subsets of X.

Given: For subsets A, B in P(X); ARB if and only if

$A \subset B$ ...(*i*)

Is R reflexive? Let A $\in$ P(X).

Putting B = A in (*i*), we have $A \subset A$

which is true. ($\because$ Every set is a subset of itself)

$\therefore$ By (*i*), ARA and hence R is reflexive.

Is R symmetric? Let A $\in$ P(X) and B $\in$ P(X) and ARB

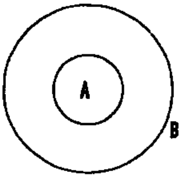
$\therefore$ By (*i*), $A \subset B$

$\therefore$ B is not a subset of A

(See the adjoining figure)

$\therefore$ B is not related to A.

Hence R is not symmetric.

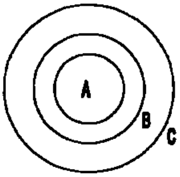


Is R transitive? Let A $\in$ P(X), B $\in$ P(X) and C $\in$ P(X) such that ARB and BRC.

$\therefore$ By (*i*), $A \subset B$ and $B \subset C$

$\therefore$ $A \subset C$ $\therefore$ By (*i*), ARC. $\therefore$ R is transitive.

$\therefore$ R is reflexive and transitive but not symmetric. Hence R is not an equivalence relation on the set P(X).



9. Given a non-empty set X, consider the binary operation $*$: P(X) $\times$ P(X) $\to$ P(X) given by A $*$ B = A $\cap$ B $\forall$ A, B in P(X), where P(X) is the power set of X. Show that X is the

identity element for this operation and X is the only invertible element in P(X) with respect to the operation *.

Sol. We know that P(X), the power set of X is the set of all subsets of the set X.

Given: The binary operation *:

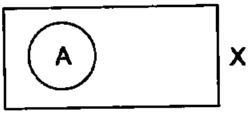
P(X) × P(X) → P(X) defined as $A * B = A \cap B$ ...(*i*)

for all A, B in P(X).

To show that the set X ∈ P(X) is the identity element for the binary operation * defined in (*i*)

Let $A \in P(X)$ ($\Rightarrow A \subset X$)

By (*i*) $A * X = A \cap X = A$ (can be seen from the adjoining figure)



similarly $X * A = X \cap A = A$

$\therefore$ $A * X = A = X * A$ $\forall A \in P(X)$

$\therefore$ set X is the identity element for the binary operation *. ...(*ii*)

$(a * e = a = e * a)$

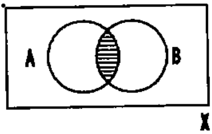
To find invertible elements in P(X) w.r.t. binary operation *.

Let $A \in P(X)$ be an invertible element of P(X).

Therefore there exists $B \in P(X)$.

(By def. of inverse)

such that $A * B = E$



$\therefore$ By (*i*), $A \cap B = X$ [by (*ii*)]

and this equation can hold true only when A = X and also B = X. ($\because$ Only $X \cap X = X$)

$\therefore$ A = X is the only invertible element of P(X) (having B = X as its inverse).

10. Find the number of all onto functions from the set {1, 2, 3, ..., *n*} to itself.

Sol. The given set is {1, 2, 3, ..., *n*} = A (say)

This set A is a finite set having only *n* elements.

We know that every **onto** function from a **finite** set A → A is **one-one** and conversely every one-one function from finite set A → A is also **onto.**

$\therefore$ Number of **onto** functions *f* from A to A here

= Number of **one-one** functions *f* from A to A ...(*i*)

Since *f* is one-one, therefore image of $1 \in A$ can be any one of the *n* elements of A.

Now image of $2 \in A$ can be any one of the remaining $(n - 1)$ elements of A ($\because$ *f* is one-one $\Rightarrow$ No two elements have the same image *i.e.*, repetition is not allowed.)

Similarly, image of $3 \in A$ can be any one of the remaining $(n - 2)$ elements of A and so on.

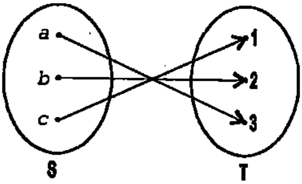
$\therefore$ Total number of **onto** functions from A → A

(*i.e.*, one-one functions from A → A). [By (*i*)]

$= n(n - 1)(n - 2) \ldots 1$

$= n\,!$. [By Multiplication Rule of Principle of counting]

11. Let S = {a, b, c} and T = {1, 2, 3}. Find F^{-1} of the following functions F from S to T, if it exists.
(*i*) F = {(a, 3), (b, 2), (c, 1)} (*ii*) F = {(a, 2), (b, 1), (c, 1)}.



Sol. (*i*) **Given:** S = {a, b, c} and T = {1, 2, 3}.

Given: Function F : S → T is

F = {(a, 3), (b, 2), (c, 1)}

⇒ F(a) = 3, F(b) = 2, F(c) = 1.

This function F is one-one because distinct element a, b, c have distinct images 3, 2, 1.

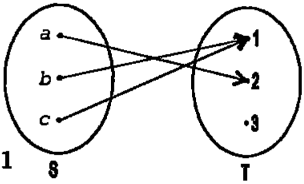
F is onto because range set

= {3, 2, 1} = {1, 2, 3} = co-domain T.

∴ F is one-one and onto function and hence F^{-1} exists and is given by

$$F^{-1}(3) = a,\ F^{-1}(2) = b, \text{ and } F^{-1}(1) = c$$

∴ Function F^{-1} = {(3, a), (2, b), (1, c)}



(*ii*) **Given:** S = {a, b, c} and

T = {1, 2, 3}

Given: Function

F = {(a, 2), (b, 1), (c, 1)}

⇒ F(a) = 2, F(b) = 1 and F(c) = 1

∴ F is not one-one because F(b) = F(c)(= 1) but $b \neq c$ *i.e.*, two elements b and c of the domain have same image 1.

∴ F is not onto because range set of F is {2, 1} ≠ co-domain

T = {1, 2, 3}. ∴ F^{-1} does not exist.

[∵ F^{-1} exists iff F is one-one and onto]

12. Consider the binary operations $*$: R × R → R and o : R × R → R defined as $a * b = |a - b|$ and $a \circ b = a$, ∀ $a, b \in$ R. Show that $*$ is commutative but not associative, o is associative but not commutative. Further, show that ∀ a, b, $c \in$ R, $a * (b \circ c) = (a * b) \circ (a * c)$. [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

Sol. Given: Binary operation $*$: R × R → R is defined as

$$a * b = |a - b| \quad ...(i)$$

for all $a, b \in$ R.

Is $*$ commutative?

Interchanging a and b in (*i*), we have

$$b * a = |b - a| = |-(a - b)| = |a - b| \quad ...(ii)$$

[∵ $|-t| = |t|$ ∀ $t \in$ R]

From (*i*) and (*ii*), we have $a * b = b * a$ ∀ $a, b \in$ R.

∴ $*$ is commutative.

Is $*$ associative? Let $a, b, c \in$ R.

By (*i*), $(a * b) * c = |a - b| * c = ||a - b| - c|$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * |b - c| = |a - |b - c||$ ...(*iv*)

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

For example, $(2 * 3) * 4 = |2 - 3| * 4 = 1 * 4 = |1 - 4| = 3$

Again $2 * (3 * 4) = 2 * |3 - 4| = 2 * 1 = |2 - 1| = 1$

$\therefore$ $(2 * 3) * 4 \neq 2 * (3 * 4)$

$\therefore$ Binary operation $*$ is not associative.

Also given: o is a binary operation from R × R → R defined as

$a \text{ o } b = a, \quad \forall\, a, b \in$ R ...(*v*)

Is o commutative? Interchanging a and b in (*v*), we have $b \text{ o } a = b$ ...(*vi*)

From (*v*) and (*vi*) $a \text{ o } b \neq b \text{ o } a$

$\therefore$ o is not commutative.

Is o associative? Let $a, b, c \in$ R.

By (*v*), $(a \text{ o } b) \text{ o } c = a \text{ o } c = a$ ...(*vii*)

Again by (*v*), $a \text{ o } (b \text{ o } c) = a \text{ o } b = a$ ...(*viii*)

From (*vii*) and (*viii*) $(a \text{ o } b) \text{ o } c = a \text{ o } (b \text{ o } c)$

$\therefore$ o is associative.

Now we are to prove that $a * (b \text{ o } c) = (a * b) \text{ o } (a * c)$

L.H.S. $= a * (b \text{ o } c) = a * b$ [By (*v*)]

$= |a - b|$ ...(*ix*) [By (*i*)]

Again R.H.S. $= (a * b) \text{ o } (a * c) = |a - b| \text{ o } |a - c|$ [By (*i*)]

$= |a - b|$ ...(*x*) [By (*v*)]

From (*ix*) and (*x*), we have L.H.S. = R.H.S.

i.e., $a * (b \text{ o } c) = (a * b) \text{ o } (a * c)$ ...(*xi*)

$\therefore$ We can say that the operation $*$ distributes itself over the operation o.

Now we are to examine if binary operation o distributes over binary operation $*$.

i.e., Is eqn. (*xi*) true on interchanging $*$ and o.

i.e., if $a \text{ o } (b * c) = (a \text{ o } b) * (a \text{ o } c)$ ...(*xii*)

L.H.S. $= a \text{ o } (b * c) = a \text{ o } |b - c|$ [By (*i*)]

$= a$ [By (*v*)]

Again, R.H.S. $= (a \text{ o } b) * (a \text{ o } c) = a * a$ [By (*v*)]

$= |a - a| = 0$ [By (*i*)]

$\therefore$ L.H.S. $\neq$ R.H.S.

$\therefore$ Eqn. (*xii*) is not true. *i.e.,* o does not distribute over $*$.

13. Given a non-empty set X, let $*$: P(X) × P(X) → P(X) be defined as A $*$ B = (A – B) ∪ (B – A), ∀ A, B ∈ P(X). Show that the empty set φ is the identity for the operation $*$ and all the elements A of P(X) are invertible with A^{-1} = A.

Sol. $A * B = (A - B) \cup (B - A) \forall A, B \in P(X)$ ...(*i*) (given)

Replacing B by ϕ in (*i*),

$$A * \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$$

$\Rightarrow$ $A * \phi = A \forall A \in P(X)$ $\therefore$ ϕ is the identity for $*$

Again replacing B by A in eqn. (*i*), we have

$$A * A = (A - A) \cup (A - A) = \phi \cup \phi = \phi$$

$\Rightarrow$ $A * A = \phi$ $\Rightarrow$ A is invertible and $A^{-1} = A$.

Remark. We know that $(A - B) \cup (B - A)$ is called **symmetric difference** of sets A and B is denoted by $A \Delta B$.

14. Define a binary operation $*$ on the set {0, 1, 2, 3, 4, 5} as

$$a * b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6 & \text{if } a+b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a.

Sol. Given: set {0, 1, 2, 3, 4, 5} = A (say)

Given: binary operation $*$ defined on the set A is defined as

$$a * b = \begin{cases} a+b, & \textit{if } a+b<6 & \text{...}(i) \\ a+b-6 & \textit{if } a+b \geq 6 & \text{...}(ii) \end{cases}$$

The composition (operation) Table for this binary operation $*$ defined by (*i*) and (*ii*) is being given below:

$\overset{*}{\rightarrow}$	0	1	2	3	4	5
0 $\rightarrow$	$0 + 0 = 0$ by (*i*)	$0 + 1 = 1$ by (*i*)	$0 + 2 = 2$ by (*i*)	$0 + 3 = 3$ by (*i*)	$0 + 4 = 4$ by (*i*)	$0 + 5 = 5$ by (*i*)
1	$1 + 0 = 1$ by (*i*)	$1 + 1 = 2$ by (*i*)	$1 + 2 = 3$ by (*i*)	$1 + 3 = 4$ by (*i*)	$1 + 4 = 5$ by (*i*)	$1 + 5 - 6 = 0$ by (*ii*)
2	$2 + 0 = 2$ by (*i*)	$2 + 1 = 3$ by (*i*)	$2 + 2 = 4$ by (*i*)	$2 + 3 = 5$ by (*i*)	$2 + 4 - 6 = 0$ by (*ii*)	$2 + 5 - 6 = 1$ by (*ii*)
3	$3 + 0 = 3$ by (*i*)	$3 + 1 = 4$ by (*i*)	$3 + 2 = 5$ by (*i*)	$3 + 3 - 6 = 0$ by (*ii*)	$3 + 4 - 6 = 1$ by (*ii*)	$3 + 5 - 6 = 2$ by (*ii*)
4	$4 + 0 = 4$ by (*i*)	$4 * 1 = 4 + 1 = 5$	$4 * 2 = 4 + 2-6 = 0$ by (*ii*) $(\because 4+2=6)$	$4 * 3 = 4 + 3-6 =1$ by (*ii*)	$4 * 4$ $4 + 4 - 6 = 2$ by (*ii*)	$4 * 5 = 4 + 5 - 6 = 3$ by (*ii*)
5	$5 * 0 = 5 + 0 = 5$	$5 * 1 = 5 + 1 - 6 = 0$ by (*ii*) $(\because 5+1= 6)$	$5 * 2 = 5 + 2-6 = 1$ by (*ii*) $(\because 5+2>6)$	$5 * 3 = 5 + 3 - 6 = 2$ by (*ii*)	$5 * 4 = 5 + 4 - 6 = 3$ by (*ii*) $(\because 5+4>6)$	$5 * 5 = 5 + 5 - 6 = 4$ by (*ii*)

Now the row headed by $*$ coincides with the row headed by 0.

∴ (e =) $0 \in A$ is the identity element for the binary operation $*$ here.

($\because a * e = a = e * a \quad \forall a \in A$)

Now we are to find inverse of each element $a \neq 0 \in A$.

In the row headed by $a = 1$, identity 0 occurs at last place and the entry vertically above it is $5 = 6 - 1 = 6 - a$

$\because$ Inverse of $a = 1$ is $6 - a$ (=5) $| a * b = e$

In the row headed by $a = 2$, identity 0 occurs at 5th place and the entry vertically above it is $4 = 6 - 2 = 6 - a$

∴ Inverse of $a = 2$ is $6 - a$ (=4) $| a * b = e$

In the row headed by $a = 3$, identity 0 occurs at 4th place and the entry vertically above it is $3 = 6 - 3 = 6 - a$

In the row headed by $a = 4$, identity 0 occurs at third place and the entry vertically above it is $2 = 6 - 4 = 6 - a$ ($a * b = e$)

In the row headed by $a = 5$, identity 0 occurs at second place and the entry vertically above it is $1 = 6 - 5 = 6 - a$.

∴ Inverse of each $a \neq 0 \in A$ is $(6 - a) \in A$.

Remarks: The reader is strongly suggested to do every question of binary operation on a finite set by the help of composition Table.

15. Let A = {– 1, 0, 1, 2}, B = {– 4, – 2, 0, 2} and $f, g : A \to B$ be functions defined by $f(x) = x^2 - x$, $x \in A$ and

$$g(x) = 2\left|x - \frac{1}{2}\right| - 1, \ x \in A.$$

Are f and g equal? Justify your answer.

Sol. Given: Set A = {– 1, 0, 1, 2} and set B = {– 4, – 2, 0, 2}.

Function $f : A \to B$ is defined by $f(x) = x^2 - x$, $x \in A$ ...(*i*)

and $g : A \to B$ is defined by $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in A$ ...(*ii*)

We know that two functions f and g are said to be **equal** if f and g have same domain (which is same here namely set A) and $f(x) = g(x)$ for all $x \in A$.

Here A = {– 1, 0, 1, 2}

From (*i*), $f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$

From (*ii*), $g(-1) = 2\left|-1 - \frac{1}{2}\right| - 1 = 2\left|-\frac{3}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1$

$= 3 - 1 = 2 \qquad \therefore f(-1) = g(-1) \ (= 2)$

From (*i*), $f(0) = 0^2 - 0 = 0 - 0 = 0$

From (*ii*), $g(0) = 2\left|-\frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\therefore$ $f(0) = g(0) = (= 0)$

From (*i*), $f(1) = 1^2 - 1 = 1 - 1 = 0$

From (*ii*), $g(1) = 2\left|1-\frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\therefore$ $f(1) = g(1)\ (= 0)$

From (*i*), $f(2) = 2^2 - 2 = 4 - 2 = 2$

and $g(2) = 2\left|2-\frac{1}{2}\right| - 1 = 2\left|\frac{3}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1$

$= 3 - 1 = 2$

$\therefore$ $f(2) = g(2)\ (= 2)$

$\therefore$ $f(x) = g(x)$ for all $x \in$ their common domain A.

$\therefore$ $f = g$.

16. Let A = {1, 2, 3}. Then number of relation containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 **(B) 2** **(C) 3** **(D) 4.**

Sol. Given: Set A = {1, 2, 3}

$\therefore$ A × A = {1, 2, 3} × {1, 2, 3}

A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} ...(*i*)

This set A × A has 3 × 3 = 9 elements.

We know that every relation on the set A is a subset of A × A.

We know that a reflexive relation on the set A must contain all elements of the type $(a, a)\ \forall\ a \in$ A. ...(*ii*)

$\therefore$ A relation on this set A containing (1, 2) and (1, 3) (given) which is reflexive and symmetric but not transitive is

R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)} ...(*iii*)

↓ ↓ ↓ (given)

(for reflexive) (given) For symmetric

This relation R has 7 elements out of 9 elements of A × A.

Now element (2, 3) of A × A can't be included in R above because then (3, 2) will also have to be included in R.

($\because$ R is symmetric (given))

and then R will become = A × A and hence will be an equivalence relation and hence will be transitive also.

The relation R given by (*iii*) is not transitive because (2, 1) ∈ R and (1, 3) ∈ R but (2, 3) ∉ R.

$\therefore$ Relation R given by (*ii*) is the only one required relation.

$\therefore$ Option (A) is the correct answer.

17. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4.

Sol. Given: Set A = {1, 2, 3}

$\therefore$ $A \times A = \{1, 2, 3\} \times \{1, 2, 3\}$

$$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\} \quad ...(i)$$

This set A × A has 3 × 3 = 9 elements.

We know that every relation on the set A is a subset of A × A.

We know that a reflexive relation on the set A must contain all elements of the type $(a, a)\ \forall\ a \in A$. ...(*ii*)

We know (and can observe from eqn. (*i*)) that A × A is the largest equivalence relation on the set A (of course containing (1, 2) also) ...(*iii*)

Let us write another equivalence relation R on the set A containing (1, 2).

This equivalence relation

$$R = \{\underbrace{(1, 1), (2, 2), (3, 3)}_{\text{(for reflexive)}}, \underbrace{(1, 2)}_{\text{(given)}}, \underbrace{(2, 1)}_{\text{For symmetric}}\} \quad ...(iv)$$

Now (1, 3) can't be included in R because then (3, 1) will also has to be taken in R ($\because$ R being equivalence relation is symmetric also) and then (3, 2) will also has to be taken in R.

($\because$ $(3, 1) \in R, (1, 2) \in R) \Rightarrow (3, 2) \in R$ because R being an equivalence relation is transitive also)

Now if $(3, 2) \in R$, then (2, 3) has also to be included in R because R being equivalence relation is symmetric.

Hence relation R given by (*iv*) containing 5 elements of A × A will also include the remaining 4 elements of A × A namely (1, 3), (3, 1), (3, 2), (2, 3) and hence relation R given by (*iv*) will become A × A.

Therefore A × A and relation R given by (*iv*) are the only two equivalence relations on the set A containing (1, 2).

$\therefore$ Option (B) is the correct answer.

18. Let $f : R \to R$ be the Signum Function defined as

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

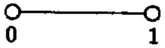
and $g : R \to R$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then, does *fog* and *gof* coincide in (0, 1]?

Sol. **Given:** $f : \mathrm{R} \rightarrow \mathrm{R}$ defined as $f(x) = \begin{cases} 1, & x > 0 & \quad ...(i) \\ 0, & x = 0 & \quad ...(ii) \\ -1, & x < 0 & \quad ...(iii) \end{cases}$

and $g : \mathrm{R} \rightarrow \mathrm{R}$ defined as $g(x) = [x]$ ...(*iv*)
where $[x]$ denotes the greatest integer less than or equal to x.
The given interval is (0, 1].
On open interval (0, 1);

$$(fog)x = f(g(x)) = f([x]) \quad \text{[By } (iv)] \qquad = f(0)$$

($\because$ We know that on open interval (0, 1), $[x] = 0$



$\therefore$ $(fog)x = 0 \ \forall \ x$ in open interval (0, 1) ...(*v*)
Again $(fog)1 = f(g(1)) = f([1])$ [By (*iv*)]
$= f(1) = 1$ ($\because$ By (*i*), $f(x) = 1$ for $x > 0$) ...(*vi*)
Now on (0, 1),
$(gof)x = g(f(x)) = g(1)$ ($\because$ By (*i*), $f(x) = 1$ for $x > 0$) [By (*i*)]
$= [1]$ [By (*iv*)] $= 1$
$\therefore$ $(gof)x = 1$ for all x in (0, 1] ...(*vii*)
The two functions *fog* and *gof* have the same domain (0, 1] but $(fog)x$ (given by (*v*) and (*vi*) $\neq$ $(gof)x$ (given by (*vii*)) $\forall \ x$ in (0, 1]
$\therefore$ $fog \neq gof$ on (0, 1] *i.e.*, the two functions don't coincide on (0, 1].

19. Number of binary operations on the set {*a*, *b*} is
(A) 10 (B) 16 (C) 20 (D) 8.

Sol. The given set {*a*, *b*} (= A (say)) has 2 (= *n*) elements.
$\therefore$ Number of binary operations on this set A

$$= n^{(n^2)} \text{ (Formula)} \qquad = 2^{(2^2)} = 2^4 = 16$$

$\therefore$ Option (B) is the correct answer.