2 Inverse Trigonometric Functions

Lesson at a Glance

1. **Inverse of a Function:** The inverse of a function f exists iff f is one-one and onto. The inverse of f is denoted by f^{-1}.
   Also if $f : X \to Y$, then $f^{-1} : Y \to X$
   and $f^{-1}of = I_X$ and $fof^{-1} = I_Y$.
2. **Properties of Inverse Trigonometric Functions**

A. (a) (i) $\sin^{-1}(\sin\theta) = \theta$ (ii) $\cos^{-1}(\cos\theta) = \theta$
(iii) $\tan^{-1}(\tan\theta) = \theta$ (iv) $\cot^{-1}(\cot\theta) = \theta$
(v) $\sec^{-1}(\sec\theta) = \theta$ (vi) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$.

(b) (i) $\text{cosec}^{-1} x = \sin^{-1}\frac{1}{x}, |x| \geq 1$

(ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}, |x| \geq 1$

(iii) $\cot^{-1} x = \tan^{-1}\frac{1}{x}, x > 0.$

B. (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}, |x| \geq 1.$

C. (i) $\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$
(ii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$
(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x, |x| \geq 1$
(iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]$
(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \geq 1$
(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R.$

3. **Formulae of Addition and Subtraction of Inverse Trigonometric Functions**

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}, x > 0, y > 0$ and $xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}, xy > -1$

(iii) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx}$

(iv) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$

(v) $\cos^{-1} x - \cos^{-1} y = \cos^{-1} (xy + \sqrt{1-x^2}\sqrt{1-y^2}), x \leq y$

(vi) $\sin^{-1} x + \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2})$

(vii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} - y\sqrt{1-x^2})$.

4. Formulae on Conversion of one Inverse Trigonometric Functions into another

(i) $2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2} = \tan^{-1} \frac{2x}{1-x^2}$

(ii) $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{x}{\sqrt{1-x^2}}, 0 < x < 1$

(iii) $\cos^{-1} x = \sin^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{\sqrt{1-x^2}}{x}, 0 < x < 1$

(iv) $\tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1+x^2}} = \sin^{-1} \frac{x}{\sqrt{1+x^2}}, x > 0$.

5. Domains and Ranges of inverse Trigonometric functions in Table form:

Inverse T-function	Domain	Range or (Principal value)
$\sin^{-1}$	[–1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	[–1, 1]	$[0, \pi]$
$\tan^{-1}$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	R	$(0, \pi)$
$\operatorname{cosec}^{-1}$	R – (–1, 1)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	R – (–1, 1)	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

TEXTBOOK QUESTIONS SOLVED

Exercise 2.1 *(Page No. 41-42)*

Find the principal values of the following:

1. $\sin^{-1} \left(-\frac{1}{2}\right)$.

Sol. Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$, then $\sin y = -\frac{1}{2}$

Since the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

therefore, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *i.e.*, y is in fourth quadrant $(-\theta)$ or in first quadrant. Also $\sin y$ is negative, therefore, y lies in fourth quadrant and y is negative (*i.e.*, $-\theta$).

Now $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\frac{1}{2}$ $\quad (\because \sin^{-1}(-x) = -\sin^{-1}x)$

$$= -\sin^{-1}\sin\frac{\pi}{6} = -\frac{\pi}{6}$$

$\therefore$ Principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $\left(-\frac{\pi}{6}\right)$.

2. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Sol. Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$, then $\cos y = \frac{\sqrt{3}}{2}$

Since the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$, therefore, $y \in [0, \pi]$ *i.e.*, y is in first or second quadrant. Also $\cos y$ is positive, therefore, y lies in first quadrant.

Now $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \cos^{-1}\cos\frac{\pi}{6} = \frac{\pi}{6}$

$\therefore$ Principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

3. cosec^{-1} **(2).**

Sol. Let $\theta = \text{cosec}^{-1} 2$ $\quad \therefore$ θ is in first quadrant because $x = 2 > 0$.
($\because$ If $x > 0$, then value of each inverse function lies in first quadrant.)

$$\therefore \quad \theta = \text{cosec}^{-1} 2 = \text{cosec}^{-1}\text{cosec}\frac{\pi}{6} = \frac{\pi}{6}.$$

4. $\tan^{-1}(-\sqrt{3})$.

Sol. Let $\tan^{-1}(-\sqrt{3}) = y$, then $\tan y = -\sqrt{3}$

Since the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, therefore, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *i.e.*, y is in fourth quadrant $(-\theta)$ or y is in first quadrant. Also $\tan y$ is negative, therefore, y lies in fourth quadrant and y is negative (*i.e.*, $-\theta$).

Now $\tan^{-1}(-\sqrt{3}) = -\tan^{-1}\sqrt{3}$ $(\because \tan^{-1}(-x) = -\tan^{-1}x)$

$$= -\tan^{-1}\tan\frac{\pi}{3} = -\frac{\pi}{3}$$

$\therefore$ Principal value of $\tan^{-1}(-\sqrt{3})$ is $\left(-\frac{\pi}{3}\right)$.

5. $\cos^{-1}\left(-\frac{1}{2}\right)$.

Sol. Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$, then $\cos y = -\frac{1}{2}$

Since the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$, therefore, $y \in [0, \pi]$ *i.e.*, y is in first or second quadrant. Also $\cos y$ is negative, therefore, y lies in second quadrant (*i.e.*, $y = \pi - \theta$).

Now $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2}$ $(\because \cos^{-1}(-x) = \pi - \cos^{-1}x)$

$$= \pi - \cos^{-1}\cos\frac{\pi}{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$\therefore$ Principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

6. $\tan^{-1}(-1)$.

Sol. Let $\theta = \tan^{-1}(-1) \therefore \theta$ lies between $-\frac{\pi}{2}$ and 0 $(\because x = -1 < 0)$

[**Note.** For $x < 0$, values of $\sin^{-1}x$, $\tan^{-1}x$ and $\operatorname{cosec}^{-1}x$ lies between $-\frac{\pi}{2}$ and 0.]

$\therefore \tan^{-1}(-1) = -\tan^{-1}1 = -\tan^{-1}\tan\frac{\pi}{4} = -\frac{\pi}{4}$

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

Sol. Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$, then $\sec y = \frac{2}{\sqrt{3}}$

Since the range of the principal value branch of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$, therefore, $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$ *i.e.*, y is in first quadrant or second quadrant. Also $\sec y$ is positive, therefore, y lies in first quadrant.

Now, $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \sec^{-1}\left(\sec\frac{\pi}{6}\right) = \frac{\pi}{6}$

$\therefore$ Principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

8. $\cot^{-1}(\sqrt{3})$.

Sol. Let $\theta = \cot^{-1}(\sqrt{3})$

$\therefore$ θ is in first quadrant because $x = \sqrt{3} > 0$.

$\therefore$ $\theta = \cot^{-1}\sqrt{3} = \cot^{-1}\cot\frac{\pi}{6} = \frac{\pi}{6}$.

9. $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$.

Sol. Let $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\therefore$ θ lies between $\frac{\pi}{2}$ and π $\left(\because\ x = -\frac{1}{2} < 0\right)$

(**Note.** For $x < 0$, value of $\cos^{-1}x$, $\cot^{-1}x$ and $\sec^{-1}x$ lies between $\frac{\pi}{2}$ and π.)

$\therefore$ $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\frac{1}{\sqrt{2}}$

$= \pi - \cos^{-1}\cos\frac{\pi}{4} = \pi - \frac{\pi}{4} = \frac{4\pi - \pi}{4} = \frac{3\pi}{4}$.

10. $\operatorname{cosec}^{-1}(-\sqrt{2})$.

Sol. Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = y$, then $\operatorname{cosec} y = -\sqrt{2}$

Since the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $- \{0\}$, therefore, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$. Also cosec y is negative, therefore, y lies in fourth quadrant $(-\theta)$ and y is negative.

Now, $\operatorname{cosec}^{-1}(-\sqrt{2}) = -\operatorname{cosec}^{-1}\sqrt{2}$ $\quad (\because \operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x)$

$= -\operatorname{cosec}^{-1}\operatorname{cosec}\frac{\pi}{4} = -\frac{\pi}{4}$

$\therefore$ Principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is $\left(-\frac{\pi}{4}\right)$.

Find the value of the following:

11. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

Sol. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$= \tan^{-1}1 + \pi - \cos^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{2}$

$= \tan^{-1}\tan\frac{\pi}{4} + \pi - \cos^{-1}\cos\frac{\pi}{3} - \sin^{-1}\sin\frac{\pi}{6}$

$= \frac{\pi}{4} + \pi - \frac{\pi}{3} - \frac{\pi}{6} = \frac{3\pi + 12\pi - 4\pi - 2\pi}{12}$

$$= \frac{9\pi}{12} = \frac{3\pi}{4}$$

12. $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

Sol. $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \cos^{-1}\cos\frac{\pi}{3} + 2\sin^{-1}\sin\frac{\pi}{6}$

$$= \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}.$$

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ **(B)** $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ **(D)** $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Sol. Option (B) is the correct answer.

(By definition of principal value for $y = \sin^{-1} x$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$)

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ **is equal to**

(A) π **(B)** $-\frac{\pi}{3}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{2\pi}{3}$.

Sol. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$$= \tan^{-1}\sqrt{3} - (\pi - \sec^{-1} 2) \quad (\because \sec^{-1}(-x) = \pi - \sec^{-1} x)$$

$$= \tan^{-1}\tan\frac{\pi}{3} - \pi + \sec^{-1}\sec\frac{\pi}{3}$$

$$= \frac{\pi}{3} - \pi + \frac{\pi}{3} = \frac{\pi - 3\pi + \pi}{3} = -\frac{\pi}{3}$$

$\therefore$ Option (B) is the correct answer.

Exercise 2.2 *(Page No. 47-48)*

Prove the following:

1. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Sol. **To prove:** $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Put $\sin\theta = x$ ($\Rightarrow \theta = \sin^{-1} x$)

$\therefore \sin 3\theta = 3x - 4x^3 \quad \Rightarrow \quad 3\theta = \sin^{-1}(3x - 4x^3)$

Putting $\theta = \sin^{-1} x$, $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$.

2. $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$.

Sol. To prove: $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x),\ x \in \left[\frac{1}{2}, 1\right]$

Let $\cos^{-1}x = \theta$, then $x = \cos\theta$

We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$

$\Rightarrow 3\theta = \cos^{-1}(4x^3 - 3x) \Rightarrow \quad 3\cos^{-1}x = \cos^{-1}(4x^3 - 3x).$

3. $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}.$

Sol. To prove: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$$\text{L.H.S.} = \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{48+77}{264-14} = \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2} = \text{R.H.S.}$$

4. $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}.$

Sol. To prove: $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

$$\text{L.H.S.} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \quad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{28+3}{21-4} = \tan^{-1}\frac{31}{17} = \text{R.H.S.}$$

Write the following functions in the simplest form:

5. $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},\ x \neq 0.$

Sol. Put $x = \tan\theta$ so that $\theta = \tan^{-1}x$

$$\therefore \quad \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1} x.$$

6. $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}}$, $|x| > 1$.

Sol. To simplify $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}}$, put $x = \sec\theta$ (See Note (*iii*) below) ($\Rightarrow \theta = \sec^{-1} x$)

$$= \tan^{-1}\frac{1}{\sqrt{\sec^2\theta - 1}} = \tan^{-1}\left(\frac{1}{\sqrt{\tan^2\theta}}\right)$$

$|\because \sec^2\theta - \tan^2\theta = 1 \Rightarrow \sec^2\theta - 1 = \tan^2\theta$

$$= \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \tan^{-1}(\cot\theta)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1} x.$$

Very useful Note: (*i*) For $\sqrt{a^2 - x^2}$, put $x = a\sin\theta$

(*ii*) For $\sqrt{a^2 + x^2}$, put $x = a\tan\theta$

and (*iii*) For $\sqrt{x^2 - a^2}$, put $x = a\sec\theta$.

7. $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$, $x < \pi$.

Sol. $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}} = \tan^{-1}\sqrt{\dfrac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}$

$[\because 1 - \cos 2\theta = 2\sin^2\theta$ and $1 + \cos 2\theta = 2\cos^2\theta]$

$$= \tan^{-1}\sqrt{\tan^2\frac{x}{2}} = \tan^{-1}\tan\frac{x}{2} = \frac{x}{2}.$$

8. $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$, $0 < x < \pi$.

Sol. The given expression $= \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Dividing the numerator and denominator by $\cos x$,

$$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}\tan x}\right) = \tan^{-1}\tan\left(\frac{\pi}{4} - x\right)$$

$$= \frac{\pi}{4} - x.$$

9. $\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}$, $|x| < a$.

Sol. To simplify $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$, put $x = a\sin\theta$;

(See note (*i*) below solution of Q. No. 7)

$$= \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2(1-\sin^2\theta)}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2\cos^2\theta}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a}$$

$$\left[\because x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}\right]$$

10. $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$, $a > 0$, $\left(-\frac{a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}\right)$.

Sol. $\tan^{-1}\left\{\frac{3a^2x - x^3}{a^3 - 3ax^2}\right\}$

(Dividing the numerator and denominator by a^3, to make the first term in denominator as 1)

$$= \tan^{-1}\left(\frac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right).$$

Put $\frac{x}{a} = \tan\theta$.

$\therefore$ The given expression $= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}\frac{x}{a}.$$

Find the values of each of the following:

11. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$.

Sol. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right] = \tan^{-1}\left[2\cos\left(2\sin^{-1}\sin\frac{\pi}{6}\right)\right]$

$$= \tan^{-1}\left[2\cos\left(2.\frac{\pi}{6}\right)\right] = \tan^{-1}\left[2\cos\frac{\pi}{3}\right]$$

$$= \tan^{-1}\left(2\times\frac{1}{2}\right) = \tan^{-1}1 = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

12. $\cot(\tan^{-1}a + \cot^{-1}a)$

Sol. $\cot(\tan^{-1}a + \cot^{-1}a)$

$$= \cot\frac{\pi}{2} = 0. \qquad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

13. $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$, $|x| < 1, y > 0$ **and** $xy < 1$.

Sol. Put $x = \tan\theta$ and $y = \tan\phi$, then the given expression

$$= \tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right)$$

$$= \tan\left(\frac{1}{2}\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} + \frac{1}{2}\cos^{-1}\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)$$

$$= \tan\left[\frac{1}{2}\sin^{-1}(\sin 2\theta) + \frac{1}{2}\cos^{-1}(\cos 2\phi)\right]$$

$$= \tan\left[\frac{1}{2}(2\theta) + \frac{1}{2}(2\phi)\right] = \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{x+y}{1-xy}.$$

14. **If** $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, **then find the value of** x.

Sol. **Given :** $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1 = \sin\frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}\frac{1}{5} = \cos^{-1}\frac{1}{5} \left(\because \sin^{-1}t + \cos^{-1}t = \frac{\pi}{2}\right)$$

$$\Rightarrow \quad x = \frac{1}{5}.$$

15. **If** $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, **then find the value of** x.

Sol. **Given:** $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1} \frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} = \frac{\pi}{4} \left(\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}\right)$$

Multiplying by L.C.M. = $(x - 2)(x + 2)$,

$$\Rightarrow \quad \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \tan \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x^2+2x-x-2+x^2-2x+x-2}{x^2-4-(x^2-1)} = 1$$

$$\Rightarrow \quad \frac{2x^2-4}{x^2-4-x^2+1} = 1 \Rightarrow \frac{2x^2-4}{-3} = 1$$

$$\Rightarrow \quad 2x^2 - 4 = -3 \Rightarrow 2x^2 = 4 - 3 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{2} \quad \therefore \quad x = \pm\sqrt{\frac{1}{2}} = \pm \frac{1}{\sqrt{2}}.$$

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$.

Sol. We know that $\sin^{-1}(\sin x) = x$. Therefore, $\sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \frac{2\pi}{3}$.

But $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which is the principal value branch of $\sin^{-1}$.

Now, $\sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \sin^{-1}\left(\sin \frac{3\pi-\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right]$

$= \sin^{-1}\left(\sin \frac{\pi}{3}\right) = \frac{\pi}{3}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\therefore \sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \frac{\pi}{3}$.

17. $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$.

Sol. We know that $\tan^{-1}(\tan x) = x$. Therefore, $\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \frac{3\pi}{4}$.

But $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ which is the principal value branch of $\tan^{-1}$.

Now, $\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}\left(\tan \frac{4\pi-\pi}{4}\right) = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right]$

$= \tan^{-1}\left[-\tan \frac{\pi}{4}\right] = -\tan^{-1}\tan \frac{\pi}{4}$

$= -\frac{\pi}{4}$ and $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\quad \therefore \tan^{-1}\left(\tan \frac{3\pi}{4}\right) = -\frac{\pi}{4}$.

18. $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$.

Sol. Let $\sin^{-1}\frac{3}{5}=x$ and $\cot^{-1}\frac{3}{5}=y$

$\Rightarrow x$ and y both lie in first quadrant because $\frac{3}{5}>0$ and also $\frac{3}{2}>0$ and hence cos x must be positive.

and $\sin x=\frac{3}{5}$ and $\cot y=\frac{3}{2}$

$\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$

$\Rightarrow \tan x=\frac{\sin x}{\cos x}=\frac{3}{4}$ and $\tan y=\frac{2}{3}$

$\therefore \tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\tan(x+y)$

$$=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}=\frac{\frac{17}{12}}{\frac{1}{2}}=\frac{17}{6}.$$

19. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ **is equal to**

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$.

Sol. We know that $(x=)\cos\frac{7\pi}{6}=\cos\left(7\times\frac{180^\circ}{6}\right)=\cos 210^\circ$ is negative. ($\because$ 210° lies in third quadrant)

$\therefore$ Value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ must lie between $\frac{\pi}{2}$ and π.

$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(2\pi-\frac{7\pi}{6}\right)\right)$ $\mid \because \cos(2\pi-\theta)=\cos\theta$

$$=2\pi-\frac{7\pi}{6}=\frac{12\pi-7\pi}{6}=\frac{5\pi}{6}$$

$\therefore$ Option (B) is the correct answer.

20. $\sin\left(\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right)$ **is equal to**

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1.

Sol. $\sin\left(\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right)=\sin\left(\frac{\pi}{3}+\sin^{-1}\frac{1}{2}\right)$ $\because \sin^{-1}(-x)=-\sin^{-1}x$

$=\sin\left(\frac{\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{6}\right)\right)$

$= \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\left(\frac{2\pi+\pi}{6}\right) = \sin\frac{3\pi}{6} \qquad = \sin\frac{\pi}{2} = 1.$

$\therefore$ Option (D) is the correct answer.

21. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ **is equal to**

(A) π **(B)** $-\frac{\pi}{2}$ **(C)** 0 **(D)** $2\sqrt{3}$.

Sol. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$

$\tan^{-1}\sqrt{3} - (\pi - \cot^{-1}\sqrt{3}) \qquad \because \cot^{-1}(-x) = \pi - \cot^{-1}x$

$\tan^{-1}\tan\frac{\pi}{3} - \left(\pi - \cot^{-1}\left(\cot\frac{\pi}{6}\right)\right)$

$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{6}\right) = \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi-5\pi}{6}$

$= -\frac{3\pi}{6} = -\frac{\pi}{2}$. $\therefore$ Option (B) is the correct answer.

MISCELLANEOUS EXERCISE *(Page No.: 51-52)*

Find the value of the following:

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Sol. Here $(x) = \cos\frac{13\pi}{6} = \cos\frac{12\pi+\pi}{6} = \cos\left(2\pi+\frac{\pi}{6}\right)$

$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} > 0.$

$\therefore$ Value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ lies in first quadrant.

$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\frac{\sqrt{3}}{2} = \cos^{-1}\cos\frac{\pi}{6} = \frac{\pi}{6}.$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$.

Sol. Here $(x) = \tan\frac{7\pi}{6} = \tan\frac{6\pi+\pi}{6} = \tan\left(\pi+\frac{\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} > 0$

$\therefore$ Value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ lies in first quadrant.

$\therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\frac{1}{\sqrt{3}} = \tan^{-1}\tan\frac{\pi}{6} = \frac{\pi}{6}.$

3. Prove that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$.

Sol. Let $\sin^{-1}\frac{3}{5} = \theta$

$\Rightarrow$ θ lies in first quadrant $\left(\because \frac{3}{5} > 0\right)$ and $\sin\theta = \frac{3}{5}$.

$\therefore$ $\cos\theta$ is positive and $= \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$$\therefore \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

We know that $\quad \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2\times\frac{3}{4}}{1-\frac{9}{16}}$

or $\tan 2\theta = \dfrac{\frac{3}{2}}{\frac{7}{16}} = \dfrac{3}{2}\times\dfrac{16}{7} = \dfrac{24}{7}$ or $2\theta = \tan^{-1}\dfrac{24}{7}$

Putting $\theta = \sin^{-1}\frac{3}{5}$, $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$.

4. Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$.

Sol. Let $\sin^{-1}\frac{8}{17} = \alpha \Rightarrow \alpha$ is in first quadrant. $\left(\because \frac{8}{17} > 0\right)$

and $\quad \sin\alpha = \frac{8}{17}$

$\therefore$ $\cos\alpha$ is positive and $= \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{64}{289}}$

$$= \sqrt{\frac{289-64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$$

$$\therefore \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}$$

Again let $\sin^{-1}\frac{3}{5} = \beta \Rightarrow \beta$ is in first quadrant. $\left(\because \frac{3}{5} > 0\right)$

and $\quad \sin\beta = \frac{3}{5}$

$\therefore$ $\cos\beta$ is also positive and $= \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$$\therefore \qquad \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

We know that $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Putting values of $\tan\alpha$ and $\tan\beta$, $= \dfrac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15}\cdot\frac{3}{4}}$

Multiplying by L.C.M. = 60, $= \dfrac{32 + 45}{60 - 24} = \dfrac{77}{36}$

i.e., $\tan(\alpha + \beta) = \dfrac{77}{36}$

$\therefore$ $\alpha + \beta = \tan^{-1}\dfrac{77}{36}$

Putting values of α and β, $\sin^{-1}\dfrac{8}{17} + \sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{77}{36}$.

5. Prove that $\cos^{-1}\dfrac{4}{5} + \cos^{-1}\dfrac{12}{13} = \cos^{-1}\dfrac{33}{65}$.

Sol. Let $\cos^{-1}\dfrac{4}{5} = \alpha$ $\Rightarrow$ α is in first quadrant. $\left(\because \dfrac{4}{5} > 0\right)$

and $\cos\alpha = \dfrac{4}{5}$

$\therefore$ $\sin\alpha$ is also positive and $= \sqrt{1 - \cos^2\alpha}$

$$= \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

Again let $\cos^{-1}\dfrac{12}{13} = \beta$

$\Rightarrow$ β is in first quadrant. $\left(\because \dfrac{12}{13} > 0\right)$

and $\cos\beta = \dfrac{12}{13}$.

$\therefore$ $\sin\beta$ is also positive and $= \sqrt{1 - \cos^2\beta}$

$$= \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{169 - 144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

We know that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Putting values, $= \dfrac{4}{5}\left(\dfrac{12}{13}\right) - \dfrac{3}{5}\left(\dfrac{5}{13}\right)$

or $\cos(\alpha + \beta) = \dfrac{48}{65} - \dfrac{15}{65} = \dfrac{33}{65}$

$\therefore$ $\alpha + \beta = \cos^{-1}\dfrac{33}{65}$

Putting values of α and β, $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$.

6. Prove that $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$.

Sol. Let $\cos^{-1}\frac{12}{13} = \alpha \Rightarrow \alpha$ is in first quadrant. $\left(\because \frac{12}{13} > 0\right)$

and $\cos\alpha = \frac{12}{13}$.

$\therefore$ $\sin\alpha$ is also positive and $= \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{144}{169}}$

$$= \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

Let $\sin^{-1}\frac{3}{5} = \beta \Rightarrow \beta$ is in first quadrant. $\left(\because \frac{3}{5} > 0\right)$

and $\sin\beta = \frac{3}{5}$.

$\therefore$ $\cos\beta$ is also positive and $= \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{9}{25}}$

$$= \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

We know that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Putting values, $\sin(\alpha+\beta) = \frac{5}{13}\left(\frac{4}{5}\right) + \frac{12}{13}\left(\frac{3}{5}\right) = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$

$\therefore$ $\alpha + \beta = \sin^{-1}\frac{56}{65}$

Putting values of α and β, $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$.

7. Prove that $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$.

Sol. Let $\sin^{-1}\frac{5}{13} = x$ and $\cos^{-1}\frac{3}{5} = y$

$\Rightarrow$ x and y both lie in first quadrant because $\frac{5}{13} > 0$ and $\frac{3}{5} > 0$

and hence $\cos x$ and $\sin y$ are both positive

and $\sin x = \frac{5}{13}$ and $\cos y = \frac{3}{5}$

$\Rightarrow$ $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

and $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$$\Rightarrow \quad \tan x = \frac{\sin x}{\cos x} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$

$$\text{and} \quad \tan y = \frac{\sin y}{\cos y} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

$$\text{Now,} \quad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}}$$

$$= \frac{\frac{21}{12}}{\frac{4}{9}} = \frac{7}{4} \times \frac{9}{4} = \frac{63}{16}$$

$$\Rightarrow \quad \tan^{-1} \frac{63}{16} = x + y$$

Putting values of x and y, $\tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$.

8. Prove that $\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$.

Sol. L.H.S. $= \left(\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7}\right) + \left(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8}\right)$

$$= \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \cdot \frac{1}{7}} \right) + \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \cdot \frac{1}{8}} \right)$$

[$\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$ if $x > 0$, $y > 0$, and $xy < 1$.

Here for first sum, $xy = \frac{1}{5} \times \frac{1}{7} = \frac{1}{35} < 1$ and for second sum $xy = \frac{1}{3} \times \frac{1}{8} = \frac{1}{24} < 1$.]

$$= \tan^{-1} \left(\frac{\frac{7+5}{35}}{\frac{35-1}{35}} \right) + \tan^{-1} \left(\frac{\frac{8+3}{24}}{\frac{24-1}{24}} \right) = \tan^{-1} \frac{12}{34} + \tan^{-1} \frac{11}{23}$$

$$= \tan^{-1} \frac{6}{17} + \tan^{-1} \frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\cdot\frac{11}{23}}\right) \qquad \left[\because\ xy = \frac{6}{17}\times\frac{11}{23} = \frac{66}{391} < 1\right]$$

Multiplying NUM and DEN by 17 × 23

$$= \tan^{-1}\left(\frac{138+187}{391-66}\right) = \tan^{-1}\left(\frac{325}{325}\right)$$

$$= \tan^{-1} 1 = \tan^{-1} \tan \frac{\pi}{4} = \frac{\pi}{4} = \text{R.H.S.}$$

9. Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in [0, 1]$.

Sol. Let $\tan^{-1}\sqrt{x} = \theta$, then $\sqrt{x} = \tan\theta$ $\therefore x = \tan^2\theta$

$$\therefore \text{R.H.S.} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x} = \frac{1}{2}\cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2}(2\theta) = \theta = \tan^{-1}\sqrt{x}.$$

L.H.S.

10. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$.

Sol. We know that

$$1 + \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} = \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

Similarly, $1 - \sin x = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2$

$$\therefore \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2}.$$

11. Prove that $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x,$

$\dfrac{-1}{\sqrt{2}} \le x \le 1.$

Sol. L.H.S. $= \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Put $x = \cos 2\theta$ $\left(\Rightarrow\ 2\theta = \cos^{-1}x \ \Rightarrow\ \theta = \dfrac{1}{2}\cos^{-1}x\right)$

$\therefore$ L.H.S. $= \tan^{-1}\left(\dfrac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$

Dividing every term in NUM and DEN by $\sqrt{2}\cos\theta$,

$= \tan^{-1}\left(\dfrac{1-\tan\theta}{1+\tan\theta}\right) = \tan^{-1}\left(\dfrac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)$

$= \tan^{-1}\tan\left(\dfrac{\pi}{4}-\theta\right) = \dfrac{\pi}{4} - \theta$

$= \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x =$ R.H.S.

12. Prove that $\dfrac{9\pi}{8} - \dfrac{9}{4}\sin^{-1}\dfrac{1}{3} = \dfrac{9}{4}\sin^{-1}\dfrac{2\sqrt{2}}{3}.$

Sol. L.H.S. $= \dfrac{9\pi}{8} - \dfrac{9}{4}\sin^{-1}\dfrac{1}{3}$

$= \dfrac{9}{4}\left(\dfrac{\pi}{2} - \sin^{-1}\dfrac{1}{3}\right)$

$= \dfrac{9}{4}\cos^{-1}\dfrac{1}{3}$ $\left(\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2} \Rightarrow \dfrac{\pi}{2} - \sin^{-1}x = \cos^{-1}x\right)$

$\Rightarrow$ L.H.S. $= \dfrac{9}{4}\theta$ ...(*i*) where $\theta = \cos^{-1}\dfrac{1}{3}$

$\therefore$ θ is in first quadrant $\left(\because \dfrac{1}{3} > 0\right)$ and $\cos\theta = \dfrac{1}{3}$

$\therefore$ $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\dfrac{1}{9}} = \sqrt{\dfrac{8}{9}} = \sqrt{\dfrac{4\times 2}{9}} = \dfrac{2}{3}\sqrt{2}$

$\therefore$ $\theta = \sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right)$

Putting this value of θ in (*i*), L.H.S. $= \frac{9}{4} \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

$= \text{R.H.S.}$

13. Solve the equation $2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$.

Sol. The given equation is

$$2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2 \cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right) \left[\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}\right]$$

$$\Rightarrow \quad \frac{2 \cos x}{\sin^2 x} = \frac{2}{\sin x}$$

Dividing both sides by $\frac{2}{\sin x}$, we have $\frac{\cos x}{\sin x} = 1$

$$\therefore \quad \cot x = 1 = \cot \frac{\pi}{4}$$

$$\therefore \quad x = \frac{\pi}{4}.$$

14. Solve the equation $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tan^{-1} x, (x > 0)$.

Sol. Put $x = \tan \theta$

∴ The given equation becomes $\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}\right] = \frac{1}{2}\theta$$

$$\Rightarrow \quad \tan^{-1} \tan\left(\frac{\pi}{4} - \theta\right) = \frac{\theta}{2}$$

$$\Rightarrow \quad \frac{\pi}{4} - \theta = \frac{\theta}{2} \quad \Rightarrow \quad \theta + \frac{\theta}{2} = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{3\theta}{2} = \frac{\pi}{4} \quad \Rightarrow \quad 12\theta = 2\pi \quad \Rightarrow \theta = \frac{2\pi}{12} = \frac{\pi}{6}$$

$$\therefore \quad x = \tan\theta = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

15. $\sin (\tan^{-1} x)$, $| x | < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ **(B)** $\frac{1}{\sqrt{1-x^2}}$ **(C)** $\frac{1}{\sqrt{1+x^2}}$ **(D)** $\frac{x}{\sqrt{1+x^2}}$.

Sol. $\sin (\tan^{-1} x) = \sin\theta$ where $\theta = \tan^{-1} x$ ($\Rightarrow$ $x = \tan\theta$)

$$= \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}}$$

$[\because \operatorname{cosec}^2\theta - \cot^2\theta = 1 \Rightarrow \operatorname{cosec}^2\theta = 1 + \cot^2\theta]$

Putting $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{x}$,

$$\sin(\tan^{-1} x) = \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{\frac{x^2+1}{x^2}}} = \frac{x}{\sqrt{x^2+1}}$$

$\therefore$ Option (D) is the correct answer.

16. $\sin^{-1}(1-x) - 2\sin^{-1} x = \dfrac{\pi}{2}$, then x is equal to

(A) $0, \dfrac{1}{2}$ **(B) $1, \dfrac{1}{2}$** **(C) 0** **(D) $\dfrac{1}{2}$.**

Sol. The given equation is $\sin^{-1}(1-x) - 2\sin^{-1} x = \dfrac{\pi}{2}$ ...(*i*)

Put $\sin^{-1} x = \theta$ $\therefore$ $x = \sin\theta$ ...(*ii*)

$\therefore$ Equation (*i*) becomes $\sin^{-1}(1-x) - 2\theta = \dfrac{\pi}{2}$

$\Rightarrow$ $\sin^{-1}(1-x) = \dfrac{\pi}{2} + 2\theta$

$\Rightarrow$ $1 - x = \sin\left(\dfrac{\pi}{2} + 2\theta\right) = \cos 2\theta = 1 - 2\sin^2\theta$

Putting $\sin\theta = x$ from (*ii*), $1 - x = 1 - 2x^2$

or $-x = -2x^2$ or $2x^2 - x = 0$ or $x(2x-1) = 0$

$\therefore$ Either $x = 0$ or $2x - 1 = 0$ *i.e.,* $2x = 1$

i.e., $x = \dfrac{1}{2}$.

Let us test these roots

Putting $x = 0$ in (*i*), $\sin^{-1} 1 - 2\sin^{-1} 0 = \dfrac{\pi}{2}$

or $\dfrac{\pi}{2} - 0 = \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} = \dfrac{\pi}{2}$ which is true.

$\therefore$ $x = 0$ is a root.

Putting $x = \dfrac{1}{2}$ in (*i*), $\sin^{-1}\dfrac{1}{2} - 2\sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{2}$

or $-\sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{2}$ [$\because$ $t - 2t = -t$]

or $-\dfrac{\pi}{6} = \dfrac{\pi}{2}$ $\left[\because \sin^{-1}\dfrac{1}{2} = \sin^{-1}\sin\dfrac{\pi}{6} = \dfrac{\pi}{6}\right]$ which is impossible.

$\therefore$ $x = \dfrac{1}{2}$ is rejected.

$\therefore$ Option (C) is the correct answer.

17. $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$ **is equal to**

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{\pi}{4}$ (D) $-\dfrac{3\pi}{4}$.

Sol. $\tan^{-1}\dfrac{x}{y} - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$

$$= \tan^{-1}\left[\frac{\dfrac{x}{y} - \left(\dfrac{x-y}{x+y}\right)}{1 + \dfrac{x}{y}\left(\dfrac{x-y}{x+y}\right)}\right] \quad \left(\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}\right)$$

Multiplying both numerator and denominator by $y(x + y)$

$$= \tan^{-1}\left[\frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}\right] = \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1} 1 = \tan^{-1}\tan\frac{\pi}{4} = \frac{\pi}{4}$$

$\therefore$ Option (C) is the correct answer.