

4

Determinants

Lesson at a Glance

1. (i) If $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is a determinant of order 2×2 , then
value of Δ is $ad - bc$
= (Product of diagonal elements
- Product of non-diagonal elements)

(ii) Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

be a determinant of order 3×3 .

We can expand Δ along any of three rows or along any of three columns.

Expanding Δ along first row,

$$\begin{aligned} \Delta &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2). \end{aligned}$$

2. Properties of Determinants:

- (i) The value of a determinant remains unchanged if its rows and columns are interchanged.
i.e. if A is a square matrix of order n , then $|A^T| = |A|$
- (ii) If two rows or columns of a determinant are interchanged, then the sign of the determinant is changed.
- (iii) If any two rows or columns of a determinant are identical (or proportional) then its value is zero.
- (iv) If each element of a row or column of a determinant Δ is multiplied by a constant k , then its value gets multiplied by k i.e. it becomes $k \Delta$.
- (v) If each element of a row (or column) of a determinant is expressed as the sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.
- (vi) If to each element of a row (or column) of a determinant be added equimultiples of the corresponding elements of one or more rows (or columns), the value of the determinant remains unchanged.
3. If A is a square matrix of order n and k is any real number, then
 $|kA| = k^n |A|$
Also $|I_n| = 1$

If $n = 3$, then $|kA| = |k^3 A|$ and $|I_3| = 1$

4. If A and B are two square matrices of same order, then $|AB| = |A| |B|$
5. **Area of a triangle** whose vertices are (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , is

given by modulus of $\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

6. Points $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$, are **collinear** iff the area

of ΔABC is zero i.e. iff $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

7. **Singular Matrix.** A square matrix A is said to be singular if $|A| = 0$.
8. **Non-Singular Matrix.** A square matrix A is said to be non-singular if $|A| \neq 0$.
9. **Adjoint of a Square Matrix.** The adjoint of a square matrix $A = [a_{ij}]$ is the transpose of the matrix $[A_{ij}]$ obtained by replacing each element of A by its Cofactor in $|A|$.

(For definition of Co-factor of an element a_{ij} of $|A|$, see Def. 1 and 2 in the note given at the beginning of solutions exercise 4.4, page 165)

$$\text{e.g., If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{then adj. } A = \text{transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

10. **Properties of adjoint of a square matrix.**

(i) If A is a square matrix of order $n \times n$, then $A \cdot \text{Adj. } A = \text{Adj. } A \cdot A = |A| I_n$.

(ii) If A is a square matrix of order $n \times n$, then $|\text{Adj. } A| = |A|^{n-1}$

For example, if A is of order 3×3 ; then $|\text{Adj. } A| = |A|^{3-1} = |A|^2$

11. **Inverse of a Square Matrix.** Let A be a square matrix of order n . If there exists a square matrix B of order n such that $AB = BA = I_n$, then B is called the inverse of A and we write $B = A^{-1}$.

$\therefore AB = BA = I_n$ becomes $AA^{-1} = A^{-1}A = I_n$.

12. **When does a square matrix has an inverse?** A square matrix A has an inverse i.e. is invertible iff A is non-singular i.e. iff $|A| \neq 0$.

13. Formula for finding A^{-1}

If a square matrix A is non-singular, then

$$A^{-1} = \frac{\text{Adj.}A}{|A|} \quad \text{also, } |A^{-1}| = \frac{1}{|A|}$$

14. Reversal Law. If A and B are invertible square matrices of same order then AB is also invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

15. Criterion for consistency of a system of n non-homogenous linear equations (non-homogenous \Rightarrow constant term of at least one equation is non-zero.)

Let $AX = B$ ($B \neq 0$) be the matrix form of these n linear non-homogenous equations in n unknowns.

Case-I $|A| \neq 0$, then the system is consistent (i.e. has a solution) and has a unique solution given by $X = A^{-1}B$.

Case-II $|A| = 0$

Sub-case-I $(\text{Adj.}A)B \neq 0$, then the system is **inconsistent** i.e. has no solution.

Sub-case-II $(\text{Adj.}A)B = 0$, then either the system is consistent and has infinitely many solutions or inconsistent (i.e. has no solution).

TEXTBOOK QUESTIONS SOLVED**Exercise 4.1 (Page No. 108-109)**

Evaluate the following determinants in Exercises 1 and 2:

$$1. \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$$

$$\text{Sol. Determinant } \begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1) - 4(-5) \\ = -2 + 20 = 18.$$

$$2. \quad (i) \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} \quad (ii) \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

$$\text{Sol. } (i) \text{ Determinant } \begin{vmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \cos \theta (\cos \theta) \\ - (-\sin \theta) (\sin \theta) \\ = \cos^2 \theta + \sin^2 \theta = 1.$$

$$(ii) \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) \\ - (x + 1)(x - 1) \\ = (x^3 + 1) - (x^2 - 1) = x^3 - x^2 + 2.$$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$.

Sol. Given: Matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

$$\therefore 2A = 2 \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 2 \times 2 \\ 2 \times 4 & 2 \times 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 2(4) - 4(8) = 8 - 32 = -24$$

...(i)

$$\text{R.H.S.} = 4|A| = 4 \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4(2 - 8) = 4(-6) = -24$$

...(ii)

From (i) and (ii), we have L.H.S. = R.H.S.

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$.

Sol. $3A = 3 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

$$\therefore \text{L.H.S.} = |3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$$

Expanding along first column = $3[36 - 0] = 3 \times 36 = 108$.

$$\text{Also R.H.S.} = 27|A| = 27 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$$

Expanding along first column

$$\begin{aligned} &= 27 [1(4 - 0)] = 27 \times 4 \\ &= 108 = \text{L.H.S.} \end{aligned} \quad \therefore |3A| = 27|A|$$

5. Evaluate the determinants:

$$(i) \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$(iii) \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$$

$$(iv) \begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

Sol. (i) Given determinant is $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ and is of order 3.

Expanding along first row

$$\begin{aligned} &= 3 \begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2) \begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix} \\ &= 3(0 - 5) + 1(0 - (-3)) - 2(0 - 0) \\ &= -15 + 3 - 0 = -12. \end{aligned}$$

(ii) Given determinant is $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$ and is of order 3.

Expanding along first row

$$\begin{aligned} &= 3 \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4) \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ &= 3(1 + 6) + 4(1 - (-4)) + 5(3 - 2) \\ &= 3(7) + 4(5) + 5(1) = 21 + 20 + 5 = 46. \end{aligned}$$

(iii) Given determinant is $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ and is of order 3.

Expanding along first row

$$\begin{aligned} &= 0 \begin{vmatrix} -1 & -3 \\ 3 & 0 \end{vmatrix} - 1 \begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2 \begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} \\ &= 0(0 + 9) - (0 - 6) + 2(-3 - 0) = 0 + 6 - 6 = 0. \end{aligned}$$

(iv) Given determinant is $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ and is of order 3.

Expanding along first row

$$\begin{aligned} &= 2 \begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2) \begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} \\ &= 2(0 - 5) + (0 + 3) - 2(0 - 6) = -10 + 3 + 12 = 5. \end{aligned}$$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$.

Sol. Matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$ $\therefore \text{Det } A \text{ i.e., } |A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$

Expanding along first row

$$\begin{aligned}
 &= 1 \begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1 \begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\
 &= (-9 - (-12)) - (-18 - (-15)) - 2(8 - 5) \\
 &= -9 + 12 - (-18 + 15) - 2(3) = 3 - (-3) - 6 \\
 &= 3 + 3 - 6 = 0
 \end{aligned}$$

Note. Such a matrix A for which $|A| = 0$ is called a **singular matrix**.

7. Find values of x , if

$$(i) \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix} \qquad (ii) \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$$

Sol. (i) Given: $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$$\Rightarrow 2 - 20 = 2x^2 - 24 \Rightarrow -18 = 2x^2 - 24$$

$$\Rightarrow -2x^2 = -24 + 18 = -6$$

Dividing by -2 , $x^2 = 3$

Taking square roots, $x = \pm \sqrt{3}$.

(ii) Given: $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$$\Rightarrow 10 - 12 = 5x - 6x \Rightarrow -2 = -x$$

Dividing by -1 , $2 = x$ i.e., $x = 2$.

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

- (A) 6 (B) ± 6 (C) -6 (D) 0.

Sol. Given: $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

$$\Rightarrow x^2 - 36 = 36 - 36 \Rightarrow x^2 - 36 = 0 \Rightarrow x^2 = 36$$

Taking square roots, $x = \pm 6$. \therefore Option (B) is the correct answer.

Exercise 4.2 (Page No. 119-121)

Using the properties of determinants and without expanding in Exercises 1 to 5, prove that:

$$1. \begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0.$$

Sol. On $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$, operate $C_1 \rightarrow C_1 + C_2$

$$= \begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix} = 0. \quad (\because C_1 \text{ and } C_3 \text{ are identical})$$

$$2. \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0.$$

Sol. On $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$, operate $C_1 \rightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0.$$

(\because All entries of one column here first are zero)

Note: The reader can do the above problem by operating $R_1 \rightarrow R_1 + R_2 + R_3$ also.

$$3. \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0.$$

Sol. On $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$, operate $C_3 \rightarrow C_3 - C_1 = \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix}$

Taking 9 common from third column = $9 \begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix} = 9(0) = 0.$

[Because two columns (one and three) are identical]

$$4. \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Sol. The given determinant is $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = \begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ba \\ 1 & ab & ac+bc \end{vmatrix}$

Operate $C_3 \rightarrow C_3 + C_2 = \begin{vmatrix} 1 & bc & ab+bc+ac \\ 1 & ca & ab+bc+ac \\ 1 & ab & ab+bc+ac \end{vmatrix}$

Taking $(ab + bc + ac)$ common from C_3 ,

$$= (ab + bc + ac) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$$= (ab + bc + ac) 0 = 0. \quad (\because C_1 \text{ and } C_3 \text{ are identical})$$

$$5. \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}.$$

$$\text{Sol. L.H.S.} = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Operate $R_1 \rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} b+c+c+a+a+b & q+r+r+p+p+q & y+z+z+x+x+y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Taking 2 common from R_1

$$= 2 \begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Operate $R_1 \rightarrow R_1 - R_2$ (to get single letter entries as required in the determinant on R.H.S.)

$$= 2 \begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Now operate $R_3 \rightarrow R_3 - R_1$ (to get single letter entries as required in the determinant on R.H.S.)

$$= 2 \begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Now operate $R_2 \rightarrow R_2 - R_3$ (objective being same as in the above two operations)

$$= 2 \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

$$\text{Interchanging } R_2 \text{ and } R_3, = -2 \begin{vmatrix} b & q & y \\ a & p & x \\ c & r & z \end{vmatrix}$$

Interchanging R_1 and R_2 ,

$$= -(-2) \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{R.H.S.}$$

By using properties of determinants, in Exercise 6 to 14, show that:

$$6. \quad \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0.$$

$$\text{Sol. Let } \Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \quad \dots(i)$$

Taking (-1) common from each row, we have

$$\Delta = (-1)^3 \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

Interchanging rows and columns in the determinant on R.H.S.,

$$\Delta = - \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \quad (\because (-1)^3 = -1)$$

$$\Rightarrow \Delta = -\Delta \quad (\text{By } (i))$$

Shifting $-\Delta$ from R.H.S. to L.H.S., $\Delta + \Delta = 0$ or $2\Delta = 0$

$$\therefore \Delta = \frac{0}{2} = 0.$$

Note. 1. We can also do this question by taking (-1) common from each column.

2. When you are asked to prove that a determinant is equal to zero or two determinants are equal, then it is to be proved so only without expanding.

3. It may be remarked that the determinant of Q. No. 6 above is determinant of a skew symmetric matrix of order 3.

$$7. \quad \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$

$$\text{Sol. L.H.S.} = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

Taking a, b, c common from R_1, R_2, R_3 respectively,

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Operate $R_1 \rightarrow R_1 + R_2$ (to create two zeros in a line (here first row))

$$= abc \begin{vmatrix} 0 & 0 & 2c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Expanding along first row (\because There are two zeros in it)

$$= abc \cdot 2c \begin{vmatrix} a & -b \\ a & b \end{vmatrix} = abc \cdot 2c (ab + ab)$$

$$= abc \cdot 2c \cdot 2ab = 4a^2b^2c^2 = \text{R.H.S.}$$

Note. Whenever we are asked to find the value of a determinant by using "Properties of Determinants", we must create two zeros in a line (Row or Column).

$$8. (i) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$(ii) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

$$\text{Sol. (i) L.H.S.} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Operating $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Expanding along first column

$$= 1 \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} (b-a) & (b-a)(b+a) \\ (c-a) & (c-a)(c+a) \end{vmatrix}$$

Taking out $(b-a)$ common from first row and $(c-a)$ common from second row

$$\begin{aligned}
 &= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \\
 &= (b-a)(c-a)(c+a-b-a) = (b-a)(c-a)(c-b) \\
 &= -(a-b)(c-a)(-(b-c)) = (a-b)(b-c)(c-a).
 \end{aligned}$$

Remark. For expanding a determinant of order 3 we should make all entries except one entry of a row or column as zeros (i.e., we should make two entries as zeros) and then expand the determinant along this row or column. For doing so, the ideal situation is that all entries of a row or column are 1 each.

If each entry of a **column** is 1, then, to create two zeros, subtract **first row** from each of the remaining two **rows**.

If each entry of a **row** is 1, then to create two zeros, subtract **first column** from each of the remaining two **columns**.

$$(ii) \text{ L.H.S.} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Here all entries of a row are 1 each.

So operate $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$ (to create two zeros in a line (here first row))

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$

$$\text{Expanding along first row,} = 1 \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$

(Forming factors)

$$= \begin{vmatrix} (b-a) & (c-a) \\ (b-a)(b^2+a^2+ab) & (c-a)(c^2+a^2+ac) \end{vmatrix}$$

Taking $(b-a)$ common from C_1 and $(c-a)$ common from C_2 ,

$$\begin{aligned}
 &= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b^2+a^2+ab & c^2+a^2+ac \end{vmatrix} \\
 &= (b-a)(c-a) (c^2+a^2+ac - b^2 - a^2 - ab) \\
 &= (b-a)(c-a)(c^2 - b^2 + ac - ab) \\
 &= (b-a)(c-a) [(c-b)(c+b) + a(c-b)] \\
 &= (b-a)(c-a)(c-b)(c+b+a) \\
 &= -(a-b)(c-a) [-(b-c)] (a+b+c) \\
 &= (a-b)(b-c)(c-a)(a+b+c) = \text{R.H.S.}
 \end{aligned}$$

$$9. \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx).$$

$$\text{Sol. L.H.S.} = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

Multiplying R_1, R_2, R_3 by x, y, z respectively (to make each entry of third column same here (xyz))

$$= \frac{1}{xyz} \begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$$

$$\text{Taking } xyz \text{ common from } C_3, = \frac{xyz}{xyz} \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$$

Now all entries of a column are same. So operate $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$ to create two zeros in a column.

$$= \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 - x^2 & y^3 - x^3 & 0 \\ z^2 - x^2 & z^3 - x^3 & 0 \end{vmatrix}$$

$$\text{Expanding along third column} = 1 \begin{vmatrix} y^2 - x^2 & y^3 - x^3 \\ z^2 - x^2 & z^3 - x^3 \end{vmatrix}$$

$$(\text{Forming factors}) = \begin{vmatrix} (y-x)(y+x) & (y-x)(y^2+x^2+xy) \\ (z-x)(z+x) & (z-x)(z^2+x^2+zx) \end{vmatrix}$$

Taking $(y-x)$ common from R_1 and $(z-x)$ common from R_2

$$\begin{aligned} &= (y-x)(z-x) \begin{vmatrix} y+x & y^2+x^2+xy \\ z+x & z^2+x^2+zx \end{vmatrix} \\ &= (y-x)(z-x) [(y+x)(z^2+x^2+zx) - (z+x)(y^2+x^2+xy)] \\ &= (y-x)(z-x) [yz^2+yx^2+xyz+xz^2+x^3+x^2z \\ &\quad - zy^2-zx^2-xyz-xy^2-x^3-x^2y] \\ &= (y-x)(z-x) [yz^2-zy^2+xz^2-xy^2] \\ &= (y-x)(z-x) [yz(z-y) + x(z^2-y^2)] \\ &= (y-x)(z-x) [yz(z-y) + x(z-y)(z+y)] \\ &= (y-x)(z-x)(z-y) [yz + x(z+y)] \\ &= -(x-y)(z-x) [-(y-z)] (yz + xz + xy) \\ &= (x-y)(y-z)(z-x)(xy + yz + zx) = \text{R.H.S.} \end{aligned}$$

$$10. (i) \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2.$$

$$(ii) \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k).$$

Sol. (i) L.H.S. = $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

Here sum of entries of each **column** is same ($= 5x + 4$), so let us operate $R_1 \rightarrow R_1 + R_2 + R_3$ to make all entries of first row equal ($= 5x + 4$).

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Taking $(5x + 4)$ common from R_1 ,

$$= (5x+4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us operate $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$ to create two zeros in a zero.

$$= (5x+4) \begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$$

Expanding along first row

$$= (5x+4) \cdot 1 \begin{vmatrix} 4-x & 0 \\ 0 & 4-x \end{vmatrix} \\ = (5x+4)(4-x)^2 = \text{R.H.S.}$$

Remark. We could also start here by operating $C_1 \rightarrow C_1 + C_2 + C_3$.

$$(ii) \text{L.H.S.} = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Here sum of entries of each **row** is same ($= 3y + k$), so let us operate $C_1 \rightarrow C_1 + C_2 + C_3$ to make all entries of first column equal ($= 3y + k$)

$$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$$

Taking $(3y+k)$ common from C_1 ,

$$= (3y+k) \begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$$

Now each entry of one (here first) column is 1, so let us operate $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ to create two zeros in a column,

$$= (3y+k) \begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix}$$

$$\text{Expanding along first column,} = (3y+k) \cdot 1 \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix}$$

$$= (3y+k)k^2 = k^2(3y+k) = \text{R.H.S.}$$

Remark. We could also start here by operating

$$R_1 \rightarrow R_1 + R_2 + R_3.$$

$$11. \quad (i) \quad \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$

$$(ii) \quad \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3.$$

$$\text{Sol.} \quad (i) \quad \text{L.H.S.} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Here sum of entries of each **column** is same ($= a+b+c$), so let us operate $R_1 \rightarrow R_1 + R_2 + R_3$ to make all entries of first row equal ($= a+b+c$)

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1 ,

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us

operate $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$ to create two zeros in a row,

$$= (a + b + c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b - c - a & 0 \\ 2c & 0 & -c - a - b \end{vmatrix}$$

Expanding along first row

$$\begin{aligned} &= (a + b + c) \cdot 1 \cdot \begin{vmatrix} -b - c - a & 0 \\ 0 & -c - a - b \end{vmatrix} \\ &= (a + b + c) [(-b - c - a)(-c - a - b)] \\ &= (a + b + c)(-)(b + c + a)(-)(c + a + b) \\ &= (a + b + c)^3 = \text{R.H.S.} \end{aligned}$$

Remark. Here we can't operate $C_1 \rightarrow C_1 + C_2 + C_3$ because sum of entries of each row is not same.

$$(ii) \text{ L.H.S.} = \begin{vmatrix} x + y + 2z & x & y \\ z & y + z + 2x & y \\ z & x & z + x + 2y \end{vmatrix}$$

Here sum of entries of each row is same ($= 2x + 2y + 2z = 2(x + y + z)$), so let us operate $C_1 \rightarrow C_1 + C_2 + C_3$ to make all entries of first column equal ($= 2(x + y + z)$)

$$= \begin{vmatrix} 2(x + y + z) & x & y \\ 2(x + y + z) & y + z + 2x & y \\ 2(x + y + z) & x & z + x + 2y \end{vmatrix}$$

Taking $2(x + y + z)$ common from C_1 ,

$$= 2(x + y + z) \begin{vmatrix} 1 & x & y \\ 1 & y + z + 2x & y \\ 1 & x & z + x + 2y \end{vmatrix}$$

Now each entry of one (here first) column is 1, so let us operate $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ to create two zeros in a column

$$= 2(x + y + z) \begin{vmatrix} 1 & x & y \\ 0 & x + y + z & 0 \\ 0 & 0 & x + y + z \end{vmatrix}$$

Expanding along first column

$$\begin{aligned} &= 2(x + y + z) \cdot 1 \cdot \begin{vmatrix} x + y + z & 0 \\ 0 & x + y + z \end{vmatrix} \\ &= 2(x + y + z) [(x + y + z)^2 - 0] \\ &= 2(x + y + z)^3 = \text{R.H.S.} \end{aligned}$$

$$12. \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 - x^3)^2.$$

$$\text{Sol. L.H.S.} = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Here sum of entries of each **column** is same ($= 1 + x + x^2$), so let us operate $R_1 \rightarrow R_1 + R_2 + R_3$ to make all entries of first row equal ($= 1 + x + x^2$)

$$= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Taking $(1 + x + x^2)$ common from R_1 ,

$$= (1 + x + x^2) \begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Now each entry of one (here first) row is 1, so let us operate $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$ to create two zeros in a row.

$$= (1 + x + x^2) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

Expanding along first row

$$= (1 + x + x^2) \cdot 1 \begin{vmatrix} 1-x^2 & x-x^2 \\ x^2-x & 1-x \end{vmatrix}$$

$$= (1 + x + x^2) \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ -x(1-x) & (1-x) \end{vmatrix}$$

$$= (1 + x + x^2) [(1-x)^2(1+x) + x^2(1-x)^2]$$

$$= (1 + x + x^2) (1-x)^2 (1+x+x^2) = (1+x+x^2)^2 (1-x)^2$$

$$= [(1+x+x^2)(1-x)]^2 \quad (\because A^2B^2 = (AB)^2)$$

$$= (1-x+x-x^2+x^2-x^3)^2 = (1-x^3)^2 = \text{R.H.S.}$$

Remark. For the above question, we could also operate $C_1 \rightarrow C_1 + C_2 + C_3$ because sum of entries of each row is also same and ($= 1 + x + x^2$).

$$13. \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3.$$

Sol. Operating $C_1 \rightarrow C_1 - b C_3, C_2 \rightarrow C_2 + a C_3$ in L.H.S. as suggested by the factor $(1 + a^2 + b^2)$ in R.H.S.

$$\text{L.H.S.} = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$[\because 2b - b(1 - a^2 - b^2) = 2b - b + a^2b + b^3$$

$$= b + a^2b + b^3 = b(1 + a^2 + b^2)]$$

Taking out $(1 + a^2 + b^2)$ common from C_1 and C_2

$$= (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1 - a^2 - b^2 \end{vmatrix}$$

Operating $R_3 \rightarrow R_3 - bR_1$ (to create another zero in first column)

$$= (1 + a^2 + b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1 - a^2 + b^2 \end{vmatrix}$$

Expanding along C_1

$$= (1 + a^2 + b^2)^2 \cdot 1 \begin{vmatrix} 1 & 2a \\ -a & 1 - a^2 + b^2 \end{vmatrix}$$

$$= (1 + a^2 + b^2)^2 (1 - a^2 + b^2 + 2a^2) = (1 + a^2 + b^2)^3.$$

$$14. \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

Sol. Multiplying C_1, C_2, C_3 by a, b, c respectively and in return dividing the determinant by abc ,

$$\Delta = \frac{1}{abc} \begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

Taking out a, b, c common from R_1, R_2, R_3 respectively,

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} \quad \text{Operating } C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

Exercise 4.3 (Page No. 122-123)

1. Find the area of the triangle with vertices at the points given in each of the following:

- (i) (1, 0), (6, 0), (4, 3) (ii) (2, 7), (1, 1), (10, 8)
 (iii) (-2, -3), (3, 2), (-1, -8).

Sol. (i) Area of the triangle having vertices at (1, 0), (6, 0), (4, 3)

$$= \text{Modulus of } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

Expanding along first row,

$$= \frac{1}{2} [1(0 - 3) - 0(6 - 4) + 1(18 - 0)]$$

$$\text{i.e., Area of triangle} = \text{modulus of } \frac{1}{2} (-3 + 18)$$

$$= \left| \frac{15}{2} \right| = \frac{15}{2} \text{ sq. units}$$

(∵ Modulus of a positive number is number itself)

- (ii) Area of the triangle having vertices at (2, 7), (1, 1), (10, 8).

$$= \text{Modulus of } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

Expanding along first row

$$= \frac{1}{2} [2(1 - 8) - 7(1 - 10) + 1(8 - 10)]$$

$$= \frac{1}{2} [2(-7) - 7(-9) - 2] = \frac{1}{2} (-14 + 63 - 2)$$

$$= \frac{1}{2} (63 - 16)$$

$$\text{i.e., Area of triangle} = \left| \frac{47}{2} \right| = \frac{47}{2} \text{ sq. units.}$$

(∵ Modulus of a positive real number is number itself)

- (iii) Area of the triangle having vertices at

(-2, -3), (3, 2), (-1, -8) is

$$\text{Modulus of } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [-2(2 + 8) - (-3)(3 + 1) + 1(-24 + 2)]$$

$$= \frac{1}{2} [-2(10) + 3(4) - 22] = \frac{1}{2} (-20 + 12 - 22)$$

$$= \frac{1}{2}(-42 + 12) = \frac{1}{2}(-30) = -15$$

\therefore Area of triangle = Modulus of -15 i.e., $= |-15|$
 $= 15$ sq. units

(\because Modulus of a negative real number is negative of itself)

2. Show that the points $A(a, b + c)$, $B(b, c + a)$, $C(c, a + b)$ are collinear.

Sol. The given points are $A(a, b + c)$, $B(b, c + a)$, $C(c, a + b)$.

\therefore Area of triangle ABC is modulus of $\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2} \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

Expanding along first row,

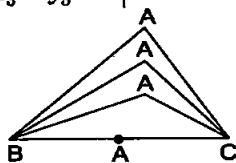
$$= \frac{1}{2} [a(c + a - a - b) - (b + c)(b - c) + 1(b(a + b) - c(c + a))]$$

$$= \frac{1}{2} [a(c - b) - (b^2 - c^2) + (ab + b^2 - c^2 - ac)]$$

$$= \frac{1}{2} (ac - ab - b^2 + c^2 + ab + b^2 - c^2 - ac) = \frac{1}{2}(0) = 0$$

i.e., Area of $\triangle ABC = 0$

\therefore Points A, B, C are collinear (See above figure).



3. Find values of k if area of triangle is 4 sq. units and vertices are:

(i) $(k, 0)$, $(4, 0)$, $(0, 2)$ (ii) $(-2, 0)$, $(0, 4)$, $(0, k)$.

Sol. (i) Given: Area of the triangle whose vertices are $(k, 0)$, $(4, 0)$, $(0, 2)$ is 4 sq. units.

$$\Rightarrow \text{Modulus of } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$$

$$\Rightarrow \text{Modulus of } \frac{1}{2} \begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 4$$

$$\text{Expanding along first row, } \left| \frac{1}{2} \{k(0 - 2) - 0 + 1(8 - 0)\} \right| = 4$$

$$\Rightarrow \left| \frac{1}{2}(-2k + 8) \right| = 4 \Rightarrow |-k + 4| = 4$$

$$\Rightarrow -k + 4 = \pm 4$$

[\because If $x \in \mathbb{R}$ and $|x| = a$ where $a \geq 0$, then $x = \pm a$]

Taking positive sign, $-k + 4 = 4$

$$\Rightarrow -k = 0 \quad \Rightarrow k = 0$$

Taking negative sign, $-k + 4 = -4$

$$\Rightarrow -k = -8 \quad \Rightarrow k = 8 \quad \text{Hence} \quad k = 0, k = 8.$$

- (ii) **Given:** Area of the triangle whose vertices are $(-2, 0)$, $(0, 4)$, $(0, k)$ is 4 sq. units.

$$\Rightarrow \text{Modulus of } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$$

$$\Rightarrow \text{Modulus of } \frac{1}{2} \begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4$$

$$\text{Expanding along first row, } \left| \frac{1}{2} \{-2(4-k) - 0 + 1(0-0)\} \right| = 4$$

$$\Rightarrow \left| \frac{1}{2} (-8 + 2k) \right| = 4 \Rightarrow |-4 + k| = 4$$

$$\Rightarrow -4 + k = \pm 4$$

(\because If $|x| = a$ where $a \geq 0$, then $x = \pm a$)

Taking positive sign, $-4 + k = 4 \Rightarrow k = 4 + 4 = 8$

Taking negative sign, $-4 + k = -4 \Rightarrow k = 0$

Hence, $k = 0, k = 8.$

4. (i) **Find the equation of the line joining (1, 2) and (3, 6) using determinants.**
 (ii) **Find the equation of the line joining (3, 1) and (9, 3) using determinants.**

Sol. (i) Let $P(x, y)$ be any point on the line joining the points (1, 2) and (3, 6).

\therefore Three points are collinear.

$$\begin{array}{c} \text{+-----+-----+-----+} \\ P(x, y) \quad (1, 2) \quad P(x, y) \quad (3, 6) \quad P(x, y) \end{array}$$

\therefore Area of triangle that could be formed by them is zero.

$$\Rightarrow \frac{1}{2} \begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0 \quad \left[\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right]$$

Multiplying both sides by 2, and expanding the determinant on left hand side along first row,

$$x(2-6) - y(1-3) + 1(6-6) = 0$$

$$\Rightarrow -4x + 2y = 0. \text{ Dividing by } -2, 2x - y = 0$$

or $-y = -2x$ i.e., $y = 2x$ which is the required equation of the line.

- (ii) Let $P(x, y)$ be any point on the line joining the points (3, 1) and (9, 3).

\therefore Three points are collinear.

$$\begin{array}{ccccccc} & | & & | & & | & & | \\ P(x, y) & (3, 1) & P(x, y) & (9, 3) & P(x, y) & & & \end{array}$$

\therefore Area of triangle that could be formed by them is zero.

$$\Rightarrow \frac{1}{2} \begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$$

Multiplying both sides by 2 and expanding the determinant on left hand side along first row,

$$x(1 - 3) - y(3 - 9) + 1(9 - 9) = 0$$

$$\Rightarrow -2x + 6y = 0$$

Dividing by -2 , $x - 3y = 0$ which is the required equation of the line.

5. If area of triangle is 35 sq. units with vertices $(2, -6)$, $(5, 4)$ and $(k, 4)$. Then k is

(A) 12 (B) -2 (C) $-12, -2$ (D) $12, -2$.

Sol. Given: Area of triangle having vertices $(2, -6)$, $(5, 4)$ and $(k, 4)$ is 35 sq. units.

$$\therefore \text{Modulus of } \left(\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} \right) = 35 \quad (\text{Given})$$

Expanding along first row,

$$\left| \frac{1}{2} \{2(4 - 4) - (-6)(5 - k) + 1(20 - 4k)\} \right| = 35$$

$$\Rightarrow \left| \frac{1}{2} \{0 + 30 - 6k + 20 - 4k\} \right| = 35$$

$$\Rightarrow \left| \frac{1}{2} (50 - 10k) \right| = 35 \Rightarrow |25 - 5k| = 35$$

$$\Rightarrow 25 - 5k = \pm 35$$

[\because If $|x| = a$ where $a \geq 0$, then $x = \pm a$]

$$\text{Taking positive sign, } 25 - 5k = 35 \Rightarrow -5k = 10$$

$$\Rightarrow k = \frac{-10}{5} = -2$$

$$\text{Taking negative sign, } 25 - 5k = -35$$

$$\Rightarrow -5k = -60 \Rightarrow k = 12$$

Thus, $k = 12, -2$ \therefore Option (D) is the correct answer.

Exercise 4.4 (Page No. 126)

Note. Minor (M_{ij}) and Cofactor (A_{ij}) of an element a_{ij} of a determinant Δ are defined **not for the value** of the element but for (i, j) th position of the element.

Def. 1. Minor M_{ij} of an element a_{ij} of a determinant Δ is the determinant obtained by omitting its i th row and j th column in which element a_{ij} lies.

Def. 2. Cofactor A_{ij} of an element a_{ij} of Δ is defined as

$A_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the minor of a_{ij} .

1. Write minors and cofactors of the elements of the following determinants:

$$(i) \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Sol. (i) Let $\Delta = \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

$$M_{11} = \text{Minor of } a_{11} = |3| = 3;$$

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^{1+1} (3) = (-1)^2 3 = 3$$

(Omit first row and first column of Δ)

$$M_{12} = \text{Minor of } a_{12} = |0| = 0$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^{1+2} (0) = (-1)^3 \cdot 0 = 0$$

$$M_{21} = \text{Minor of } a_{21} = |-4| = -4,$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^{2+1} (-4) = (-1)^3 (-4) = 4$$

$$M_{22} = \text{Minor of } a_{22} = |2| = 2,$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^{2+2} 2 = (-1)^4 2 = 2$$

(ii) Let $\Delta = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$$M_{11} = \text{Minor of } a_{11} = |d| = d,$$

$$A_{11} = (-1)^{1+1} d = (-1)^2 d = d$$

$$M_{12} = \text{Minor of } a_{12} = |b| = b,$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 b = -b$$

$$M_{21} = \text{Minor of } a_{21} = |c| = c,$$

$$A_{21} = (-1)^{2+1} c = (-1)^3 c = -c$$

$$M_{22} = \text{Minor of } a_{22} = |a| = a,$$

$$A_{22} = (-1)^{2+2} a = (-1)^4 a = a.$$

2. Write Minors and Cofactors of the elements of the following determinants:

$$(i) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

Sol. (i) Let $\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$\therefore M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 1 = 1$$

$$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$$

(Omitting first row and second column of Δ)

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 0 = 0$$

$$M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$$

$$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 0 = 0$$

$$M_{21} = \text{Minor of } a_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0,$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 0 = 0$$

$$M_{22} = \text{Minor of } a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 1 = 1$$

$$M_{23} = \text{Minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$$

$$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 0 = 0$$

$$M_{31} = \text{Minor of } a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0,$$

$$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 0 = 0$$

$$M_{32} = \text{Minor of } a_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0,$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 0 = 0,$$

$$M_{33} = \text{Minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 1 = 1.$$

$$(ii) \text{ Let } \Delta = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 - (-1) = 10 + 1 = 11,$$

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 11 = 11$$

$$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6,$$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 6 = -6$$

$$M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3,$$

$$A_{13} = (-1)^{1+3} M_{13} = (-1)^4 3 = 3$$

$$M_{21} = \text{Minor of } a_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4,$$

$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$$

$$M_{22} = \text{Minor of } a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2,$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 2 = 2$$

$$M_{23} = \text{Minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$$

$$A_{23} = (-1)^{2+3} M_{23} = (-1)^5 1 = -1$$

$$M_{31} = \text{Minor of } a_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20,$$

$$A_{31} = (-1)^{3+1} M_{31} = (-1)^4 (-20) = -20$$

$$M_{32} = \text{Minor of } a_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13,$$

$$A_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-13) = 13$$

$$M_{33} = \text{Minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5,$$

$$A_{33} = (-1)^{3+3} M_{33} = (-1)^6 5 = 5.$$

Note. Two Most Important Results

1. Sum of the products of the elements of any row or column of a determinant Δ with their corresponding factors is $= \Delta$.
i.e., $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ etc.
2. Sum of the products of the elements of any row or column of a determinant Δ with the cofactors of any other row or column of Δ is zero.

For example, $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$.

3. Using Cofactors of elements of second row, evaluate

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}.$$

Sol. $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Elements of second row of Δ are $a_{21} = 2$, $a_{22} = 0$, $a_{23} = 1$

$$A_{21} = \text{Cofactor of } a_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} \quad (\because A_{ij} = (-1)^{i+j} M_{ij})$$

↓ ↓

(determinant obtained by omitting second row and first column of Δ)
 $= (-1)^3 (9 - 16) = -(-7) = 7$

$$A_{22} = \text{Cofactor of } a_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = (-1)^4 (15 - 8) = 7$$

$$A_{23} = \text{Cofactor } a_{23} = (-1)^{2+3} \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^5 (10 - 3) = -7$$

Now by Result I of Note after the solution of Q. No. 2,

$$\begin{aligned} \Delta &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= 2(7) + 0(7) + 1(-7) = 14 - 7 = 7. \end{aligned}$$

Remark. The above method of finding the value of Δ is equivalent to expanding Δ along second row.

4. Using Cofactors of elements of third column, evaluate

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}.$$

Sol.
$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Here elements of third column of Δ are

$$a_{13} = yz, a_{23} = zx, a_{33} = xy$$

$$\begin{aligned} A_{13} &= \text{Cofactor of } a_{13} = (-1)^{1+3} \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} \\ &= (-1)^4 (z - y) = z - y \end{aligned}$$

↓

(determinant obtained by omitting first row and third column of Δ)

$$A_{23} = \text{Cofactor of } a_{23} = (-1)^{2+3} \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = (-1)^5 (z - x) = -(z - x)$$

$$A_{33} = \text{Cofactor of } a_{33} = (-1)^{3+3} \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = (-1)^6 (y - x) = y - x$$

Now by Result I of Note after the solution of Q. NO. 2,

$$\begin{aligned} \Delta &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \\ &= yz(z - y) + zx[-(z - x)] + xy(y - x) \\ &= yz^2 - y^2z - z^2x + zx^2 + xy^2 - x^2y \\ &= (yz^2 - y^2z) + (xy^2 - xz^2) + (zx^2 - x^2y) \\ &= yz(z - y) + x(y^2 - z^2) - x^2(y - z) \\ &= -yz(y - z) + x(y + z)(y - z) - x^2(y - z) \\ &= (y - z)[-yz + xy + xz - x^2] \\ &= (y - z)[-y(z - x) + x(z - x)] \\ &= (y - z)(z - x)(-y + x) = (x - y)(y - z)(z - x) \end{aligned}$$

Remark. The above method of finding the value of Δ is equivalent to expanding Δ along third column.

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactor of a_{ij} , then value of Δ is given by

(A) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

(B) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(C) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$

(D) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

Sol. Option (D) is correct answer as given in Result I of Note after solution of Q. No. 2 and used in the solution of Q. No. 3 and 4 above.

Remark. The values of expressions given in options (A) and (C) are each equal to zero as given in Result II of Note after solution of Q. No. 2.

Exercise 4.5 (Page No. 131-132)

Find adjoint of each of the matrices in Exercises 1 and 2.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Sol. Here $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$\therefore A_{11} = \text{Cofactor of } a_{11} = (-1)^2 4 = 4,$$

$$A_{12} = \text{Cofactor of } a_{12} = (-1)^3 3 = -3$$

$$A_{21} = \text{Cofactor of } a_{21} = (-1)^3 2 = -2,$$

$$A_{22} = \text{Cofactor of } a_{22} = (-1)^4 1 = 1$$

$$\therefore \text{adj. } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$$

Remark. For writing the Cofactors of the elements of a determinant of order 2, assign a **positive sign** to the Cofactors of **diagonal** elements and a **negative sign** to the Cofactors of **non-diagonal** elements.

2. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

Sol. Here $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{vmatrix}$

$$\begin{aligned} \therefore A_{11} &= + \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3, A_{12} = - \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} \\ &= - (2 + 10) = -12, \end{aligned} \quad \text{(See Note 2, below)}$$

$$A_{13} = + \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6, A_{21} = - \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1) = 1,$$

$$\begin{aligned} A_{22} &= + \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5, A_{23} = - \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} \\ &= -(-2) = 2, \end{aligned}$$

$$A_{31} = + \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11,$$

$$\begin{aligned} A_{32} &= - \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5 - 4) = -1, A_{33} = + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \\ &= 3 + 2 = 5 \end{aligned}$$

$$\therefore \text{adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}.$$

Note. 1. Adjoint of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **is** $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

i.e., To write adjoint of a 2×2 matrix, interchange the diagonal elements and change the signs of non-diagonal elements.

The above result can be used as a formula.

2. For writing the Cofactors of the elements of a determinant of order 3×3 , using the rule $(-1)^{i+j} M_{ij}$, the signs to be assigned to 9 cofactors are alternately + and - beginning with +.

Verify $A(\text{adj. } A) = (\text{adj. } A)A = |A| I$ **in Exercises 3 and 4:**

3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}.$

Sol. Let $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$$\therefore \text{By Note 1, above, adj. } A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$\left(\because \text{adj.} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\begin{aligned} \therefore A \cdot (\text{adj. } A) &= \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad \dots(i)$$

$$\begin{aligned} \text{Again } (\text{adj. } A) \cdot A &= \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \\ &= \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad \dots(ii)$$

$$\text{Now } |A| = \begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix} = 2(-6) - 3(-4) = -12 + 12 = 0$$

Again $|A| I = |A| I_2$ (I is I_2 because A is of order 2×2)

$$= 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \dots(iii)$$

From (i), (ii) and (iii), $A (\text{adj. } A) = (\text{adj. } A) A = |A| I$.

$$4. \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}.$$

$$\text{Sol. Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \quad \therefore |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$$

Let A_{ij} denote Cofactor of a_{ij}

(For rule of signs to be assigned see Note 2 at the end of solution of Q. No. 2).

$$\therefore A_{11} = + \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = + (0 + 0) = 0,$$

$$A_{12} = - \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = - (9 + 2) = -11,$$

$$A_{13} = + \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = + (0 - 0) = 0,$$

$$A_{21} = - \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = - (-3 - 0) = 3,$$

$$A_{22} = + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = + (3 - 2) = 1,$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = - (0 + 1) = -1,$$

$$A_{31} = + \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = + (2 - 0) = 2,$$

$$A_{32} = - \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = - (-2 - 6) = 8,$$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = + (0 + 3) = 3$$

$$\therefore \text{adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}' = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\begin{aligned} \therefore A (\text{adj. } A) &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0-0-0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \dots(i) \end{aligned}$$

$$\begin{aligned} \text{Now } (\text{adj. } A) A &= \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0-0+0 & 0+2+9 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \dots(ii) \end{aligned}$$

$$\text{Now } |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$$

Expanding along first row

$$= 1(0 - 0) - (-1)(9 + 2) + 2(0 - 0) = 0 + 11 + 0 = 11$$

Again $|A| I = |A| I_3$ ($\because A$ is 3×3 , therefore I must be I_3)

$$= 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \dots(iii)$$

From (i), (ii) and (iii)

$$A (\text{adj. } A) = (\text{adj. } A) A = |A| I.$$

Find the inverse of the matrix (if it exists) given in Exercises 5 to 11.

$$5. \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}.$$

$$\text{Sol. Let } A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 - (-8) = 6 + 8 = 14 \neq 0$$

\therefore Matrix A is non-singular and hence A^{-1} exists.

$$\text{We know that } \text{adj. } A = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix} \left(\because \text{Adj. } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\text{We know that } A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{1}{14} \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}.$$

$$6. \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}.$$

$$\text{Sol. Let } A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = -2 - (-15) = -2 + 15 = 13 \neq 0$$

$\therefore A^{-1}$ exists.

$$\text{We know that } \text{adj. } A = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix} \left(\because \text{Adj. } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{adj. } A) = \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}.$$

$$7. \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

$$\text{Sol. Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad \therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix}$$

Expanding along first row

$$= 1(10 - 0) - 2(0 - 0) + 3(0 - 0) = 10 \neq 0$$

$\therefore A^{-1}$ exists.

$$A_{11} = + \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = + (10 - 0) = 10,$$

$$A_{12} = - \begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = - (0 - 0) = 0,$$

$$A_{13} = + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = (0 - 0) = 0,$$

$$A_{21} = - \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -(10 - 0) = -10,$$

$$A_{22} = + \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = (5 - 0) = 5,$$

$$A_{23} = - \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = -(0 - 0) = 0,$$

$$A_{31} = + \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = (8 - 6) = 2,$$

$$A_{32} = - \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -(4 - 0) = -4,$$

$$A_{33} = + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = +(2 - 0) = 2$$

$$\therefore \text{adj. } A = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}' = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}.$$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}.$

Sol. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ $\therefore |A| = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{vmatrix}$

Expanding along first row $|A| = 1(-3 - 0) - 0 + 0 = -3 \neq 0$

$$A_{11} = + \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = (-3 - 0) = -3,$$

$$A_{12} = - \begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = -(-3 - 0) = 3,$$

$$A_{13} = + \begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = +(6 - 15) = -9,$$

$$A_{21} = - \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = -(0 - 0) = 0,$$

$$A_{22} = + \begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = +(-1 - 0) = -1,$$

$$A_{23} = - \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -(2 - 0) = -2,$$

$$A_{31} = + \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = (0 - 0) = 0,$$

$$A_{32} = - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = -(0 - 0) = 0,$$

$$A_{33} = + \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = +(3 - 0) = 3$$

$$\therefore \text{adj. } A = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{-1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}.$$

9. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}.$

Sol. Let $|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{vmatrix}$

Expanding by first row,

$$= 2(-1) - 1(4) + 3(8 - 7) = -2 - 4 + 3 = -3 \neq 0$$

$\Rightarrow A$ is non-singular $\therefore A^{-1}$ exists.

$$A_{11} = + \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1,$$

$$A_{12} = - \begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = -4,$$

$$A_{13} = + \begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} = 8 - 7 = 1,$$

$$A_{21} = - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5,$$

$$A_{22} = + \begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} = 23,$$

$$A_{23} = - \begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} = -11,$$

$$A_{31} = \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 3,$$

$$A_{32} = - \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 12,$$

$$A_{33} = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -6$$

$$\therefore \text{adj. } A = \begin{bmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{bmatrix}' = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj. } A = -\frac{1}{3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

10. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ $\therefore |A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix}$

Expanding along first row,

$$= 1(8 - 6) - (-1)(0 + 9) + 2(0 - 6)$$

$$= 2 + 9 - 12 = -1 \neq 0$$

$\therefore A^{-1}$ exists.

$$A_{11} = + \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = (8 - 6) = 2,$$

$$A_{12} = - \begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0 + 9) = -9$$

$$A_{13} = + \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = +(0 - 6) = -6,$$

$$A_{21} = - \begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = -(-4 + 4) = 0$$

$$A_{22} = + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (4 - 6) = -2,$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -(-2 + 3) = -1$$

$$A_{31} = + \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = 3 - 4 = -1,$$

$$A_{32} = - \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3 - 0) = 3$$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = (2 - 0) = 2$$

$$\therefore \text{adj. } A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} \therefore A^{-1} &= \frac{1}{|A|} \text{adj. } A = \frac{1}{-1} \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \quad \left(\because \frac{1}{-1} = \frac{-1}{1} = -1 \right) \end{aligned}$$

$$11. \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}.$$

$$\text{Sol. Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix} \quad \therefore |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{vmatrix}$$

Expanding along first row

$$= 1(-\cos^2 \alpha - \sin^2 \alpha) - 0 + 0 = -(\cos^2 \alpha + \sin^2 \alpha)$$

$$\text{or } |A| = -1 \neq 0$$

$\therefore A^{-1}$ exists.

$$\begin{aligned} A_{11} &= + \begin{vmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{vmatrix} = (-\cos^2 \alpha - \sin^2 \alpha) \\ &= -(\cos^2 \alpha + \sin^2 \alpha) = -1 \end{aligned}$$

$$A_{12} = - \begin{vmatrix} 0 & \sin \alpha \\ 0 & -\cos \alpha \end{vmatrix} = -(0 - 0) = 0,$$

$$A_{13} = + \begin{vmatrix} 0 & \cos \alpha \\ 0 & \sin \alpha \end{vmatrix} = 0$$

$$A_{21} = - \begin{vmatrix} 0 & 0 \\ \sin \alpha & -\cos \alpha \end{vmatrix} = -(0 - 0) = 0,$$

$$A_{22} = + \begin{vmatrix} 1 & 0 \\ 0 & -\cos \alpha \end{vmatrix} = (-\cos \alpha - 0) = -\cos \alpha$$

$$A_{23} = - \begin{vmatrix} 1 & 0 \\ 0 & \sin \alpha \end{vmatrix} = -(\sin \alpha - 0) = -\sin \alpha,$$

$$A_{31} = + \begin{vmatrix} 0 & 0 \\ \cos \alpha & \sin \alpha \end{vmatrix} = 0 - 0 = 0$$

$$A_{32} = - \begin{vmatrix} 1 & 0 \\ 0 & \sin \alpha \end{vmatrix} = -\sin \alpha,$$

$$A_{33} = + \begin{vmatrix} 1 & 0 \\ 0 & \cos \alpha \end{vmatrix} = \cos \alpha.$$

$$\therefore \text{adj. } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj. } A = - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

($\because |A| = -1$, obtained above)

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}.$$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. Given: Matrix $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$.

$$\text{Therefore } |A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 15 - 14 = 1 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj. } A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \left(\because \text{adj. } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

Given: Matrix $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

$$\therefore |B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = 54 - 56 = -2 \neq 0$$

$$\therefore B^{-1} = \frac{1}{|B|} \text{adj. } B = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$\text{Now } AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

$$\therefore |AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 67(61) - 87(47) = 4087 - 4089 \\ = -2 \neq 0$$

$$\therefore \text{L.H.S.} = (AB)^{-1} = \frac{1}{|AB|} \text{adj. } (AB)$$

$$= \frac{1}{-2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \quad \dots(i)$$

$$\begin{aligned} \text{R.H.S.} &= B^{-1} A^{-1} = \frac{-1}{2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \quad \dots(ii) \end{aligned}$$

From (i) and (ii) we have L.H.S. = R.H.S. i.e., $(AB)^{-1} = B^{-1} A^{-1}$.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1} .

Sol. Given: $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\text{L.H.S.} = A^2 - 5A + 7I = A^2 - 5A + 7I_2$$

(I is I_2 here because A is 2×2)

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 8-15 & 5-5 \\ -5+5 & 3-10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -7+7 & 0+0 \\ 0+0 & -7+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O = \text{R.H.S.}$$

$$\Rightarrow A^2 - 5A + 7I_2 = O \quad \dots(i)$$

Hence to find A^{-1} . Multiplying both sides of eqn. (i) by A^{-1} ,

$$A^2 A^{-1} - 5A A^{-1} + 7I_2 A^{-1} = O A^{-1}$$

$$\Rightarrow A - 5I_2 + 7A^{-1} = O$$

$$[\because A^2 A^{-1} = A \cdot A A^{-1} = A I_2 = A \text{ and } A A^{-1} = I_2 \text{ and } I B = B]$$

$$\Rightarrow 7A^{-1} = -A + 5I_2$$

$$= - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\Rightarrow 7A^{-1} = \begin{bmatrix} -3+5 & -1+0 \\ 1+0 & -2+5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

Caution. Because we were to find; Hence A^{-1} i.e., A^{-1} from $A^2 - 5A + 7I = O$,

so don't use

$$A^{-1} = \frac{\text{adj. } A}{|A|} \text{ to find } A^{-1} \text{ here.}$$

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find numbers a and b such that $A^2 + aA + bI = O$.

Sol. Given: Matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Putting values of A^2 and A in $A^2 + aA + bI_2 = O$,
(Here I is I_2 because A is 2×2), we have

$$\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 11+3a+b & 8+2a+0 \\ 4+a+0 & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding entries, we have

$$11 + 3a + b = 0 \quad \dots(i)$$

$$8 + 2a = 0 \quad (\Rightarrow 2a = -8 \Rightarrow a = -4)$$

$$4 + a = 0 \quad (\Rightarrow a = -4), 3 + a + b = 0 \quad \dots(ii)$$

Value of $a = -4$ is same from both equations.

Therefore, $a = -4$ is correct.

Putting $a = -4$ in (i), $11 - 12 + b = 0$ or $b - 1 = 0$ i.e., $b = 1$

Again putting $a = -4$ in (ii), $3 - 4 + b = 0$

i.e., $-1 + b = 0$ or $b = 1$

The two values of $b = 1$ are same from both equations.

$\therefore A^2 + aA + bI = 0$ holds true when $a = -4$ and $b = 1$.

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, show that

$$A^3 - 6A^2 + 5A + 11I = O. \text{ Hence find } A^{-1}.$$

Sol. $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Performing row by column multiplication,

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \quad \dots(i)$$

$$\begin{aligned} \therefore A^3 &= A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} \end{aligned}$$

Now, putting values of A^3 , A^2 , A and I_3 in $A^3 - 6A^2 + 5A + 11I_3$
(Here I is I_3 because matrix A is of order 3×3)

$$\begin{aligned} &= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \\ &= \begin{bmatrix} 8-24+5 & 7-12+5 & 1-6+5 \\ -23+18+5 & 27-48+10 & -69+84-15 \\ 32-42+10 & -13+18-5 & 58-84+15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \\ &= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3 \times 3} \\ \therefore A^3 - 6A^2 + 5A + 11I_3 &= O_{3 \times 3} \quad \text{(proved above)} \end{aligned}$$

Hence find A^{-1} .

(See caution at the end of solution of Q. No. 13)

Now multiplying both sides by A^{-1} .

$$(A^{-1}A)A^2 - 6(A^{-1}A)A + 5(A^{-1}A) + 11A^{-1}I_3 = A^{-1} \cdot O_{3 \times 3}$$

$$\Rightarrow A^2 - 6IA + 5I + 11A^{-1} = 0 \quad (\because A^{-1}A = I \text{ and } A^{-1}O = 0)$$

$$\Rightarrow A^2 - 6A + 5I + 11A^{-1} = 0$$

$$\Rightarrow 11A^{-1} = 6A - 5I - A^2$$

$$\text{or } 11A^{-1} = 6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \quad (\text{From (i)})$$

$$\text{or } 11A^{-1} = \begin{bmatrix} 6-5-4 & 6-2 & 6-1 \\ 6+3 & 12-5-8 & -18+14 \\ 12-7 & -6+3 & 18-5-14 \end{bmatrix}$$

$$\text{or } 11A^{-1} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix} \quad \text{or } A^{-1} = \frac{1}{11} \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

16. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1} .

Sol. Given: $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\therefore A^3 = A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\text{L.H.S.} = A^3 - 6A^2 + 9A - 4I = A^3 - 6A^2 + 9A - 4I_3$$

(Here I is I_3 because A is 3×3)

Putting values

$$\begin{aligned}
 &= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \\
 &\quad + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} \\
 &\quad + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\
 &= \begin{bmatrix} 22-36 & -21+30 & 21-30 \\ -21+30 & 22-36 & -21+30 \\ 21-30 & -21+30 & 22-36 \end{bmatrix} + \begin{bmatrix} 18-4 & -9-0 & 9-0 \\ -9-0 & 18-4 & -9-0 \\ 9-0 & -9-0 & 18-4 \end{bmatrix} \\
 &= \begin{bmatrix} -14 & 9 & -9 \\ 9 & -14 & 9 \\ -9 & 9 & -14 \end{bmatrix} + \begin{bmatrix} 14 & -9 & 9 \\ -9 & 14 & -9 \\ 9 & -9 & 14 \end{bmatrix} \\
 &= \begin{bmatrix} -14+14 & 9-9 & -9+9 \\ 9-9 & -14+14 & 9-9 \\ -9+9 & 9-9 & -14+14 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = \text{R.H.S.} \\
 \therefore A^3 - 6A^2 + 9A - 4I_3 = O \quad \dots(i)
 \end{aligned}$$

Hence to find A^{-1}

(See caution at the end of solution of Q. No. 13)

Multiplying both sides of (i) by A^{-1} ,

$$A^3 A^{-1} - 6A^2 A^{-1} + 9A A^{-1} - 4I_3 A^{-1} = O A^{-1}$$

or

$$A^2 - 6A + 9I_3 - 4A^{-1} = 0$$

$$[\because A^3 A^{-1} = A^2 A A^{-1} = A^2 I = A^2 \text{ etc. and also } I B = B]$$

$$\Rightarrow -4A^{-1} = -A^2 + 6A - 9I_3$$

$$\text{Dividing by } -4, A^{-1} = \frac{1}{4}A^2 - \frac{6}{4}A + \frac{9}{4}I_3 = \frac{1}{4}[A^2 - 6A + 9I_3]$$

$$\Rightarrow A^{-1} = \frac{1}{4} \left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

$$\begin{aligned} \Rightarrow A^{-1} &= \frac{1}{4} \left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \right] \\ &= \frac{1}{4} \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \end{aligned}$$

17. Let A be a non-singular matrix of order 3×3 . Then $|\text{adj. } A|$ is equal to

- (A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3|A|$.

Sol. If A is a non-singular matrix of order $n \times n$,

then $|\text{adj. } A| = |A|^{n-1}$.

Putting $n = 3$, $|\text{adj. } A| = |A|^2$

\therefore Option (B) is the correct answer.

18. If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to

- (A) $\det A$ (B) $\frac{1}{\det A}$ (C) 1 (D) 0.

Sol. We know that $AA^{-1} = I$ for every invertible matrix A .

Taking determinants on both sides, we have

$$|AA^{-1}| = |I| \Rightarrow |A| |A^{-1}| = 1$$

$$\text{Dividing by } |A|, |A^{-1}| = \frac{1}{|A|} \quad \text{i.e., } \det(A^{-1}) = \frac{1}{\det A}$$

\therefore Option (B) is the correct answer.

Exercise 4.6 (Page No. 136-137)

Examine the consistency of the system of equations in Exercises 1 to 3.

1. $x + 2y = 2$

$2x + 3y = 3$.

Sol. Given linear equations are

$$x + 2y = 2$$

$$2x + 3y = 3$$

$$\text{Their matrix form is } \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad (\Rightarrow AX = B)$$

$$\text{Comparing } A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$$

\therefore (Unique) solution and hence equations are consistent.

2. $2x - y = 5$

$x + y = 4$

Sol. Given linear equations are

$2x - y = 5$

$x + y = 4$

Their matrix form is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ ($\Rightarrow AX = B$)

Comparing $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2 - (-1) = 3 \neq 0$

 \therefore (Unique) solution and hence equations are consistent.

3. $x + 3y = 5$

$2x + 6y = 8$

Sol. Given linear equations are

$x + 3y = 5$

$2x + 6y = 8$

Their matrix form is $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ ($\Rightarrow AX = B$)

Comparing $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ $|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$

So we are to find (adj. A) B

adj. A = $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$ (\because adj. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

\therefore (adj. A) B = $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$

$\left(\because \text{The matrix } \begin{bmatrix} 6 \\ -2 \end{bmatrix} \text{ has non-zero entries} \right)$

 \therefore Given Equations are Inconsistent i.e., have no common solution.**Examine the consistency of the system of equations in Exercises 4 to 6.**

4. $x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

Sol. The given equations are

$x + y + z = 1$

...(i)

$2x + 3y + 2z = 2$

...(ii)

$ax + ay + 2az = 4$

...(iii)

Their matrix form is $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ ($\Rightarrow AX = B$)

$$\therefore \text{Matrix } A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix} \quad \therefore |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{vmatrix}$$

Expanding along first row,

$$\begin{aligned} |A| &= 1(6a - 2a) - 1(4a - 2a) + 1(2a - 3a) \\ &= 4a - 2a - a = a \end{aligned}$$

Case I. $a \neq 0$ $\therefore |A| = a \neq 0$

\therefore (Unique) solution and hence equations are consistent.

Case II. $a = 0$ $\therefore |A| = a = 0$.

Putting $a = 0$ in given equation (iii), we have $0 = 4$ which is impossible.

\therefore Given equations are inconsistent if $a = 0$.

5. $3x - y - 2z = 2$

$$2y - z = -1$$

$$3x - 5y = 3$$

Sol. The given equations are

$$3x - y - 2z = 2$$

$$2y - z = -1 \quad \text{i.e., } 0x + 2y - z = -1$$

and $3x - 5y = 3 \quad \text{i.e., } 3x - 5y + 0z = 3$

Their matrix form is $\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad (\Rightarrow AX = B)$

Comparing $A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

Expanding along first row,

$$\begin{aligned} |A| &= 3(0 - 5) - (-1)(0 + 3) + (-2)(0 - 6) \\ &= 3(-5) + 3 + 12 = -15 + 15 = 0 \end{aligned}$$

So now we are to find (adj. A) B

To find adj. A for $|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

$$A_{11} = + \begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} = (0 - 5) = -5,$$

$$A_{12} = - \begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -(0 + 3) = -3,$$

$$A_{13} = + \begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} = (0 - 6) = -6,$$

$$A_{21} = - \begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} = -(0 - 10) = 10,$$

$$A_{22} = + \begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} = (0 + 6) = 6,$$

$$A_{23} = - \begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = -(-15 + 3) = 12,$$

$$A_{31} = + \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} = (1 + 4) = 5,$$

$$A_{32} = - \begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = -(-3 - 0) = 3,$$

$$A_{33} = + \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = +(6 - 0) = 6.$$

$$\therefore \text{adj. A} = \begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\therefore (\text{adj. A}) B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 - 10 + 15 \\ -6 - 6 + 9 \\ -12 - 12 + 18 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq 0 \quad \left[\because \text{The matrix } \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \text{ has non-zero entries} \right]$$

\therefore Given equations are inconsistent.

6. $5x - y + 4z = 5$

$$2x + 3y + 5z = 2$$

$$5x - 2y + 6z = -1$$

Sol. The given equations are

$$5x - y + 4z = 5$$

$$2x + 3y + 5z = 2$$

$$5x - 2y + 6z = -1$$

Their matrix form is $\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \quad (\Rightarrow AX = B)$

$$\therefore A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \quad |A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

Expanding along first row

$$\begin{aligned} &= 5(18 + 10) - (-1)(12 - 25) + 4(-4 - 15) \\ &= 5(28) + (-13) + 4(-19) \\ &= 140 - 13 - 76 = 140 - 89 = 51 \neq 0 \end{aligned}$$

\therefore Given system of equations has a (unique) solution and hence equations are consistent.

Solve the system of linear equations, using matrix method, in Exercises 7 to 10.

7. $5x + 2y = 4$

$7x + 3y = 5.$

Sol. The given equations are

$5x + 2y = 4$

$7x + 3y = 5$

Their matrix form is $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ ($\Rightarrow AX = B$)

Comparing $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 15 - 14 = 1 \neq 0$$

\therefore Solution is unique and $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{|A|} (\text{adj. } A) \cdot B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1} \begin{bmatrix} \boxed{3} & \boxed{-2} \\ \boxed{-7} & \boxed{5} \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \left(\because \text{adj. } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Equating corresponding entries, we have $x = 2$ and $y = -3$.

8. $2x - y = -2$

$3x + 4y = 3.$

Sol. The given equations are

$2x - y = -2$

$3x + 4y = 3$

Their matrix form is $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ ($\Rightarrow AX = B$)

Comparing $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8 - (-3) = 8 + 3 = 11 \neq 0$$

\therefore Solution is unique and $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{|A|} (\text{adj. } A) \cdot B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow = \frac{1}{11} \begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Equating corresponding entries, we have $x = -\frac{5}{11}$ and $y = \frac{12}{11}$.

9. $4x - 3y = 3$

$3x - 5y = 7$.

Sol. The given equations are

$4x - 3y = 3$

$3x - 5y = 7$

Their matrix form is $\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ ($\Rightarrow AX = B$)

Comparing $A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & -3 \\ 3 & -5 \end{vmatrix} = -20 - (-9) = -20 + 9 = -11 \neq 0$$

\therefore Solution is unique and $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{|A|} (\text{adj. } A) B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -15+21 \\ -9+28 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} 6 \\ 19 \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{19}{11} \end{bmatrix}$$

Equating corresponding entries, we have $x = -\frac{6}{11}$ and $y = -\frac{19}{11}$.

10. $5x + 2y = 3$

$3x + 2y = 5$.

Sol. The given equations are

$5x + 2y = 3$

$3x + 2y = 5$

Their matrix form is $\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ ($\Rightarrow AX = B$)

Comparing $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4 \neq 0$$

∴ Solution is unique and $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{|A|} (\text{adj. } A) B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \left(\because \text{adj.} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Equating corresponding entries, we have $x = -1$ and $y = 4$.

Solve the system of linear equations, using matrix method, in Exercises 11 to 14.

11. $2x + y + z = 1$

$$x - 2y - z = \frac{3}{2}$$

$$3y - 5z = 9.$$

Sol. The given equations are

$$2x + y + z = 1$$

$$x - 2y - z = \frac{3}{2}$$

$$3y - 5z = 9 \quad \text{or} \quad 0x + 3y - 5z = 9$$

Their matrix form is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$

$$(\Rightarrow AX = B)$$

Comparing $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix},$$

Expanding along first row, $= 2(10 + 3) - 1(-5 - 0) + 1(3 - 0)$
or $|A| = 2(13) + 5 + 3 = 26 + 5 + 3 = 34 \neq 0$

∴ Solution is unique and $X = A^{-1}B = \frac{1}{|A|} (\text{adj. } A) B \dots(i)$

Let us find adj. A

$$A_{11} = + \begin{vmatrix} -2 & -1 \\ 3 & -5 \end{vmatrix} = 10 + 3 = 13,$$

$$A_{12} = - \begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} = -(-5 - 0) = 5,$$

$$A_{13} = + \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} = (3 - 0) = 3,$$

$$A_{21} = - \begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} = -(-5 - 3) = 8,$$

$$A_{22} = + \begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix} = (-10 - 0) = -10,$$

$$A_{23} = - \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = -(6 - 0) = -6,$$

$$A_{31} = + \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} = (-1 + 2) = 1,$$

$$A_{32} = - \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -(-2 - 1) = 3,$$

$$A_{33} = + \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5.$$

$$\therefore \text{Adj. A} = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}' = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$\text{Putting values in eqn. (i), } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

$$= \frac{1}{34} \begin{bmatrix} 13 + 12 + 9 \\ 5 - 15 + 27 \\ 3 - 9 - 45 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Equating corresponding entries, we have $x = 1$,

$$y = \frac{1}{2}, z = -\frac{3}{2}.$$

12. $x - y + z = 4$

$$2x + y - 3z = 0$$

$$x + y + z = 2.$$

Sol. The given equations are

$$x - y + z = 4$$

$$2x + y - 3z = 0$$

$$x + y + z = 2$$

Their matrix form is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \quad (\Rightarrow AX = B)$

Comparing $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding along first row,

$$= 1(1 + 3) - (-1)(2 + 3) + 1(2 - 1)$$

or $|A| = 4 + 5 + 1 = 10 \neq 0$

\therefore Solution is unique and $X = A^{-1}B = \frac{1}{|A|} (\text{adj. } A) B \dots(i)$

To find adj. A

$$A_{11} = + \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = (1 + 3) = 4,$$

$$A_{12} = - \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2 + 3) = -5,$$

$$A_{13} = + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (2 - 1) = 1,$$

$$A_{21} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-1 - 1) = 2,$$

$$A_{22} = + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (1 - 1) = 0,$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -(1 + 1) = -2,$$

$$A_{31} = + \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = (3 - 1) = 2,$$

$$A_{32} = - \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -(-3 - 2) = 5,$$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3.$$

$$\therefore \text{adj. } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}' = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Putting these values in eqn. (i), we have

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4-0+6 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

Equating corresponding entries, we have

$$x = 2, \quad y = -1, \quad z = 1.$$

13. $2x + 3y + 3z = 5$

$$x - 2y + z = -4$$

$$3x - y - 2z = 3.$$

Sol. The given equations are

$$2x + 3y + 3z = 5$$

$$x - 2y + z = -4$$

$$3x - y - 2z = 3$$

Their matrix form is $\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} \quad (\Rightarrow \mathbf{AX} = \mathbf{B})$

Comparing $\mathbf{A} = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$$|\mathbf{A}| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$$

Expanding along first row, $|\mathbf{A}| = 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6)$

$$= 2(5) - 3(-5) + 3(5) = 10 + 15 + 15 = 40 \neq 0$$

\therefore Solution is unique and $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{|\mathbf{A}|} (\text{adj. } \mathbf{A}) \mathbf{B} \quad \dots(i)$

Let us find adj. A

$$A_{11} = + \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} = 4 + 1 = 5,$$

$$A_{12} = - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -(-2 - 3) = 5,$$

$$A_{13} = + \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = -1 + 6 = 5,$$

$$A_{21} = - \begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} = -(-6 + 3) = 3,$$

$$A_{22} = + \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -4 - 9 = -13,$$

$$A_{23} = - \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2 - 9) = 11,$$

$$A_{31} = + \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} = 3 + 6 = 9,$$

$$A_{32} = - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2 - 3) = 1,$$

$$A_{33} = + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4 - 3 = -7.$$

$$\therefore \text{adj. } A = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}' = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Putting these values in eqn. (i),

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Equating corresponding entries, we have $x = 1$, $y = 2$, $z = -1$.

14. $x - y + 2z = 7$

$$3x + 4y - 5z = -5$$

$$2x - y + 3z = 12.$$

Sol. The given equations are

$$x - y + 2z = 7$$

$$3x + 4y - 5z = -5$$

$$2x - y + 3z = 12$$

Their matrix form is $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$ ($\Rightarrow AX = B$)

Comparing, $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$$

Expanding along first row,

$$|A| = 1(12 - 5) - (-1)(9 + 10) + 2(-3 - 8) \\ = 7 + 19 - 22 = 4 \neq 0$$

\therefore Solution is unique and $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{|A|} (\text{adj. } A) \mathbf{B}$... (i)

Let us find adj. A

$$A_{11} = + \begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} = 12 - 5 = 7,$$

$$A_{12} = - \begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} = -(9 + 10) = -19,$$

$$A_{13} = + \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -3 - 8 = -11,$$

$$A_{21} = - \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} = -(-3 + 2) = 1,$$

$$A_{22} = + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1,$$

$$A_{23} = - \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -(-1 + 2) = -1,$$

$$A_{31} = + \begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} = 5 - 8 = -3,$$

$$A_{32} = - \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -(-5 - 6) = 11,$$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 4 + 3 = 7.$$

$$\therefore \text{adj. } A = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}' = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Putting values in eqn. (i),

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} \\ = \frac{1}{4} \begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Equating corresponding entries, we have $x = 2, y = 1, z = 3$.

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} . Using A^{-1} , solve the system of equations

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3.$$

Sol. Given: Matrix $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

To find A^{-1}	$ A = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$
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Expanding along first row,

$$|A| = 2(-4 + 4) - (-3)(-6 + 4) + 5(3 - 2) \\ = 0 + 3(-2) + 5 = -6 + 5 = -1 \neq 0$$

$\therefore A^{-1}$ exists and $A^{-1} = \frac{1}{|A|} (\text{adj. } A)$...(i)

To find adj. A from $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$$A_{11} = + \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = (-4 + 4) = 0,$$

$$A_{12} = - \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -(-6 + 4) = 2,$$

$$A_{13} = + \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1,$$

$$A_{21} = - \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -(6 - 5) = -1,$$

$$A_{22} = + \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -4 - 5 = -9,$$

$$A_{23} = - \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2 + 3) = -5,$$

$$A_{31} = + \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = (12 - 10) = 2,$$

$$A_{32} = - \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -(-8 - 15) = 23,$$

$$A_{33} = + \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = (4 + 9) = 13.$$

$$\therefore \text{adj. } A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

Putting this value of adj. A in (i),

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \dots(ii) \left(\because \frac{1}{-1} = -1 \right)$$

Now using (this) A^{-1} , we are to solve the equations

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

Their matrix form is $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \quad (\Rightarrow AX = B)$

Comparing $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Solution is unique and $X = A^{-1}B$ ($\because A^{-1}$ exists by (ii))

Putting values, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Equating corresponding entries, we have $x = 1, y = 2, z = 3$.

- 16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹ 70. Find cost of each item per kg by matrix method.**

Sol. Let ₹ x , ₹ y , ₹ z per kg be the prices of onion, wheat and rice respectively.

\therefore According to the given data, we have the following three equations

$$4x + 3y + 2z = 60,$$

$$2x + 4y + 6z = 90,$$

and $6x + 2y + 3z = 70.$

We know that these equations can be expressed in the matrix form as

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

or $AX = B$,

$$\text{where } A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} \quad |A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$$

Expanding along first row,

$$\begin{aligned} |A| &= 4(12 - 12) - 3(6 - 36) + 2(4 - 24) \\ &= 0 - 3(-30) + 2(-20) = 90 - 40 = 50 \neq 0 \end{aligned}$$

Hence A is non-singular

$\therefore A^{-1}$ exists.

\therefore Unique solution is $X = A^{-1}B$...(i)

$$A_{11} = + (12 - 12) = 0, \quad A_{12} = - (6 - 36) = 30,$$

$$A_{13} = + (4 - 24) = -20$$

$$A_{21} = - (9 - 4) = -5, \quad A_{22} = + (12 - 12) = 0,$$

$$A_{23} = - (8 - 18) = 10, \quad A_{31} = + (18 - 8) = 10,$$

$$A_{32} = - (24 - 4) = -20, \quad A_{33} = + (16 - 6) = 10$$

$$\therefore \text{adj. } A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}' = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Putting values of X, A^{-1} and B in (i), we have

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} \\ &= \frac{1}{50} \begin{bmatrix} -450 + 700 \\ 1800 - 1400 \\ -1200 + 900 + 700 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} \end{aligned}$$

$$\text{or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$$\Rightarrow x = 5, y = 8, z = 8.$$

\therefore The cost of onion, wheat and rice are respectively ₹ 5, ₹ 8 and ₹ 8 per kg.

MISCELLANEOUS EXERCISE (Page No.: 141-143)

1. Prove that the determinant $\begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$ is independent of θ

Sol. Let $\Delta = \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$

Expanding along first row

$$\begin{aligned} \Delta &= x \begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin \theta \begin{vmatrix} -\sin \theta & 1 \\ \cos \theta & x \end{vmatrix} + \cos \theta \begin{vmatrix} -\sin \theta & -x \\ \cos \theta & 1 \end{vmatrix} \\ &= x(-x^2 - 1) - \sin \theta (-x \sin \theta - \cos \theta) + \cos \theta (-\sin \theta + x \cos \theta) \\ &= -x^3 - x + x \sin^2 \theta + \sin \theta \cos \theta - \sin \theta \cos \theta + x \cos^2 \theta \\ &= -x^3 - x + x (\sin^2 \theta + \cos^2 \theta) = -x^3 - x + x \\ &= -x^3 \text{ which is free from } \theta \text{ i.e., independent of } \theta. \end{aligned}$$

2. Without expanding the determinants, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

Sol. L.H.S. = $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Multiplying R_1 by a , R_2 by b and R_3 by c (by looking at the partial products in third column)

$$= \frac{1}{abc} \begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

Taking abc common from C_3 , $= \frac{abc}{abc} \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$

Interchanging C_1 and C_3 , $= - \begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$

Interchanging C_2 and C_3 , $= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} (\because (-1)(-1) = 1)$
 $= \text{R.H.S.}$

3. Evaluate $\begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$.

Sol. Let $\Delta = \begin{vmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta & -\sin \alpha \\ -\sin \beta & \cos \beta & 0 \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{vmatrix}$

Expanding along R_1 ,

$$\begin{aligned} &= \cos \alpha \cos \beta \begin{vmatrix} \cos \beta & 0 \\ \sin \alpha \sin \beta & \cos \alpha \end{vmatrix} - \cos \alpha \sin \beta \begin{vmatrix} -\sin \beta & 0 \\ \sin \alpha \cos \beta & \cos \alpha \end{vmatrix} - \sin \alpha \begin{vmatrix} -\sin \beta & \cos \beta \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta \end{vmatrix} \\ &= \cos \alpha \cos \beta (\cos \alpha \cos \beta - 0) - \cos \alpha \sin \beta (-\cos \alpha \sin \beta - 0) - \sin \alpha (-\sin \alpha \sin^2 \beta - \sin \alpha \cos^2 \beta) \\ &= \cos^2 \alpha \cos^2 \beta + \cos^2 \alpha \sin^2 \beta + \sin^2 \alpha (\sin^2 \beta + \cos^2 \beta) \\ &= \cos^2 \alpha (\cos^2 \beta + \sin^2 \beta) + \sin^2 \alpha (\sin^2 \beta + \cos^2 \beta) \\ &= \cos^2 \alpha + \sin^2 \alpha = 1. \end{aligned}$$

4. If a, b and c are real numbers and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0.$$

Show that either $a + b + c = 0$ or $a = b = c$.

Sol. Given: $(\Delta =) \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Operate $R_1 \rightarrow R_1 + R_2 + R_3$

(\because Sum of entries of each **column** is same and
 $= 2a + 2b + 2c = 2(a + b + c)$)

$$\Rightarrow \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Taking out $2(a + b + c)$ common from R_1 .

$$\Rightarrow 2(a + b + c) \begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

\therefore Either $2(a + b + c) = 0$ i.e., $a + b + c = \frac{0}{2} = 0 \quad \dots(i)$

or $\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Now each entry of first row is 1.

So operate $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$ (to create two zeros in first row)

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ c+a & a+b-c-a & b+c-c-a \\ a+b & b+c-a-b & c+a-a-b \end{vmatrix} = 0$$

Expanding along R_1 , $\Rightarrow \begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix} = 0$

$$\Rightarrow (b-c)(c-b) - (b-a)(c-a) = 0$$

$$\Rightarrow bc - b^2 - c^2 + bc - bc + ab + ac - a^2 = 0$$

$$\Rightarrow -a^2 - b^2 - c^2 + ab + bc + ac = 0$$

Multiplying by -2 , $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 0$

$$\text{or } a^2 + a^2 + b^2 + b^2 + c^2 + c^2 - 2ab - 2bc - 2ac = 0$$

$$\text{or } (a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (a^2 + c^2 - 2ac) = 0$$

$$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

$$\Rightarrow a - b = 0 \text{ and } b - c = 0 \text{ and } c - a = 0$$

$$[\because x^2 + y^2 + z^2 = 0 \text{ and } x, y, z \in \mathbb{R}$$

$$\Rightarrow x = 0, y = 0 \text{ and } z = 0]$$

$$\Rightarrow a = b \text{ and } b = c \text{ and } c = a \Rightarrow a = b = c \quad \dots(ii)$$

From (i) and (ii) either $a + b + c = 0$ or $a = b = c$.

Note. We can also start doing this question by operating

$$C_1 \rightarrow C_1 + C_2 + C_3.$$

5. Solve the equation
$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0.$$

Sol. Given: The equation
$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Sum of entries of each **column** is same and $= (3x + a)$, so let us operate $R_1 \rightarrow R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Taking out $(3x + a)$ common from R_1

$$\Rightarrow (3x + a) \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

\therefore Either $3x + a = 0$ i.e., $3x = -a$ i.e., $x = -\frac{a}{3}$... (i)

or
$$\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Now each entry of first row is 1, so operate $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$ (to create two zeros in first row)

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding along first row $1(a^2 - 0) = 0$ i.e., $a^2 = 0$
 $\Rightarrow a = 0$. But this is contrary to given that $a \neq 0$.

\therefore From (i) $x = -\frac{a}{3}$ is the only solution (root).

Note. We can also start doing this question by operating $C_1 \rightarrow C_1 + C_2 + C_3$.

6. Prove that
$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Sol. L.H.S. =
$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & bc & c(a+c) \\ a(a+b) & b^2 & ac \\ ab & b(b+c) & c^2 \end{vmatrix}$$

Taking a, b, c common from C_1, C_2, C_3 respectively

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Operate $R_1 \rightarrow R_1 - R_2 - R_3$ (to create one zero in R_1)

$$= abc \begin{vmatrix} a-a-b-b & c-b-b-c & a+c-a-c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$= abc \begin{vmatrix} -2b & -2b & 0 \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Operate $C_2 \rightarrow C_2 - C_1$ (to create another zero in R_1)

$$= abc \begin{vmatrix} -2b & 0 & 0 \\ a+b & -a & a \\ b & c & c \end{vmatrix}$$

$$\text{Expanding along } R_1 = abc(-2b) \begin{vmatrix} -a & a \\ c & c \end{vmatrix}$$

$$= abc(-2b)(-ac-ac)$$

$$= abc(-2b)(-2ac) = 4a^2b^2c^2 = \text{R.H.S.}$$

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

Sol. Given: $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

We know that $(AB)^{-1} = B^{-1} A^{-1}$ (Reversal Law) ... (i)

Now A^{-1} is given, so let us find B^{-1} .

$$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

Expanding along first row,

$$|B| = 1(3-0) - 2(-1-0) + (-2)(2-0) \\ = 3 + 2 - 4 = 1 \neq 0$$

$\therefore B^{-1}$ exists.

To find adj. B

$$B_{11} = + \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = (3 - 0) = 3,$$

$$B_{12} = - \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$$

$$B_{13} = + \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2,$$

$$B_{21} = - \begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = -(2 - 4) = 2,$$

$$B_{22} = + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1,$$

$$B_{23} = - \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -(-2 - 0) = 2,$$

$$B_{31} = + \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = (0 + 6) = 6,$$

$$B_{32} = - \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = -(0 - 2) = 2,$$

$$B_{33} = + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = (3 + 2) = 5.$$

$$\therefore \text{adj. B} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{|B|} (\text{adj. B}) = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \quad (\because |B| = 1)$$

Putting values of B^{-1} and A^{-1} in eqn. (i), we have

$$\begin{aligned} (AB)^{-1} &= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 - 30 + 30 & -3 + 12 - 12 & 3 - 10 + 12 \\ 3 - 15 + 10 & -1 + 6 - 4 & 1 - 5 + 4 \\ 6 - 30 + 25 & -2 + 12 - 10 & 2 - 10 + 10 \end{bmatrix} \\ &= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \end{aligned}$$

8. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that

(i) $(\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$

(ii) $(A^{-1})^{-1} = A$.

Sol. Given: Matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \therefore |A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$

$$= 1(15 - 1) - (-2)(-10 - 1) + 1(-2 - 3)$$

$$= 14 - 22 - 5 = -13 \neq 0$$

To find adj. A

$$A_{11} = + \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 15 - 1 = 14,$$

$$A_{12} = - \begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = -(-10 - 1) = 11$$

$$A_{13} = + \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = (-2 - 3) = -5,$$

$$A_{21} = - \begin{vmatrix} -2 & 1 \\ 1 & 5 \end{vmatrix} = -(-10 - 1) = 11$$

$$A_{22} = + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = (5 - 1) = 4,$$

$$A_{23} = - \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -(1 + 2) = -3$$

$$A_{31} = + \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -2 - 3 = -5,$$

$$A_{32} = - \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -(1 + 2) = -3$$

$$A_{33} = + \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix} = 3 - 4 = -1.$$

$$\therefore \text{adj. } A (= B \text{ (say)}) = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}'$$

$$= \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{-1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} \dots(i)$$

Now let us find $(\text{adj. } A)^{-1} = B^{-1}$

$$|B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$$

Expanding along first row,

$$\begin{aligned} |B| &= 14(-4 - 9) - 11(-11 - 15) - 5(-33 + 20) \\ &= 14(-13) - 11(-26) - 5(-13) \\ &= -182 + 286 + 65 = -182 + 351 = 169 \neq 0 \end{aligned}$$

To find $\text{adj. } B$

$$B_{11} = + \begin{vmatrix} 4 & -3 \\ -3 & -1 \end{vmatrix} = (-4 - 9) = -13,$$

$$B_{12} = - \begin{vmatrix} 11 & -3 \\ -5 & -1 \end{vmatrix} = -(-11 - 15) = 26,$$

$$B_{13} = + \begin{vmatrix} 11 & 4 \\ -5 & -3 \end{vmatrix} = +(-33 + 20) = -13,$$

$$B_{21} = - \begin{vmatrix} 11 & -5 \\ -3 & -1 \end{vmatrix} = -(-11 - 15) = 26,$$

$$B_{22} = + \begin{vmatrix} 14 & -5 \\ -5 & -1 \end{vmatrix} = (-14 - 25) = -39,$$

$$B_{23} = - \begin{vmatrix} 14 & 11 \\ -5 & -3 \end{vmatrix} = -(-42 + 55) = -13,$$

$$B_{31} = + \begin{vmatrix} 11 & -5 \\ 4 & -3 \end{vmatrix} = -33 + 20 = -13,$$

$$B_{32} = - \begin{vmatrix} 14 & -5 \\ 11 & -3 \end{vmatrix} = -(-42 + 55) = -13,$$

$$B_{33} = + \begin{vmatrix} 14 & 11 \\ 11 & 4 \end{vmatrix} = (56 - 121) = -65.$$

$$\therefore \text{adj. } B = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}' = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

Taking -13 common from this matrix $\text{adj. } B$

$$= -13 \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\therefore B^{-1} \text{ i.e., } (\text{adj. } A)^{-1} = \frac{1}{|B|} \text{adj. } B$$

$$\begin{aligned}
 &= \frac{1}{169} (-13) \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \\
 &= \frac{-1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \quad \dots(ii)
 \end{aligned}$$

Now let us find $(A^{-1})^{-1} = A$ (say) where

$$C = A^{-1} = \frac{-1}{13} \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} \quad [\text{By (i)}]$$

$$\text{or} \quad C = A^{-1} = \begin{bmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

$$\begin{aligned}
 \therefore |C| &= |A^{-1}| = -\frac{14}{13} \left(-\frac{4}{169} - \frac{9}{169} \right) \\
 &\quad - \left(-\frac{11}{13} \right) \left(-\frac{11}{169} - \frac{15}{169} \right) + \frac{5}{13} \left(\frac{-33}{169} + \frac{20}{169} \right) \\
 &= -\frac{14}{13} \left(-\frac{13}{169} \right) + \frac{11}{13} \left(-\frac{26}{169} \right) + \frac{5}{13} \left(-\frac{13}{169} \right) \\
 &= \frac{14}{169} - \frac{22}{169} - \frac{5}{169} = \frac{14-22-5}{169} = -\frac{13}{169} = -\frac{1}{13} \neq 0
 \end{aligned}$$

Let us now find adj. C i.e., adj. (A^{-1})

$$C_{11} = + \begin{vmatrix} -\frac{4}{13} & \frac{3}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = \left(-\frac{4}{169} - \frac{9}{169} \right) = \frac{-13}{169} = \frac{-1}{13}$$

$$C_{12} = - \begin{vmatrix} -\frac{11}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{1}{13} \end{vmatrix} = - \left(-\frac{11}{169} - \frac{15}{169} \right) = \frac{26}{169} = \frac{2}{13}$$

$$C_{13} = + \begin{vmatrix} -\frac{11}{13} & -\frac{4}{13} \\ \frac{5}{13} & \frac{3}{13} \end{vmatrix} = + \left(-\frac{33}{169} + \frac{20}{169} \right) = \frac{-13}{169} = \frac{-1}{13}$$

$$C_{21} = - \begin{vmatrix} -\frac{11}{13} & \frac{5}{13} \\ \frac{3}{13} & \frac{1}{13} \end{vmatrix} = - \left(-\frac{11}{169} - \frac{15}{169} \right) = \frac{26}{169} = \frac{2}{13}$$

$$C_{22} = + \begin{vmatrix} -\frac{14}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{1}{13} \end{vmatrix} = -\frac{14}{169} - \frac{25}{169} = -\frac{39}{169} = -\frac{3}{13}$$

$$C_{23} = - \begin{vmatrix} -\frac{14}{13} & -\frac{11}{13} \\ \frac{5}{13} & \frac{3}{13} \end{vmatrix} = - \left(-\frac{42}{169} + \frac{55}{169} \right) = -\frac{13}{169} = -\frac{1}{13}$$

$$C_{31} = + \begin{vmatrix} -\frac{11}{13} & \frac{5}{13} \\ -\frac{4}{13} & \frac{3}{13} \end{vmatrix} = -\frac{33}{169} + \frac{20}{169} = -\frac{13}{169} = -\frac{1}{13}$$

$$C_{32} = - \begin{vmatrix} -\frac{14}{13} & \frac{5}{13} \\ -\frac{11}{13} & \frac{3}{13} \end{vmatrix} = - \left(-\frac{42}{169} + \frac{55}{169} \right) = -\frac{13}{169} = -\frac{1}{13}$$

$$C_{33} = + \begin{vmatrix} -\frac{14}{13} & -\frac{11}{13} \\ -\frac{11}{13} & -\frac{4}{13} \end{vmatrix} = \frac{56}{169} - \frac{121}{169} = -\frac{65}{169} = -\frac{5}{13}$$

$$\begin{aligned} \therefore \text{adj. } C = \text{adj. } (A^{-1}) &= \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix} \end{aligned}$$

Taking $-\frac{1}{13}$ common,

$$\Rightarrow \text{adj. } C = \text{adj. } (A^{-1}) = -\frac{1}{13} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \quad \dots(iii)$$

From (ii) and (iii), we can say that $(\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$
 (\because R.H.Sides of eqns. (ii) and (iii) same)

Hence first part is verified.

$$\text{Again } (A^{-1})^{-1} = C^{-1} \quad (\because C = A^{-1})$$

$$= \frac{1}{|C|} \text{adj. } C = \frac{1}{\left(-\frac{1}{13}\right)} \left(-\frac{1}{13}\right) \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \quad (\text{By (iii)})$$

$$\Rightarrow (A^{-1})^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A \quad (\text{given})$$

Hence second part is verified.

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$.

Sol. Let $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Operate $R_1 \rightarrow R_1 + R_2 + R_3$ (because sum of entries of each column is same and $= 2x + 2y = 2(x + y)$)

$$= \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Taking out $2(x + y)$ common from R_1 ,

$$= 2(x+y) \begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Now each entry of first row is 1, so let us operate $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$ (to create two zeros in first row)

$$= 2(x+y) \begin{vmatrix} 1 & 0 & 0 \\ y & x+y-y & x-y \\ x+y & x-x-y & y-x-y \end{vmatrix}$$

$$= 2(x+y) \begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

$$\begin{aligned} \text{Expanding along } R_1, \Delta &= 2(x+y) \cdot 1 \begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix} \\ &= 2(x+y) (-x^2 + y(x-y)) = 2(x+y) (-x^2 + xy - y^2) \\ &= -2(x+y) (x^2 + y^2 - xy) = -2(x^3 + y^3) \\ &\quad [\because x^3 + y^3 = (x+y)(x^2 + y^2 - xy)] \end{aligned}$$

Remark. This question can also be done by operating

$$C_1 \rightarrow C_1 + C_2 + C_3.$$

10. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}.$

Sol. Let $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Each entry of one column (here first) is 1.

Sol let us operate $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ (to create two zeros in first column)

$$= \begin{vmatrix} 1 & x & y \\ 0 & x+y-x & 0 \\ 0 & 0 & x+y-y \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding along first column, $\Delta = 1 \begin{vmatrix} y & 0 \\ 0 & x \end{vmatrix} = xy.$

Using properties of determinants in Exercises 11 to 15, prove that:

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma).$

Sol. L.H.S. = $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$

Operate $C_3 \rightarrow C_3 + C_1$ to make all entries of third column equal

$$= \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$$

Taking out $(\alpha + \beta + \gamma)$ common from C_3 ,

$$= (\alpha + \beta + \gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix}$$

Now each entry of one column (here third is 1), so let us operate $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ to create two zeros in third column

$$= (\alpha + \beta + \gamma) \begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta - \alpha & \beta^2 - \alpha^2 & 0 \\ \gamma - \alpha & \gamma^2 - \alpha^2 & 0 \end{vmatrix}$$

Expanding along third column,

$$\begin{aligned} &= (\alpha + \beta + \gamma) \begin{vmatrix} \beta - \alpha & \beta^2 - \alpha^2 \\ \gamma - \alpha & \gamma^2 - \alpha^2 \end{vmatrix} \\ &= (\alpha + \beta + \gamma) \begin{vmatrix} (\beta - \alpha) & (\beta - \alpha)(\beta + \alpha) \\ (\gamma - \alpha) & (\gamma - \alpha)(\gamma + \alpha) \end{vmatrix} \end{aligned}$$

Taking $(\beta - \alpha)$ common from R_1 and $(\gamma - \alpha)$ common from R_2

$$\begin{aligned} &= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha) \begin{vmatrix} 1 & \beta + \alpha \\ 1 & \gamma + \alpha \end{vmatrix} \\ &= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma + \alpha - \beta - \alpha) \\ &= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma - \beta) \\ &= (\alpha + \beta + \gamma) [-(\alpha - \beta)] (\gamma - \alpha) [-(\beta - \gamma)] \\ &= (\alpha - \beta)(\beta - \gamma) (\gamma - \alpha) (\alpha + \beta + \gamma) \\ &= \text{R.H.S.} \end{aligned}$$

$$12. \begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x).$$

$$\text{Sol. L.H.S.} = \begin{vmatrix} x & x^2 & 1 + px^3 \\ y & y^2 & 1 + py^3 \\ z & z^2 & 1 + pz^3 \end{vmatrix}$$

(Here each entry of third column is the sum of two entries, so we can write this determinant as sum of two determinants, the first two columns being same in both determinants)

$$\begin{aligned}
 &= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \\
 &\quad \quad \quad \uparrow \quad \quad \quad \uparrow \\
 &= \Delta_1 + \Delta_2 \qquad \qquad \qquad \dots(i)
 \end{aligned}$$

$$\text{Now } \Delta_2 = \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

In an effort to make Δ_2 similar to Δ_1 , taking x, y, z common from R_1, R_2, R_3 respectively and p common from C_3 ,

$$\Delta_2 = pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\text{Operate } C_1 \leftrightarrow C_3, = -pxyz \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$$

$$\text{Operate } C_1 \leftrightarrow C_2, \Delta_2 = pxyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = pxyz\Delta_1$$

$$\text{Putting this value of } \Delta_2 \text{ in (i), L.H.S.} = \Delta_1 + pxyz\Delta_1 = (1 + pxyz)\Delta_1 \qquad \dots(ii)$$

$$\text{Now } \Delta_1 = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

Operate $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ (to create two zeros in third column)

$$= \begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 0 \end{vmatrix}$$

Expanding along third column

$$\Delta_1 = \begin{vmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{vmatrix} = \begin{vmatrix} (y-x) & (y-x)(y+x) \\ (z-x) & (z-x)(z+x) \end{vmatrix}$$

Taking $(y-x)$ common from R_1 and $(z-x)$ common from R_2 ,

$$\begin{aligned} \Delta_1 &= (y-x)(z-x) \begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix} \\ &= (y-x)(z-x)(z+x-y-x) \\ &= (y-x)(z-x)(z-y) = -(x-y)(z-x)(-y-z) \end{aligned}$$

or $\Delta_1 = (x-y)(y-z)(z-x)$

Putting this value of Δ_1 in (ii),

$$\text{L.H.S.} = (1 + pxyz)(x-y)(y-z)(z-x) = \text{R.H.S.}$$

$$13. \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca).$$

$$\text{Sol.} \quad \text{L.H.S.} = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Here the sum of entries of each row is same and

$$= a+b+c. \text{ So let us operate } C_1 \rightarrow C_1 + C_2 + C_3$$

$$\therefore \text{L.H.S.} = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

Taking out $(a+b+c)$ common from C_1 ,

$$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Now each entry of first column is 1. So let us operate $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ to create two zeros in first column,

$$\text{L.H.S.} = (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 3b+a-b & -b+c+a-c \\ 0 & -c+b+a-b & 3c+a-c \end{vmatrix}$$

Expanding along first column,

$$= (a+b+c) \cdot 1 \begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix}$$

$$\begin{aligned} \therefore \text{L.H.S.} &= (a+b+c) [(2b+a)(2c+a) - (a-b)(a-c)] \\ &= (a+b+c) [4bc + 2ab + 2ac + a^2 - a^2 + ac + ab - bc] \\ &= (a+b+c)(3ab + 3bc + 3ac) \end{aligned}$$

$$= 3(a + b + c)(ab + bc + ac)$$

$$= \text{R.H.S.}$$

14.
$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1.$$

Sol. L.H.S. =
$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

On looking at $a_{11} = 1$,
Operate $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ (to make entries a_{21} and a_{31} of first column as zeros)

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix} \begin{array}{l} R_2: \quad 2 \quad 3+2p \quad 4+3p+2q \\ 2R_1: \quad 2 \quad 2+2p \quad 2+2p+2q \\ \hline R_2 - 2R_1: \quad 0 \quad 1 \quad 2+p \\ \\ R_3: \quad 3 \quad 6+3p \quad 10+6p+3q \\ 3R_1: \quad 3 \quad 3+3p \quad 3+3p+3q \\ \hline R_3 - 3R_1: \quad 0 \quad 3 \quad 7+3p \end{array}$$

Expanding along first column,

$$\text{L.H.S.} = 1 \begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix} - 0 + 0$$

$$= 7 + 3p - 3(2 + p) = 7 + 3p - 6 - 3p = 1 = \text{R.H.S.}$$

15.
$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos (\alpha + \delta) \\ \sin \beta & \cos \beta & \cos (\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos (\gamma + \delta) \end{vmatrix} = 0$$

Sol. L.H.S. =
$$\begin{vmatrix} \sin \alpha & \cos \alpha & \cos (\alpha + \delta) \\ \sin \beta & \cos \beta & \cos (\beta + \delta) \\ \sin \gamma & \cos \gamma & \cos (\gamma + \delta) \end{vmatrix}$$

$$= \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta - \sin \alpha \sin \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta - \sin \beta \sin \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta - \sin \gamma \sin \delta \end{vmatrix}$$

Operate $C_3 \rightarrow C_3 + (\sin \delta) C_1$

$$= \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta - \sin \alpha \sin \delta + \sin \alpha \sin \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta - \sin \beta \sin \delta + \sin \beta \sin \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta - \sin \gamma \sin \delta + \sin \gamma \sin \delta \end{vmatrix}$$

$$= \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \cos \delta \\ \sin \beta & \cos \beta & \cos \beta \cos \delta \\ \sin \gamma & \cos \gamma & \cos \gamma \cos \delta \end{vmatrix}$$

Taking $\cos \delta$ common from C_3 ,

$$= \cos \delta \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \\ \sin \beta & \cos \beta & \cos \beta \\ \sin \gamma & \cos \gamma & \cos \gamma \end{vmatrix}$$

$$= \cos \delta (0) \quad [\because C_2 \text{ and } C_3 \text{ have become identical}]$$

$$= 0$$

$$= \text{R.H.S.}$$

16. Solve the system of the following equations: (Using matrices)

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.$$

Sol. Put $\frac{1}{x} = u, \frac{1}{y} = v, \frac{1}{z} = w$.

\therefore The given equations become

$$2u + 3v + 10w = 4, \quad 4u - 6v + 5w = 1, \quad 6u + 9v - 20w = 2$$

The matrix form of these equations is

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

It is of the form $AX = B$

$$\text{where } A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

Expanding by first row

$$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$$

$$= 2(75) - 3(-110) + 10(72) = 150 + 330 + 720 = 1200 \neq 0$$

\therefore Matrix A is non-singular

$\therefore A^{-1}$ exists. \therefore Unique solution is $X = A^{-1}B$... (i)

Now let us find adj. A.

$$\text{Now } A_{11} = +(120 - 45) = 75, \quad A_{12} = -(-80 - 30) = 110,$$

$$A_{13} = +(36 + 36) = 72, \quad A_{21} = -(-60 - 90) = 150,$$

$$A_{22} = +(-40 - 60) = -100, \quad A_{23} = -(18 - 18) = 0$$

$$A_{31} = +(15 + 60) = 75, \quad A_{32} = -(10 - 40) = 30,$$

$$A_{33} = +(-12 - 12) = -24$$

$$\therefore \text{adj. } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Putting these values of X, A^{-1} and B in (i), we have

$$\begin{aligned} \begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{1200} \begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix} = \frac{1}{1200} \begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix} \end{aligned}$$

Equating corresponding entries, $u = \frac{1}{2}$, $v = \frac{1}{3}$, $w = \frac{1}{5}$

$$\therefore \frac{1}{u} = 2, \frac{1}{v} = 3, \frac{1}{w} = 5$$

$$\text{i.e., } x = \frac{1}{u} = 2, y = \frac{1}{v} = 3, z = \frac{1}{w} = 5.$$

Choose the correct answer in Exercises 17 to 19.

17. If a, b, c , are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} \text{ is}$$

(A) 0

(B) 1

(C) x

(D) $2x$.

Sol. $\therefore a, b$ and c are in A.P. $\therefore b - a = c - b \quad \dots(i)$

$$\text{Let } \Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

Operating $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_2$, [By looking at (i)], we have

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(c-b) \end{vmatrix}$$

$$\text{Putting } c - b = b - a \text{ from (i), } \Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(b-a) \end{vmatrix}$$

$$= 0$$

($\because R_2$ and R_3 are identical)

Alternative Method

Operate $R_1 \rightarrow R_1 + R_3 - 2R_2$.

18. If x, y, z are non zero real numbers, then the inverse

of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

$$(A) \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$(B) xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$(C) \frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$(D) \frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Sol. Given: Matrix } A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \quad \therefore |A| = \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix}$$

Expanding along first row,

$$|A| = x(yz - 0) - 0 + 0 = xyz \neq 0$$

(\because It is given that x, y, z are non-zero real numbers)

$$\therefore A^{-1} \text{ exists and } A^{-1} = \frac{1}{|A|} \text{adj. } A \quad \dots(i)$$

To find adj. A

$$A_{11} = + \begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix} = yz, \quad A_{12} = - \begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = -(0-0) = 0,$$

$$A_{13} = + \begin{vmatrix} 0 & y \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0, \quad A_{21} = - \begin{vmatrix} 0 & 0 \\ 0 & z \end{vmatrix} = -(0-0) = 0,$$

$$A_{22} = + \begin{vmatrix} x & 0 \\ 0 & z \end{vmatrix} = xz, \quad A_{23} = - \begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0,$$

$$A_{31} = + \begin{vmatrix} 0 & 0 \\ y & 0 \end{vmatrix} = (0-0) = 0, \quad A_{32} = - \begin{vmatrix} x & 0 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0,$$

$$A_{33} = + \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} = xy.$$

$$\therefore \text{adj. A} = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}' = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\text{Putting values in eqn. (i), } A^{-1} = \frac{1}{xyz} \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$= \begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

$$= \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

\therefore Option (A) is the correct answer.

Remark. The answer of this Q. No. 18 should be used as a formula for one mark questions and Entrance Examinations.

For example, inverse matrix of diagonal matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ is diagonal matrix } \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

$$19. \text{ Let } A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}, \text{ where } 0 \leq \theta \leq 2\pi.$$

Then

(A) $\text{Det (A)} = 0$

(B) $\text{Det (A)} \in (2, \infty)$

(C) $\text{Det (A)} \in (2, 4)$

(D) $\text{Det (A)} \in [2, 4].$

$$\text{Sol. Given: Matrix A} = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{vmatrix}$$

Expanding along first row

$$\begin{aligned} \det A \text{ i.e., } |A| &= 1(1 + \sin^2 \theta) - \sin \theta \\ &\quad (-\sin \theta + \sin \theta) + 1(\sin^2 \theta + 1) \\ &= 1 + \sin^2 \theta + 1 + \sin^2 \theta = 2 + 2\sin^2 \theta \quad \dots(i) \end{aligned}$$

We know that $-1 \leq \sin \theta \leq 1$

$$\begin{aligned} \therefore 0 &\leq \sin^2 \theta \leq 1 \\ (\because \sin^2 \theta &\text{ can never be negative}) \end{aligned}$$

Multiplying by 2, $0 \leq 2\sin^2 \theta \leq 2$

Adding 2 to all sides $2 \leq 2 + 2\sin^2 \theta \leq 4$

$$\text{i.e.,} \quad 2 \leq \det. A \leq 4 \quad \text{(By (i))}$$

\therefore (Value of) $\det A \in$ closed interval $[2, 4]$.

Therefore option (D) is correct answer.

