8 Applications of Integrals

Lesson at a Glance

1. Area ALMB bounded by the curve $y = f(x)$, x-axis and the ordinates (i.e. vertical lines)

   $x = a$ and $x = b$ is $\left|\int_a^b y\,dx\right| = \left|\int_a^b f(x)\,dx\right|$

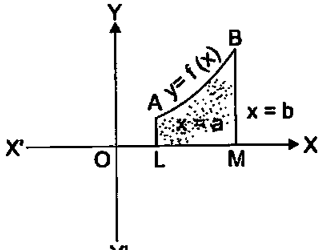


Fig. (i)

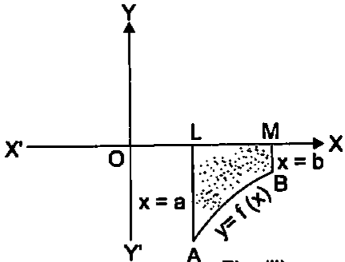


Fig. (ii)

   for both cases when $y \geq 0$ for all x in $[a,b]$ as in Fig. (*i*) or when $y \leq 0$ for all x in [a, b] as in Fig. (*ii*)

2. Area ALMB bounded by the curve $x = f(y)$, y–axis and horizontal lines $y = a$ and $y = b$ is $\left|\int_a^b x\,dy\right| = \left|\int_a^b f(y)\,dy\right|$

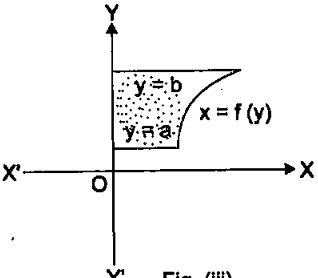


Fig. (iii)

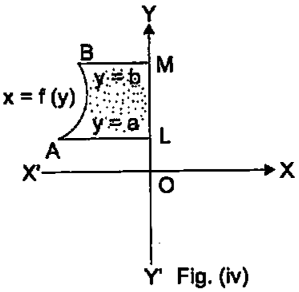


Fig. (iv)

   For both cases when $x \geq 0$ for all y in [a, b] as in Fig. (iii) or when $x \leq 0$ for all y in [a, b] as in Fig. (iv).

   Remark: "Area under the curve" means area between the curve and x-axis

Rules to find the limits of integration for finding area if limits are not given

3. To find area between curve and x-axis, find intersections of the curve with x-axis *i.e.,* put $y = 0$ in equation of the curve and find values of x.
4. To find area between curve and y-axis, find intersections of the curve with y-axis *i.e.,* put $x = 0$ in equation of the curve and find values of y.
5. To find area between the two curves, find their points of intersection *i.e.,* solve the equations of the two curves for x and y.
6. **Working Rule to calculate the area enclosed between two curves**

Step 1. Make the rough sketches of the given functions $y = f(x)$ and $y = g(x)$.

Step 2. Find out points of intersection of the two curves by solving their equations simultaneously for x and y.

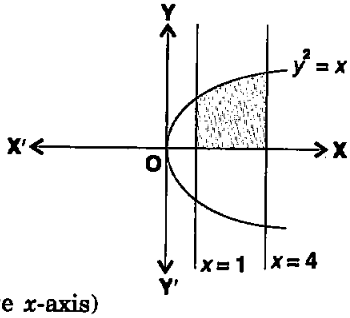
Step 3. Now mark the region whose area is required.

Step 4. Find out the limits of integration from points of intersection.

Step 5. Find the areas for the separate curves by the formulae given in (1) and (2) page 549 and then subtract them or add them as observed from the region.

TEXTBOOK QUESTIONS SOLVED

EXERCISE 8.1 *(Page No.: 365-366)*



1. **Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis.**

Sol. Equation of the curve (rightward parabola) is $y^2 = x$

$\therefore \quad y = \sqrt{x} \quad \quad ...(i)$

(For branch of the parabola above x-axis)

$\therefore$ Required area (as shown shaded in the figure)

$$= \left|\int_1^4 y\,dx\right| = \left|\int_1^4 \sqrt{x}\,dx\right| \quad (\because \text{ From } (i)\ y = \sqrt{x})$$

$$= \left|\int_1^4 x^{1/2}\,dx\right| = \left|\frac{\left(x^{\frac{3}{2}}\right)_1^4}{\frac{3}{2}}\right| = \left|\frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)\right|$$

$$= \left|\frac{2}{3}(4\sqrt{4} - 1\sqrt{1})\right| \left[\because\ x^{\frac{3}{2}} = x^{\frac{2+1}{2}} = x^{\frac{2}{2}+\frac{1}{2}} = x^{1+\frac{1}{2}} = x^1 \cdot x^{\frac{1}{2}} = x\sqrt{x}\ \right]$$

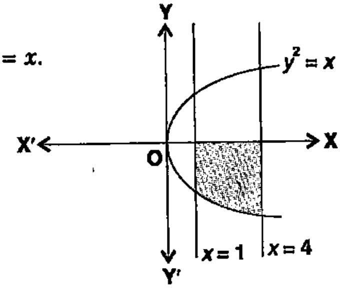
$$= \left|\frac{2}{3}(4(2) - 1(1))\right| = \frac{2}{3}(8 - 1) = \frac{2}{3} \times 7 = \frac{14}{3} \text{ sq. units.}$$

Note. $x^{\frac{3}{2}} = x\sqrt{x}$.

Remark. Equation of the curve is $y^2 = x$.

$\therefore$ $y = -\sqrt{x}$ for branch of the parabola below the x-axis.

The reader is within his or her rights to find the required area as shown shaded in the figure in the remark as $\left|\int_{x=1}^{x=4} y\,dx\right|$ taking $y = -\sqrt{x}$.



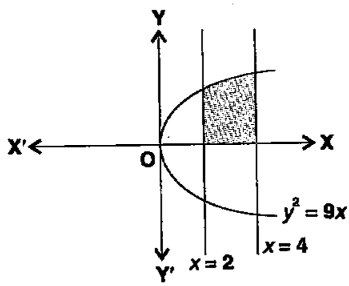
2. **Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.**

Sol. Equation of (rightward parabola) curve is $y^2 = 9x$

$\therefore$ $y = \sqrt{9x} = 3\sqrt{x}$ ...(*i*)

for branch of curve in first quadrant.

$\therefore$ Required (shaded) area bounded by curve $y^2 = 9x$, (vertical lines $x = 2$, $x = 4$), and x-axis in first quadrant



$$= \left|\int_2^4 y\,dx\right| = \left|\int_2^4 3\sqrt{x}\,dx\right| \quad \text{(By (i))}$$

$$= \left|3\int_2^4 x^{\frac{1}{2}}\,dx\right| = 3\frac{\left(x^{\frac{3}{2}}\right)_2^4}{\frac{3}{2}} = 3\left(\frac{2}{3}\right)[4^{\frac{3}{2}} - 2^{\frac{3}{2}}]$$

$$= 2(4\sqrt{4} - 2\sqrt{2}) \qquad [\because\ x^{\frac{3}{2}} = x\sqrt{x}\,]$$

$$= 2(8 - 2\sqrt{2}) = (16 - 4\sqrt{2}) \text{ sq. units.}$$

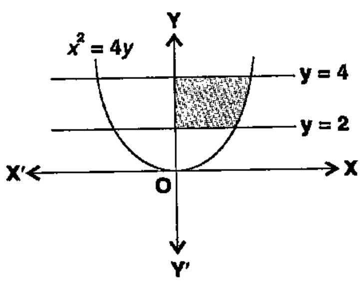
3. **Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.**

Sol. Equation of (upward parabola) curve is $x^2 = 4y$

$\therefore$ $x = \sqrt{4y} = 2\sqrt{y}$ ...(*i*)

for branch of curve in first quadrant.

$\therefore$ Required (shaded) area bounded by curve $x^2 = 4y$, (Horizontal lines $y = 2$, $y = 4$) and y-axis in first quadrant



$$= \left|\int_2^4 x\,dy\right| = \left|\int_2^4 2\sqrt{y}\,dy\right| \qquad \text{(By (i))}$$

$$= \left|2\int_2^4 y^{\frac{1}{2}}\,dy\right| = \left|2\frac{\left(y^{\frac{3}{2}}\right)_2^4}{\frac{3}{2}}\right|$$

$$= \frac{4}{3}\left|(4^{\frac{3}{2}} - 2^{\frac{3}{2}})\right| = \frac{4}{3}(4\sqrt{4} - 2\sqrt{2}) \qquad (\because\ x^{\frac{3}{2}} = x\sqrt{x})$$

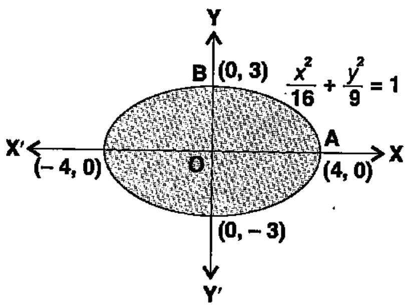
$$= \frac{4}{3}(4(2) - 2\sqrt{2}) = \left(\frac{32 - 8\sqrt{2}}{3}\right) \text{ sq. units.}$$

4. Find the area of the region bounded by the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

Sol. Equation of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \qquad ...(i)$$



Here $a^2(= 16) > b^2(= 9)$

From (i), $\frac{y^2}{9} = 1 - \frac{x^2}{16}$

$$= \frac{16 - x^2}{16}$$

$$\Rightarrow \qquad y^2 = \frac{9}{16}(16 - x^2)$$

$$\Rightarrow \qquad y = \frac{3}{4}\sqrt{16 - x^2} \qquad ...(ii)$$

for arc of ellipse in first quadrant.

Ellipse (i) is symmetrical about x-axis.

($\because$ On changing $y \to -y$ in (i), it remains unchanged).

Ellipse (i) is symmetrical about y-axis.

($\because$ On changing $x \to -x$ in (i), it remains unchanged)

Intersections of ellipse (i) with x-axis ($y = 0$)

Putting $y = 0$ in (i), $\frac{x^2}{16} = 1 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

$\therefore$ Intersections of ellipse (i) with x-axis are (4, 0) and (– 4, 0).

Intersections of ellipse (i) with y-axis ($x = 0$)

Putting $x = 0$ in (i), $\frac{y^2}{9} = 1 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$.

$\therefore$ Intersections of ellipse (i) with y-axis are (0, 3) and (0, – 3).

$\therefore$ Area of region bounded by ellipse (*i*)

$= $ Total shaded area

$= 4 \times$ Area OAB of ellipse in first quadrant

$= 4 \left| \int_0^4 y\, dx \right|$ ($\because$ At end B of arc AB of ellipse; $x = 0$ and at end A of arc AB; $x = 4$)

$$= 4 \left| \int_0^4 \frac{3}{4}\sqrt{16 - x^2}\, dx \right| \qquad \text{[By (ii)]}$$

$$= 3 \left| \int_0^4 \sqrt{4^2 - x^2}\, dx \right| = 3 \left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2}\sin^{-1}\frac{x}{4} \right]_0^4$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]$$

$$= 3 \left[\frac{4}{2}\sqrt{16 - 16} + 8\sin^{-1} 1 - (0 + 8\sin^{-1} 0) \right] = 3 \left[0 + \frac{8\pi}{2} \right]$$

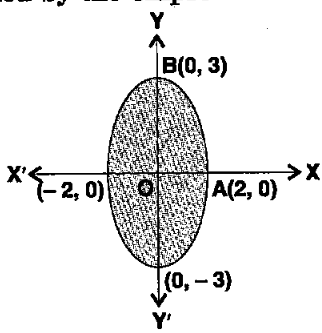
$$\left(\because \sin\frac{\pi}{2} = 1 \Rightarrow \sin^{-1} 1 = \frac{\pi}{2} \text{ and } \sin 0 = 0 \Rightarrow \sin^{-1} 0 = 0 \right)$$

$= 3(4\pi) = 12\pi$ sq. units.

Remark. We can also find area of this ellipse as $4 \left| \int_0^3 x\, dy \right|$

5. Find the area of the region bounded by the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$



Sol. Equation of the ellipse is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad ...(i)$$

Here $a^2 (= 4) < b^2 (= 9)$

From (*i*), $\frac{y^2}{9} = 1 - \frac{x^2}{4} = \frac{4 - x^2}{4}$

$$\Rightarrow \quad y^2 = \frac{9}{4}(4 - x^2) \quad \Rightarrow \quad y = \frac{3}{2}\sqrt{4 - x^2} \qquad ...(ii)$$

for arc of ellipse in first quadrant. Clearly ellipse (*i*) is symmetrical about x-axis and y-axis both.

[$\because$ On changing y to $-y$ in (*i*) or x to $-x$ in (*i*) keep it unchanged]

Intersections of ellipse (*i*) with x-axis ($y = 0$)

Putting $y = 0$ in (*i*), $\frac{x^2}{4} = 1 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\therefore$ Intersections of ellipse (*i*) with x-axis are (2, 0) and (– 2, 0)

Intersections of ellipse (*i*) with y-axis (x = 0)

Putting $x = 0$ in (*i*), $\frac{y^2}{9} = 1 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

$\therefore$ Intersections of ellipse (*i*) with y-axis are (0, 3) and (0, – 3).

$\therefore$ Area of region bounded by ellipse (*i*)

= Total shaded area

= 4 × area OAB of ellipse in first quadrant

$= 4 \left| \int_0^2 y \, dx \right|$ ($\because$ At end B of arc AB of ellipse $x = 0$ and at end A of arc AB, $x = 2$)

$$= 4 \left| \int_0^2 \frac{3}{2}\sqrt{4 - x^2}\, dx \right| \quad \text{(By (ii))}$$

$$= 4 \cdot \frac{3}{2} \left| \int_0^2 \sqrt{2^2 - x^2}\, dx \right| = 6 \left[\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} \right]_0^2$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]$$

$$= 6 \left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}1 - 0 - 2\sin^{-1}0 \right]$$

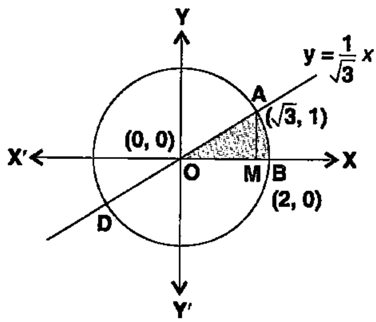
$$= 6 \left[0 + 2 \cdot \frac{\pi}{2} - 0 \right] = 6\pi \text{ sq. units.}$$

6. Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}\, y$ and the circle $x^2 + y^2 = 4$.

Sol. Step I. To draw the graphs and shade the region whose area we are to find.

Equation of the circle is

$x^2 + y^2 = 4 = 2^2$ ...(*i*)



We know that eqn. (*i*) represents a circle whose centre is (0, 0) and radius is 2.

Equation of the given line is

$$x = \sqrt{3}\, y$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}x \quad ...(ii)$$

We know that equation (*ii*) being of the form $y = mx$ where $m = \frac{1}{\sqrt{3}} = \tan 30° = \tan\theta \Rightarrow \theta = 30°$ represents a straight line passing through the origin and making angle of 30° with x-axis.

We are to find area of shaded region OAB in first quadrant (only).

Step II. Let us solve (*i*) and (*ii*) for *x* and *y* to find their points of intersection.

Putting $y = \frac{x}{\sqrt{3}}$ from (*ii*) in (*i*), $x^2 + \frac{x^2}{3} = 4$

$\Rightarrow \quad 3x^2 + x^2 = 12 \quad \Rightarrow \quad 4x^2 = 12 \quad \Rightarrow \quad x^2 = 3$

$\Rightarrow \quad x = \pm\sqrt{3}$

For $x = \sqrt{3}$, from (*ii*), $y = \frac{1}{\sqrt{3}}\sqrt{3} = 1$

For $x = -\sqrt{3}$, from (*ii*), $y = \frac{1}{\sqrt{3}}(-\sqrt{3}) = -1$

$\therefore$ The two points of intersections of circle (*i*) and line (*ii*) are $A(\sqrt{3}, 1)$ and $D(-\sqrt{3}, -1)$.

Step III. Now shaded area OAM between segment OA of line (*ii*) and *x*-axis

$= \left|\int_0^{\sqrt{3}} y\, dx\right| \qquad (\because \text{ At O, } x = 0 \text{ and at A, } x = \sqrt{3})$

$= \left|\int_0^{\sqrt{3}} \frac{1}{\sqrt{3}} x\, dx\right| \qquad$ [By (*ii*)]

$= \frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{3}{2} - 0\right) = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$ sq. units ...(*iii*)

Step IV. Now shaded area AMB between arc AB of circle and *x*-axis

$= \left|\int_{\sqrt{3}}^{2} y\, dx\right| \qquad (\because \text{ at A, } x = \sqrt{3} \text{ and at B, } x = 2)$

$= \left|\int_{\sqrt{3}}^{2} \sqrt{2^2 - x^2}\, dx\right| \qquad (\text{From } (ii),\ y^2 = 2^2 - x^2 \Rightarrow y = \sqrt{2^2 - x^2})$

$= \left(\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_{\sqrt{3}}^{2}$

$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$= \left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}1 - \left(\frac{\sqrt{3}}{2}\sqrt{4-3} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$= 0 + 2 \cdot \frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2 \cdot \frac{\pi}{3} \qquad \left[\because \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \Rightarrow \frac{\pi}{3} = \sin^{-1}\frac{\sqrt{3}}{2}\right]$

$= \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \pi - \frac{2\pi}{3} - \frac{\sqrt{3}}{2} = \frac{3\pi - 2\pi}{3} - \frac{\sqrt{3}}{2}$

$= \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units. ...(*iv*)

Step V. Required shaded area OAB

= Area OAM + Area AMB

$$= \frac{\sqrt{3}}{2} + \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \text{ sq. units.} \qquad \text{[By (iii) and (iv)]}$$

7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.

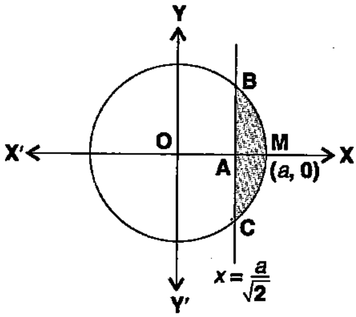
Sol. Given: Equation of the circle is

$x^2 + y^2 = a^2$ ...(*i*)

$\therefore \quad y^2 = a^2 - x^2$

$\therefore \quad y = \sqrt{a^2 - x^2}$ ...(*ii*)

for arc BM of circle in Ist quadrant.



We know that equation (*i*) represents a circle whose centre is origin (0, 0) and radius a.

Clearly, circle (*i*) is symmetrical both about x-axis and y-axis.

We also know that graph of (vertical) line $x = \frac{a}{\sqrt{2}}$ is parallel to y-axis at a distance $\frac{a}{\sqrt{2}}$ $(< a)$ to the right of origin.

$\therefore$ Area of smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$ = Area ABMC = **2** × Area ABM

$$= 2\left|\int_{\frac{a}{\sqrt{2}}}^{a} y\,dx\right|$$

[$\because$ At point B (point of vertical line BC) $x = \frac{a}{\sqrt{2}}$ and at point M, x = radius a)

$$= 2\left|\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx\right| \qquad \text{(By (ii))}$$

$$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{\frac{a}{\sqrt{2}}}^{a}$$

$$= 2\left[\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}1 - \left(\frac{\frac{a}{\sqrt{2}}}{2}\sqrt{a^2 - \frac{a^2}{2}} + \frac{a^2}{2}\sin^{-1}\frac{\frac{a}{\sqrt{2}}}{a}\right)\right]$$

$$= 2\left[0 + \frac{a^2}{2}\frac{\pi}{2} - \frac{a}{2\sqrt{2}}\sqrt{\frac{a^2}{2}} - \frac{a^2}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= 2\left[\frac{\pi a^2}{4} - \frac{a}{2\sqrt{2}}\frac{a}{\sqrt{2}} - \frac{a^2}{2}\frac{\pi}{4}\right] \left(\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}\right)$$

$$= 2\left[\frac{\pi a^2}{4} - \frac{\pi a^2}{8} - \frac{a^2}{4}\right]. \qquad [\because \sqrt{2}\,\sqrt{2} = (\sqrt{2})^2 = 2]$$

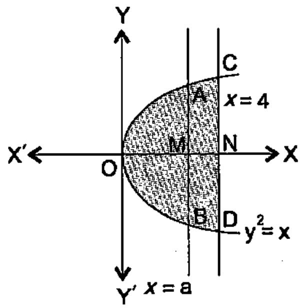
$$= 2a^2\left(\frac{\pi}{4} - \frac{\pi}{8} - \frac{1}{4}\right) = 2a^2\left(\frac{2\pi - \pi - 2}{8}\right)$$

$$= \frac{a^2}{4}(\pi - 2) = \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right) \text{ sq. units.}$$

Note. It may be clearly noted that in this question No. 7 we were not to find only area AMB or only area AMC because x-axis is not given to be a boundary of the region in question whose area is required.

We have drawn x-axis here only as a line of reference because without drawing x-axis and y-axis as lines of reference, we can't draw any graph.

8. **The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.**



Sol. Equation of the curve (rightward parabola) is

$$x = y^2 \quad i.e., \quad y^2 = x \qquad ...(i)$$

From (i), $y = \sqrt{x}$ ...(ii)

for arc OAC of parabola in first quadrant.

We know that equation (i) represents a right-ward parabola with symmetry about x-axis.

($\because$ Changing y to $-y$ in (i) keeps it unchanged)

Given: Area bounded by parabola (i) and vertical line $x = 4$ is divided into two equal parts by the vertical line $x = a$.

$\Rightarrow$ Area OAMB = Area AMBDNC.

$$\Rightarrow \quad 2\left|\int_0^a y\,dx\right| = 2\left|\int_a^4 y\,dx\right|$$

(For multipliction by 2 on each side, see **Note** above after solution of Q. No. 7)

Dividing by 2 and putting $y = \sqrt{x} = x^{\frac{1}{2}}$ from (ii),

$$\left|\int_0^a x^{\frac{1}{2}}\,dx\right| = \left|\int_a^4 x^{\frac{1}{2}}\,dx\right|$$

$$\Rightarrow \quad \frac{\left(x^{\frac{3}{2}}\right)_0^a}{\frac{3}{2}} = \frac{\left(x^{\frac{3}{2}}\right)_a^4}{\frac{3}{2}} \Rightarrow \frac{2}{3}\left[a^{\frac{3}{2}} - 0\right] = \frac{2}{3}\left[4^{\frac{3}{2}} - a^{\frac{3}{2}}\right]$$

Dividing both sides by $\frac{2}{3}$, $a^{\frac{3}{2}} = 4\sqrt{4} - a^{\frac{3}{2}}$

Transposing, $2a^{\frac{3}{2}} = 8 \Rightarrow a^{\frac{3}{2}} = 4 \Rightarrow a = 4^{\frac{2}{3}}$.

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = | x |$.

Sol. The required area is the area included between the parabola $y = x^2$ and the modulus function

$$y = | x | = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

We know that, the graph of the modulus function consists of two rays (*i.e.*, half lines $y = x$ for $x \geq 0$ and $y = -x$ for $x \leq 0$) passing through the origin and at right angles to each other. The half line $y = x$ if $x \geq 0$ has slope 1 and hence makes an angle of 45° with positive x-axis.

$y = x^2$ represents an upward parabola with vertex at origin.

The graphs of the two functions $y = x^2$ and $y = | x |$ are symmetrical about the y-axis.

[$\because$ Both equations remain unchanged on changing x to $-x$ as $| -x | = | x |$]

Let us first find the area between the parabola

$$y = x^2 \quad ...(i)$$

and the ray $y = x$ for $x \geq 0$ ...(*ii*)

To find limits of integration, let us solve (*i*) and (*ii*) for x.

Putting $y = x^2$ from (*i*) in (*ii*), we have $x^2 = x$

or $x^2 - x = 0$ or $x(x - 1) = 0 \therefore x = 0$ or $x = 1$

For $y = | x |$

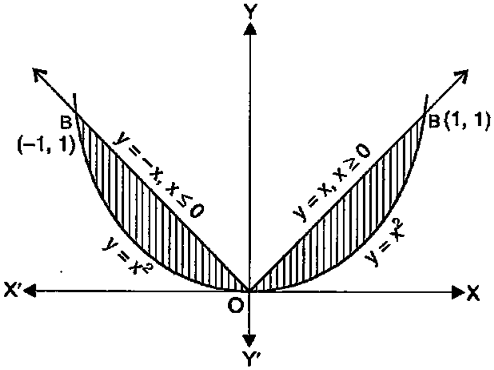
Table of values

$y = x$ if $x \geq 0$

x	0	1	2
y	0	1	2

$y = -x$ if $x \leq 0$

x	0	−1	−2
y	0	1	2



Area between parabola (*i*) and x-axis between limits

$x = 0$ and $x = 1$

$$= \int_0^1 y\,dx = \int_0^1 x^2\,dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3} \qquad ...(iii)$$

Area between ray (*ii*) and x-axis,

$$= \int_0^1 y\,dx = \int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2} \qquad ...(iv)$$

∴ Required shaded area in first quadrant

= Area between ray $y = x$ for $x \geq 0$ and x-axis

– Area between parabola (*i*) and x-axis in first quadrant

= Area given by (*iv*) – Area given by (*iii*)

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units

Similarly, shaded area in second quadrant $= \frac{1}{6}$ sq. units.

∴ Total area of shaded region in the above figure

$= \frac{1}{6} + \frac{1}{6} = 2 \times \frac{1}{6} = \frac{1}{3}$ sq. units.

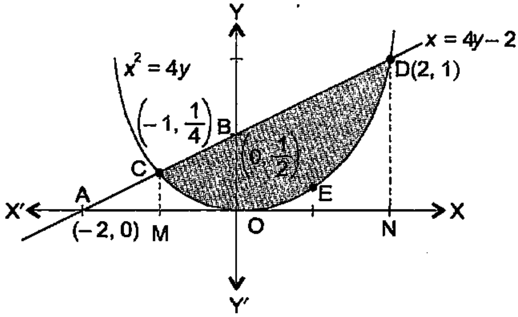
10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Sol. Step I. Graphs and region of Integration.

Equation of the given curve is $x^2 = 4y$ ...(*i*)

We know that eqn. (*i*) reprsents an upward parabola symmetrical about y-axis

[∵ on changing x to $-x$ in (*i*), eqn. (*i*) remains unchanged]



Equation of the given line is

$$x = 4y - 2 \qquad ...(ii)$$

$\Rightarrow \quad x + 2 = 4y \qquad \Rightarrow \quad y = \frac{x+2}{4}$

Table of values for $x = 4y - 2$

x	0	-2
y	$\frac{1}{2}$	0

We are to find the area of the shaded region shown in the adjoining figure.

Step II. To find points of intersections of curve (*i*) and line (*ii*), let us solve (*i*) and (*ii*) for *x* and *y*.

Putting $y = \frac{x^2}{4}$ from (*i*) in (*ii*),

$$x = 4 \cdot \frac{x^2}{4} - 2 \Rightarrow x = x^2 - 2 \Rightarrow -x^2 + x + 2 = 0$$

or $\quad x^2 - x - 2 = 0$

$\Rightarrow \quad x^2 - 2x + x - 2 = 0$ or $x(x - 2) + (x - 2) = 0$

or $\quad (x - 2)(x + 1) = 0$

$\therefore \quad$ Either $x - 2 = 0$ or $x + 1 = 0$

i.e., $\quad x = 2$ or $x = -1$

For $x = 2$, from (*i*), $\quad y = \frac{x^2}{4} = \frac{4}{4} = 1 \quad \therefore \quad (2, 1)$

For $x = -1$, from (*i*), $y = \frac{x^2}{4} = \frac{1}{4} \quad \therefore \quad \left(-1, \frac{1}{4}\right)$.

$\therefore$ The two points of intersection of parabola (*i*) and line (*ii*) are $C\left(-1, \frac{1}{4}\right)$ and D(2, 1).

Step III. Area CMOEDN between parabola (*i*) and *x*-axis

$$= \left|\int_{-1}^{2} y\,dx\right| = \left|\int_{-1}^{2} \frac{x^2}{4}\,dx\right| \qquad \left(\because \text{From } (i)\ y = \frac{x^2}{4}\right)$$

$$= \left|\frac{\left(x^3\right)_{-1}^{2}}{12}\right| = \left|\frac{1}{12}(2^3 - (-1)^3)\right| = \frac{1}{12}(8 - (-1))$$

$$= \frac{1}{12}(8 + 1) = \frac{9}{12} = \frac{3}{4} \text{ sq. units} \qquad ...(iii)$$

Step IV. Area of trapezium CMND between line (*ii*) and *x*-axis

$$= \left|\int_{-1}^{2} y\,dx\right| = \left|\int_{-1}^{2} \frac{x+2}{4}\,dx\right| = \left|\frac{1}{4}\int_{-1}^{2} (x+2)\,dx\right|$$

$$= \frac{1}{4}\left|\left(\frac{x^2}{2} + 2x\right)_{-1}^{2}\right| = \frac{1}{4}\left|\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right|$$

$$= \frac{1}{4}\left|2 + 4 - \frac{1}{2} + 2\right| = \frac{1}{4}\left|8 - \frac{1}{2}\right|$$

$$= \frac{1}{4}\left|\frac{16-1}{2}\right| = \frac{1}{4}\left(\frac{15}{2}\right) = \frac{15}{8} \text{ sq. units.} \quad ...(iv)$$

∴ Required shaded area

= Area given by (*iv*) – Area given by (*iii*)

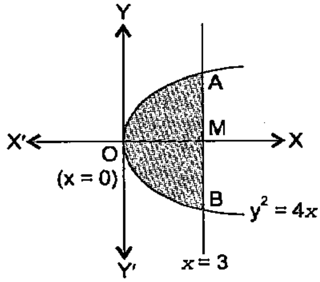
= Area of trapezium CMND – Area (CMOEDN)

$$= \frac{15}{8} - \frac{3}{4} = \frac{15-6}{8} = \frac{9}{8} \text{ sq. units.}$$

11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Sol. Equation of the (parabola) curve is

$y^2 = 4x$ ...(*i*)



$\therefore y = \sqrt{4x} = 2x^{\frac{1}{2}}$ ...(*ii*)

for arc OA of parabola in first quadrant.

We know that equation (*i*) represents a rightward parabola with symmetry about x-axis.

(∵ Changing y to $-y$ in (*i*), keeps it unchanged)

∴ Required shaded area OAMB.

(See Note after solution of example 7)

= 2(Area OAM)

$$= 2\left|\int_0^3 y\,dx\right| = 2\left|\int_0^3 2x^{\frac{1}{2}}\,dx\right| \quad \text{(By (ii))}$$

$$= 4\left|\frac{\left(x^{\frac{3}{2}}\right)_0^3}{\frac{3}{2}}\right| = 4\,.\,\frac{2}{3}\,[3^{\frac{3}{2}} - 0] = \frac{8}{3}3\sqrt{3} = 8\sqrt{3} \text{ sq. units.}$$

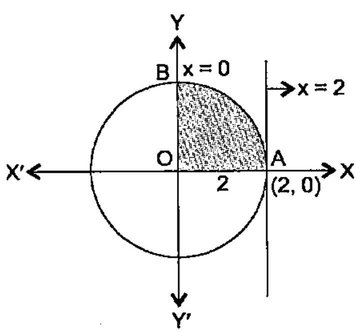
12. Choose the correct answer:

Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(A) π **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{4}$.

Sol. Equation of the circle is

$x^2 + y^2 = 4 = 2^2$ ...(*i*)



We know that equation (*i*) represents a circle whose centre is origin and radius is 2.

∴ $y^2 = 2^2 - x^2$

∴ $y = \sqrt{2^2 - x^2}$ ...(*ii*)

for arc AB of the circle in first quadrant.

∴ Required area lying in the first quadrant bounded by the circle $x^2 + y^2 = 4$ and the (vertical) lines $x = 0$ and (tangent line) $x = 2$.

$$= \left| \int_0^2 y\, dx \right| = \left| \int_0^2 \sqrt{2^2 - x^2}\, dx \right| \quad \text{By } (ii))$$

$$= \left| \left(\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} \right)_0^2 \right|$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]$$

$$= \frac{2}{2}\sqrt{4-4} + 2\sin^{-1} 1 - (0 + 2\sin^{-1} 0)$$

$$= 0 + 2 \cdot \frac{\pi}{2} - 0 - 0 = \pi \text{ sq. units.}$$

$[\because \sin 0 = 0 \Rightarrow \sin^{-1} 0 = 0]$

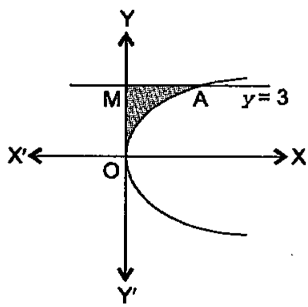
∴ Option (A) is the correct answer.

13. Choose the correct answer:

Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

(A) 2 **(B) $\frac{9}{4}$**

(C) $\frac{9}{3}$ **(D) $\frac{9}{2}$.**



Sol. Equation of the curve (rightward parabola) is

$$y^2 = 4x \quad ...(i)$$

∴ Required area of the region bounded by parabola (*i*), y-axis and the (horizontal) line $y = 3$

= Area OAM

$$= \left| \int_0^3 x\, dy \right| \quad ...(ii)$$

[∵ For arc OA of the parabola (*i*), at point O, $y = 0$ and at point A, $y = 3$]

Putting $x = \frac{y^2}{4}$ from (*i*) in (*ii*), required area

$$= \left| \int_0^3 \frac{y^2}{4}\, dy \right|$$

$$= \frac{1}{4} \left| \left(\frac{y^3}{3} \right)_0^3 \right| = \frac{1}{4} \left| \frac{27}{3} - 0 \right| = \frac{9}{4} \text{ sq. units}$$

∴ Option (B) is correct answer.

Exercise 8.2 *(Page No. 371-372)*

1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

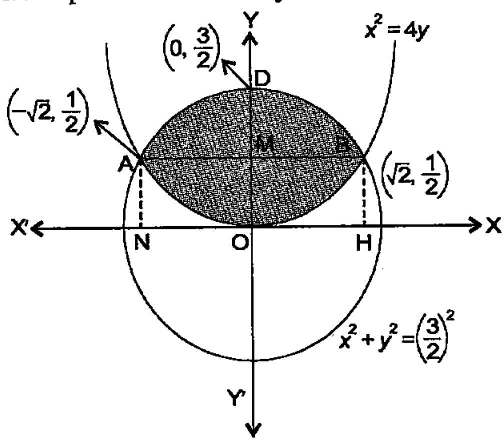
Sol. Step I. Let us draw graphs and shade the region of integration.

Given: Equation of the circle is $4x^2 + 4y^2 = 9$

Dividing by 4, $x^2 + y^2 = \frac{9}{4} = \left(\frac{3}{2}\right)^2$ ...(*i*)

We know that this equation (*i*) represents a circle whose centre is (0, 0) and radius $\frac{3}{2}$ $(x^2 + y^2 = r^2)$

Equation of parabola is $x^2 = 4y$ ...(*ii*)



(eqn. (*ii*) represents an upward parabola symmetrical about *y*-axis)

Step II. Let us solve eqns. of circle (*i*) and parabola (*ii*) for *x* and *y* to find their points of intersection.

Putting $x^2 = 4y$ from (*ii*) in (*i*), we have $4y + y^2 = \frac{9}{4}$

Multiplying by L.C.M. (= 4),

$16y + 4y^2 = 9$ or $4y^2 + 16y - 9 = 0$

$\Rightarrow \quad 4y^2 + 18y - 2y - 9 = 0 \Rightarrow 2y(2y + 9) - 1(2y + 9) = 0$

$\Rightarrow \quad (2y + 9)(2y - 1) = 0$

$\therefore$ Either $2y + 9 = 0$ or $2y - 1 = 0$

$\Rightarrow \quad 2y = -9$ or $2y = 1$

$\Rightarrow \quad y = -\frac{9}{2}$ or $y = \frac{1}{2}$

For $y = -\frac{9}{2}$, from (*i*) $x^2 = 4y = 4\left(-\frac{9}{2}\right) = -18$

which is impossible because square of a real number can never be negative.

For $y = \frac{1}{2}$, from (*i*), $x^2 = 4y = 4 \times \frac{1}{2} = 2$

$\therefore \quad x = \pm\sqrt{2}$

$\therefore$ Points of intersections of circle (*i*) and parabola (*ii*) are

$$A\left(-\sqrt{2}, \frac{1}{2}\right) \text{ and } B\left(\sqrt{2}, \frac{1}{2}\right).$$

Step III. Area OBM = Area between parabola (*ii*) and *y*-axis

$$= \left|\int_0^{\frac{1}{2}} x\,dy\right|$$

($\because$ at O, $y = 0$ and at B, $y = \frac{1}{2}$)

From (*ii*), putting $x = \sqrt{4y} = 2\sqrt{y} = 2y^{\frac{1}{2}}$,

$$\text{Area OBM} = \left|\int_0^{\frac{1}{2}} 2y^{\frac{1}{2}}\,dy\right| = 2 \cdot \frac{\left(y^{\frac{3}{2}}\right)_0^{\frac{1}{2}}}{\frac{3}{2}}$$

$$= 2 \cdot \frac{2}{3}\left[\left(\frac{1}{2}\right)^{\frac{3}{2}} - 0\right] = \frac{4}{3}\ \frac{1}{2}\ \sqrt{\frac{1}{2}} \qquad \left[\because\ x^{\frac{3}{2}} = x\sqrt{x}\right]$$

$$= \frac{2}{3}\ \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3} \qquad \ldots(iii) \qquad \left|\ \because\ \frac{x}{\sqrt{x}} = \sqrt{x}\right.$$

Step IV. Now area BDM = Area between circle (*i*) and *y*-axis

$$= \left|\int_{\frac{1}{2}}^{\frac{3}{2}} x\,dy\right| \quad \left[\because \text{ At point B, } y = \frac{1}{2} \text{ and at point D, } y = \frac{3}{2}\right]$$

From (*i*), putting $x^2 = \left(\frac{3}{2}\right)^2 - y^2$ *i.e.*, $x = \sqrt{\left(\frac{3}{2}\right)^2 - y^2}$,

$$= \left|\int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\left(\frac{3}{2}\right)^2 - y^2}\,dy\right| = \left[\frac{y}{2}\sqrt{\left(\frac{3}{2}\right)^2 - y^2} + \frac{\left(\frac{3}{2}\right)^2}{2}\sin^{-1}\frac{y}{\left(\frac{3}{2}\right)}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$\left[\because \int\sqrt{a^2 - y^2}\,dy = \frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]$$

$$= \frac{3}{4}\sqrt{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} + \frac{9}{8}\sin^{-1}\left(\frac{\frac{3}{2}}{\frac{3}{2}}\right) - \left[\frac{1}{4}\sqrt{\frac{9}{4} - \frac{1}{4}} + \frac{9}{8}\sin^{-1}\left(\frac{\frac{1}{2}}{\frac{3}{2}}\right)\right]$$

$$= \left(\frac{3}{4}\times 0\right) + \frac{9}{8}\sin^{-1} 1 - \left[\frac{1}{4}\sqrt{\frac{8}{4}} + \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= \frac{9}{8} \times \frac{\pi}{2} - \frac{1}{4}\sqrt{2} - \frac{9}{8}\sin^{-1}\frac{1}{3}$$

$$= \frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3} \quad \ldots(iv)$$

Step V. $\therefore$ Required shaded area (of circle (*i*) which is interior to parabola (*ii*)) = Area AOBDA

= 2(Area OBD) = 2[Area OBM + Area MBD]

$$= 2\left[\frac{\sqrt{2}}{3} + \left(\frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right)\right]$$

(By (*iii*)) (By (*iv*))

$$= 2\left[\sqrt{2}\left(\frac{1}{3} - \frac{1}{4}\right) + \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\sqrt{2}\left(\frac{4-3}{12}\right) + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \left(\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}\right) = \frac{\sqrt{2}}{6} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\cos^{-1}\frac{1}{3} \textbf{ sq. units.}$$

Ans $\left(\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right)$

Remark: $= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\sqrt{1 - \frac{1}{9}}$ $\quad (\because \cos^{-1} x = \sin^{-1}\sqrt{1-x^2})$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\sqrt{\frac{8}{9}} = \left(\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right) \text{ sq. units.}$$

Note: The equation $(x-\alpha)^2 + (y-\beta)^2 = r^2$ represents a circle whose centre is (α, β) and radius is r.

2. Find the area bounded by the curves $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.

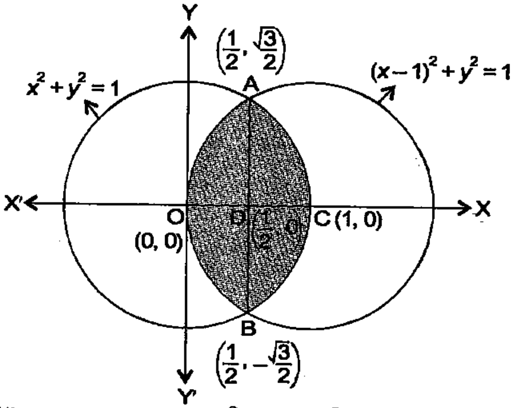
Sol. The equations of the two circles are

$$x^2 + y^2 = 1 \quad \ldots(i)$$

and $$(x - 1)^2 + y^2 = 1 \quad \ldots(ii)$$

The first circle has centre at the origin and radius 1. The second circle has centre at (1, 0) and radius 1. Both are symmetrical about the x-axis. Circle (*i*) is symmetrical about y-axis also.

For points of intersections of circles (*i*) and (*ii*), let us solve equations (*i*) and (*ii*) for x and y.



From (*i*), $y^2 = 1 - x^2$

Putting $y^2 = 1 - x^2$ in eqn. (*ii*), $(x - 1)^2 + 1 - x^2 = 1$

or $x^2 + 1 - 2x + 1 - x^2 = 1$

or $-2x + 1 = 0 \quad \therefore \quad x = \frac{1}{2}$

Putting $x = \frac{1}{2}$, $y^2 = 1 - x^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore \qquad y = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

$\therefore$ The two points of intersections of circles (*i*) and (*ii*) are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

From (*i*), $y^2 = 1 - x^2$;

$\therefore \qquad y = \sqrt{1-x^2}$ in first quadrant.

From (*ii*), $y^2 = 1 - (x - 1)^2$ and

$\therefore \qquad y = \sqrt{1-(x-1)^2}$ in first quadrant.

Required area OACBO (area enclosed between the two circles) (shown shaded)

= 2 × Area OAC

= 2 [Area OAD + Area DAC]

$$= 2\left[\int_0^{1/2} y \text{ of circle } (ii)\, dx + \int_{1/2}^{1} y \text{ of circle } (i)\, dx\right]$$

$$= 2\left[\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx + \int_{1/2}^{1} \sqrt{1-x^2}\, dx\right]$$

$$= 2\left[\left\{\frac{(x-1)\sqrt{1-(x-1)^2}}{2} + \frac{1}{2}\sin^{-1}(x-1)\right\}_0^{1/2} + \left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right\}_{1/2}^{1}\right]$$

$$\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left\{-\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\left(-\frac{1}{2}\right)\right\} - \{\sin^{-1}(-1)\} + \sin^{-1} 1 - \left\{\frac{1}{2}\sqrt{\frac{3}{4}} + \sin^{-1}\frac{1}{2}\right\}$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} \qquad = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units.}$$

3. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

Sol. Equation of the given curve is $y = x^2 + 2$ ...(*i*)

or $x^2 = y - 2$

It is an upward parabola ($\because$ An equation of the form $x^2 = ky$, $k > 0$ represents an upward parabola).

Eqn. (*i*) contains only even powers of x and hence remains unchanged on changing x to $-x$ in (*i*).

$\therefore$ The parabola (*i*) is symmetrical about y-axis.

Parabola (*i*) meets y-axis (its line of symmetry) *i.e.* $x = 0$ in (0, 2) [put $x = 0$ in (*i*) to get $y = 2$]

$\therefore$ Vertex of the parabola is (0, 2).

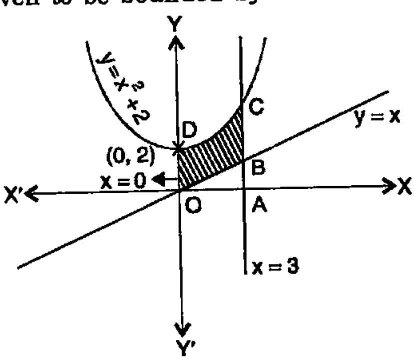
Equation of the given line is $y = x$ ...(*ii*)

We know that it is a straight line passing through the origin and having slope 1 *i.e.*, making an angle of 45° with x-axis.

Table of values for the line $y = x$

x	0	1	2
y	0	1	2

Also the required area is given to be bounded by the vertical lines $x = 0$ to $x = 3$.



$\therefore$ Limits of integration are given to be $x = 0$ to $x = 3$.

Area bounded by parabola (*i*) namely $y = x^2 + 2$, the x-axis and the ordinates $x = 0$ to $x = 3$ is the area OACD and

$$= \int_0^3 y\,dx = \int_0^3 (x^2 + 2)\,dx$$

$$= \left(\frac{x^3}{3} + 2x\right)_0^3 = (9 + 6) - 0 = 15 \qquad ...(iii)$$

Area bounded by line (*ii*) namely $y = x$, the x-axis and the ordinates $x = 0$, $x = 3$ is area OAB and $= \int_0^3 y\,dx = \int_0^3 x\,dx = \left(\frac{x^2}{2}\right)_0^3$

$= \frac{9}{2} - 0 = \frac{9}{2}$ ...(iv)

∴ Required area (shown shaded) *i.e.*, area OBCD

= area OACD − area OAB

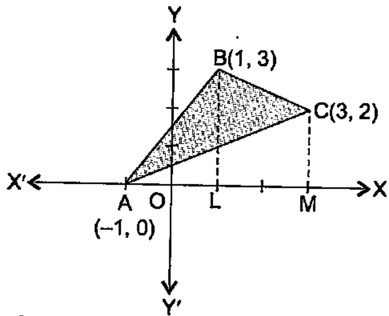
= Area given by (*iii*) − Area given by (*iv*)

$= 15 - \frac{9}{2} = \frac{21}{2}$ sq. units.

Remark: On solving Eqns (*i*) and (*ii*) for x we get imaginary values of x and hence curves (*i*) and (*ii*) don't intersect.

4. Using integration, find the area of the region bounded by the triangle whose vertices are (– 1, 0), (1, 3) and (3, 2).

Sol. Given: Vertices of triangle are A(– 1, 0), B(1, 3) and C(3, 2).



∴ Equation of line AB is

$y - 0 = \frac{3-0}{1-(-1)}(x - (-1))$

$\left(y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\right)$

or $y = \frac{3}{2}(x + 1)$

∴ Area of ΔABL = Area bounded by this line AB and x-axis

$= \left|\int_{-1}^{1} y\,dx\right|$

(∵ At point A, $x = -1$ and at point B, $x = 1$)

$= \left|\int_{-1}^{1} \frac{3}{2}(x+1)\,dx\right| = \frac{3}{2}\left|\int_{-1}^{1} (x+1)\,dx\right|$

$= \frac{3}{2}\left|\left(\frac{x^2}{2}+x\right)_{-1}^{1}\right| = \frac{3}{2}\left[\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)\right]$

$= \frac{3}{2}\left(\frac{3}{2}-\left(-\frac{1}{2}\right)\right) = \frac{3}{2}\left(\frac{3}{2}+\frac{1}{2}\right) = \frac{3}{2}\cdot\frac{4}{2} = 3$ ...(i)

Again equation of line BC is

$y - 3 = \frac{2-3}{3-1}(x - 1)$

$\Rightarrow y - 3 = -\frac{1}{2}(x - 1) \Rightarrow y = 3 - \left(\frac{x-1}{2}\right) = \frac{6 - x + 1}{2}$

$\Rightarrow \quad y = \frac{7-x}{2} = \frac{1}{2}(7 - x)$

∴ Area of trapezium BLMC = Area bounded by line BC and x-axis

$$= \left|\int_1^3 y\,dx\right| = \left|\int_1^3 \frac{1}{2}(7-x)\,dx\right|$$

$$= \frac{1}{2}\left(7x - \frac{x^2}{2}\right)_1^3 = \frac{1}{2}\left[21 - \frac{9}{2} - \left(7 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left(21 - \frac{9}{2} - 7 + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{42 - 9 - 14 + 1}{2}\right) = \frac{1}{4}(20)$$

$$= 5 \qquad \text{...}(ii)$$

Again equation of line AC is

$$y - 0 = \frac{2-0}{3-(-1)}(x - (-1)) \Rightarrow y = \frac{2}{4}(x+1)$$

$$\Rightarrow \quad y = \frac{1}{2}(x+1)$$

∴ Area of ΔACM = Area bounded by line AC and x-axis

$$= \left|\int_{-1}^3 y\,dx\right| = \left|\int_{-1}^3 \frac{1}{2}(x+1)\,dx\right| = \frac{1}{2}\left(\frac{x^2}{2} + x\right)_{-1}^3$$

$$= \frac{1}{2}\left[\frac{9}{2} + 3 - \left(\frac{1}{2} - 1\right)\right] = \frac{1}{2}\left[\frac{9}{2} + 3 - \frac{1}{2} + 1\right]$$

$$= \frac{1}{2}\left[\frac{9+6-1+2}{2}\right] = \frac{16}{4} = 4 \qquad \text{...}(iii)$$

We can observe from the figure that required area of ΔABC

= Area of ΔABL + Area of Trapezium BLMC – Area of ΔACM

= 3 + 5 – 4 = 4 sq. units.

By (*i*) By (*ii*) By (*iii*)

5. Using integration, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Sol. Equation of one side of triangle is $y = 2x + 1$ ...(*i*)

Equation of second side of triangle is $y = 3x + 1$ ...(*ii*)

Third side of triangle is $x = 4$. ...(*iii*)

It is a line parallel to y-axis at a distance 4 to right of y-axis.

Let us solve (*i*) and (*ii*) for x and y.

Eqn. (*ii*) – eqn. (*i*)

gives $x = 0$.

Put $x = 0$ in (*i*), $y = 1$.

∴ Point of intersection of lines (*i*) and (*ii*) is A(0, 1)

Putting $x = 4$ from (*iii*) in (*i*), $y = 8 + 1 = 9$

∴ Point of intersection of lines (*i*) and (*iii*) is B(4, 9).

Putting $x = 4$ from (*iii*) in (*ii*), $y = 12 + 1 = 13$.

$\therefore$ Point of intersection of lines (*ii*) and (*iii*) is C(4, 13).

Area between line (*ii*) *i.e.*, line AC and x-axis

$$= \int_0^4 y\,dx = \int_0^4 (3x+1)\,dx \quad \text{[By (ii)]}$$

$$= \left(\frac{3x^2}{2} + x\right)_0^4$$

$= 24 + 4 = 28$ sq. units ...(*iv*)

Area between line (*i*) *i.e.*, line AB and x-axis

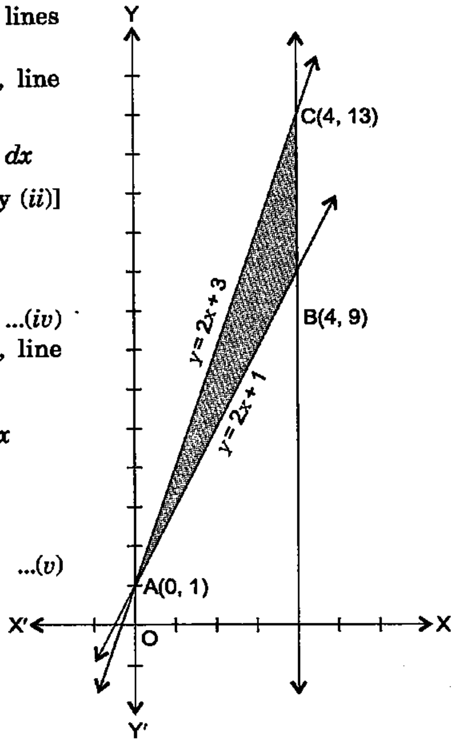
$$= \int_0^4 y\,dx = \int_0^4 (2x+1)\,dx$$

$$= \left(x^2 + x\right)_0^4 \quad \text{[By (i)]}$$

$= 16 + 4 = 20$ sq. units ...(*v*)

$\therefore$ Area of triangle ABC = Area given by (*iv*) – Area given by (*v*)

$= 28 - 20 = 8$ sq. units.



6. Choose the correct answer:

Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(A) $2(\pi - 2)$ (B) $\pi - 2$ (C) $2\pi - 1$ (D) $2(\pi + 2)$.

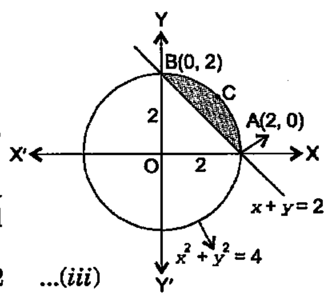
Sol. **Step I.** Equation of circle is $x^2 + y^2 = 4 = 2^2$ ...(*i*)

$\therefore \quad y^2 = 2^2 - x^2$

$\therefore \quad y = \sqrt{2^2 - x^2}$ ...(*ii*)

for arc AB of the circle in first quadrant.

We know that eqn. (*i*) represents a circle whose centre is origin and radius is 2.



Equation of the line is $x + y = 2$ ...(*iii*)

Table of values

x	0	2
y	2	0

$\therefore$ Graph of equation (*iii*) is the straight line joining the points (0, 2) and (2, 0).

The region for required area is shown as shaded in the figure.

Step II. From the graphs of circle (*i*) and straight line (*iii*), it is clear that points of intersections of circle (*i*) and straight line (*iii*) are A(2, 0) and B(0, 2).

Step III. Area OACB, bounded by circle (*i*) and coordinate axes in first quadrant

$= \left|\int_0^2 y\,dx\right| = \int_0^2 \sqrt{2^2 - x^2}\,dx$ ($\because$ From (*ii*), $y = \sqrt{2^2 - x^2}$)

$= \left(\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_0^2$

$\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$= \left(\frac{2}{2}\sqrt{4-4} + 2\sin^{-1} 1\right) - (0 + 2\sin^{-1} 0)$

$= 0 + 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$ ...(*iv*)

Step IV. Area of triangle OAB, bounded by straight line (*iii*) and co-ordinate axes

$= \left|\int_0^2 y\,dx\right| = \left|\int_0^2 (2 - x)\,dx\right|$ ($\because$ From (*iii*), $y = 2 - x$)

$= \left(2x - \frac{x^2}{2}\right)_0^2 = (4 - 2) - (0 - 0) = 2$ ...(*v*)

Step V. $\therefore$ Required shaded area

= Area OACB given by (*iv*) – Area of triangle OAB by (*v*)

= $(\pi - 2)$ sq. units.

$\therefore$ Option (B) is the correct answer.

7. Choose the correct answer:

Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

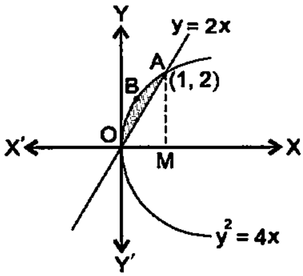
(A) $\frac{2}{3}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{3}{4}$.

Sol. Step I. Equation of one curve (parabola) is

$y^2 = 4x$ ...(*i*)

$\therefore$ $y = \sqrt{4x} = 2\sqrt{x} = 2x^{\frac{1}{2}}$ ...(*ii*)

for arc of the parabola in first quadrant.



We know that eqn. (*i*) represents a rightward parabola symmetrical about *x*-axis.

Equation of second curve (line) is $y = 2x$ ...(*iii*)

We know that $y = 2x$ represents a straight line passing through the origin.

We are required to find the area of the shaded region.

II. Let us solve (*i*) and (*iii*) for x and y.

Putting $y = 2x$ from (*iii*) in (*i*), we have

$$4x^2 = 4x \Rightarrow 4x^2 - 4x = 0 \Rightarrow 4x(x-1) = 0$$

∴ Either $4x = 0$ or $x - 1 = 0$

i.e., $x = \frac{0}{4} = 0$ or $x = 1$

When $x = 0$, from (*ii*), $y = 0$ ∴ point is O(0, 0)

When $x = 1$, from (*ii*), $y = 2x = 2$ ∴ point is A(1, 2)

∴ Points of intersections of circle (*i*) and line (*ii*) are O(0, 0) and A(1, 2).

III. Area OBAM = Area bounded by parabola (*i*) and x-axis

$$= \left|\int_0^1 y\,dx\right| = \left|\int_0^1 2x^{\frac{1}{2}}\,dx\right| \quad [\because \text{ From } (ii)\ y = 2x^{\frac{1}{2}}]$$

$$= 2\frac{\left(x^{\frac{3}{2}}\right)_0^1}{\frac{3}{2}} = \frac{4}{3}(1-0) = \frac{4}{3} \qquad ...(iv)$$

IV. Area of ΔOAM = Area of bounded by line (*iii*) and x-axis

$$= \left|\int_0^1 y\,dx\right| = \left|\int_0^1 2x\,dx\right| \quad (\because \text{ From } (iii)\ y = 2x)$$

$$= 2\left(\frac{x^2}{2}\right)_0^1 = \left(x^2\right)_0^1 = 1 - 0 = 1 \qquad ...(v)$$

V. ∴ Required shaded area OBA

= Area OBAM – Area of Δ OAM

$$= \frac{4}{3} - 1 = \frac{4-3}{3} = \frac{1}{3} \text{ sq. units.}$$

(By (*iv*)) (By (*v*))

∴ Option (B) is the correct answer.

MISCELLANEOUS EXERCISE *(Page No.: 375-376)*

1. Find the area under the given curves and given lines:

(*i*) $y = x^2$, $x = 1$, $x = 2$ and x-axis.

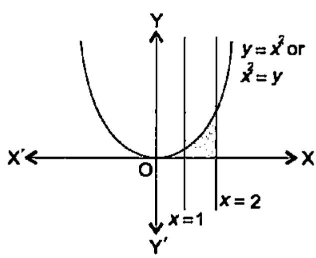
(*ii*) $y = x^4$, $x = 1$, $x = 5$ and x-axis.

Sol. (*i*) Equation of the curve (parabola) is

$y = x^2$ *i.e.,* $x^2 = y$ ...(*i*)

It is an upward parabola symmetrical about y-axis.

[∵ Changing x to $-x$ in (*i*) keeps it unchanged]



Required area bounded by curve (*i*) $y = x^2$, vertical lines

$x = 1, x = 2$ and x-axis

$$= \left| \int_1^2 y\,dx \right| = \left| \int_1^2 x^2\,dx \right| \qquad \text{(By (i))}$$

$$= \left| \left(\frac{x^3}{3} \right)_1^2 \right| = \left| \frac{8}{3} - \frac{1}{3} \right| = \frac{7}{3} \text{ sq. units.}$$

(ii) Equation of the curve is $y = x^4$ ...(i)

$\Rightarrow$ $y \geq 0$ for all real x ($\because$ Power of x is Even (4))

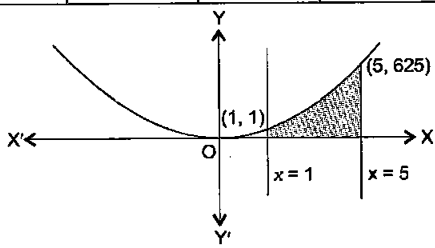
Curve (i) is symmetrical about y-axis.

[$\because$ On changing x to $-x$ in (i), eqn. (i) remains unchanged]

Clearly, curve (i) passes through the origin because for $x = 0$, from (i) $y = 0$.

Table of values for curve $y = x^4$ for $x = 1$ to $x = 5$ (given)

x	1	2	3	4	5
y	1	$2^4 = 16$	$3^4 = 81$	$4^4 = 256$	$5^4 = 625$



Required shaded area between the curve $y = x^4$, vertical lines $x = 1, x = 5$ and x-axis

$$= \left| \int_1^5 y\,dx \right| = \left| \int_1^5 x^4\,dx \right| \qquad \text{(By (i))}$$

$$= \left| \left(\frac{x^5}{5} \right)_1^5 \right| = \frac{5^5}{5} - \frac{1^5}{5} = \frac{3125 - 1}{5} = \frac{3124}{5}$$

$$= \frac{3124 \times 2}{10} = 624.8 \text{ sq. units.}$$

2. Find the area between the curves $y = x$ and $y = x^2$.

Sol. Step I. To draw the graphs and region of integration.

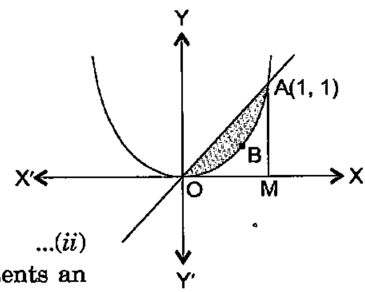
Equation of one curve (straight line) is

$y = x$ ...(i)

We know that graph of eqn. (i) is a straight line passing through origin.

Equation of second curve (parabola) is

$y = x^2$ or $x^2 = y$ ...(ii)



We know that equation (ii) represents an upward parabola with symmetry about y-axis.

Step II. Let us find points of intersections of curves (*i*) and (*ii*) by solving them for x and y.

Putting $y = x$ from (*i*) in (*ii*), we have

$$x = x^2 \quad \text{or} \quad x - x^2 = 0 \quad \text{or} \quad x(1 - x) = 0$$

$\therefore$ Either $x = 0$ or $1 - x = 0$ *i.e.*, $x = 1$.

When $x = 0$, from (*i*) $y = 0$ $\therefore$ Point is O(0, 0)

When $x = 1$, from (*i*), $y = 1$ $\therefore$ Point is A(1, 1)

$\therefore$ Points of intersections of line (*i*) and parabola (*ii*) are O(0, 0) and A(1, 1).

Step III. Area of triangle OAM

= Area bounded by line (*i*) and x-axis

$$= \left|\int_0^1 y\,dx\right| = \left|\int_0^1 x\,dx\right| \qquad (\because \text{ From } (i)\ y = x)$$

$$= \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2} \qquad ...(iii)$$

Step IV. Area OBAM = Area bounded by parabola (*ii*) and x-axis

$$= \left|\int_0^1 y\,dx\right| = \left|\int_0^1 x^2\,dx\right| \qquad (\because \text{ From } (ii)\ y = x^2)$$

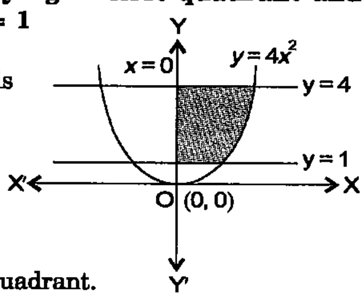
$$= \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3} - 0 = \frac{1}{3} \qquad ...(iv)$$

$\therefore$ Required shaded area OBA between line (*i*) and parabola (*ii*)

= Area of Δ OAM – Area OBAM

$$= \underset{(\text{By } (iii))}{\frac{1}{2}} - \frac{1}{3} = \frac{3-2}{6} = \underset{(\text{By } (iv))}{\frac{1}{6}} \text{ sq. units.}$$

3. Find the area of the region lying in first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.



Sol. Equation of the (parabola) curve is

$$y = 4x^2$$

$$\Rightarrow \quad x^2 = \frac{y}{4} \qquad ...(i)$$

$$\therefore \quad x = \sqrt{\frac{y}{4}} = \frac{\sqrt{y}}{2} \qquad ...(ii)$$

For branch of parabaola in first quadrant.

We know that this equation represents an upward parabola with symmetry about y-axis.

$\therefore$ Required shaded area of the region lying in first quadrant bounded by parabola (*i*), $x = 0$ ($\Rightarrow$ y-axis) and the horizontal lines $y = 1$ and $y = 4$ is

$$\left|\int_1^4 x\,dy\right| = \left|\int_1^4 \frac{\sqrt{y}}{2}\,dy\right| \qquad \text{(By (ii))}$$

$$= \frac{1}{2}\left|\int_1^4 y^{\frac{1}{2}}\,dy\right| = \frac{1}{2}\left|\frac{\left(y^{\frac{3}{2}}\right)_1^4}{\frac{3}{2}}\right| = \frac{1}{2}\cdot\frac{2}{3}\,[4^{\frac{3}{2}} - 1^{\frac{3}{2}}]$$

$$= \frac{1}{3}(4\sqrt{4} - 1) = \frac{1}{3}(8-1) = \frac{7}{3} \text{ sq. units.}$$

4. Sketch the graph of $y = |x + 3|$ and evaluate $\int_{-6}^{0} |x+3|\,dx$.

Sol. Equation of the given curve is

$$y = |x + 3| \qquad ...(i)$$

We know that from (*i*),

$$y = |x + 3| \geq 0 \text{ for all real } x.$$

∴ Graph of curve is only above the x-axis *i.e.,* in first and second quadrants only.

From (*i*), $y = |x + 3| = x + 3$ if $x + 3 \geq 0$ *i.e.,* if $x \geq -3$ ...(*ii*)

and $y = |x + 3| = -(x + 3)$ if $x + 3 \leq 0$ *i.e.,* if $x \leq -3$ ...(*iii*)

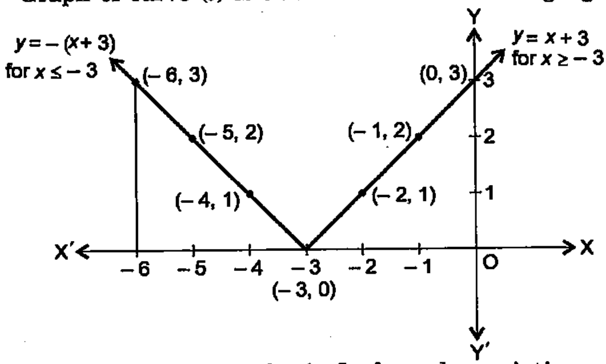
Table of values for $y = x + 3$ for $x \geq -3$

x	-3	-2	-1	0
y	0	1	2	3

Table of values for $y = -(x + 3)$ for $x \leq -3$

x	-3	-4	-5	-6
y	0	1	2	3

∴ Graph of curve (*i*) is as shown in the following figure.



∴ Graph of $y = |x + 3|$ is L-shaped consisting of two rays above the x-axis at right angles to each other.

Now, $\int_{-6}^{0} |x+3|\,dx = \int_{-6}^{-3} |x+3|\,dx + \int_{-3}^{0} |x+3|\,dx$

[∵ On putting expression $x + 3$ within modulus equal to zero, we get $x = -3$ and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$]

$= \int_{-6}^{-3} -(x+3)\,dx \quad + \quad \int_{-3}^{0} (x+3)\,dx$

(By (*iii*) because on $(-6,-3), x<-3 \Rightarrow x+3<0$) (By (*ii*) because on $(-3,0), x>-3 \Rightarrow x+3>0$)

$$= -\left(\frac{x^2}{2}+3x\right)_{-6}^{-3} + \left(\frac{x^2}{2}+3x\right)_{-3}^{0}$$

$$= -\left[\frac{9}{2}-9-(18-18)\right] + \left[0-\left(\frac{9}{2}-9\right)\right]$$

$$= -\frac{9}{2}+9+0+0-\frac{9}{2}+9 = 18-\frac{18}{2} = 18-9 = 9 \text{ sq. units.}$$

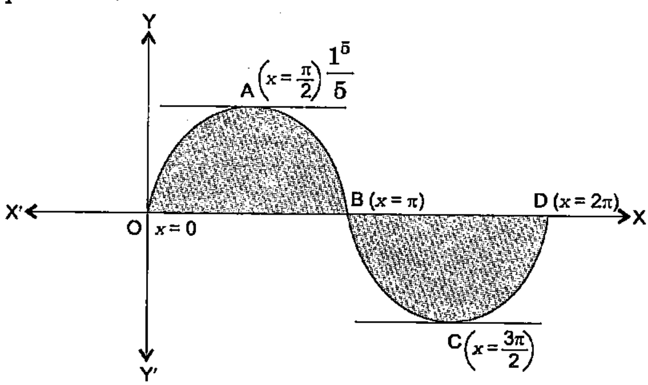
5. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

Sol. Equation of the curve is $y = \sin x$ ...(*i*)

Let us draw the graph of $y = \sin x$ from $x = 0$ to $x = 2\pi$

Now we know that $y = \sin x \geq 0$ for $0 \leq x \leq \pi$ *i.e.*, in first and second quadrants

and $y = \sin x \leq 0$ for $\pi \leq x \leq 2\pi$ *i.e.*, in third and fourth quadrants.



To find points where tangent is parallel to x-axis, put $\frac{dy}{dx} = 0$.

$\Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}, x = \frac{3\pi}{2}$

Table of values for curve $y = \sin x$ between $x = 0$ and $x = 2\pi$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	1	0	-1	0

[$\because$ $\sin n\pi = 0$ for every integer n

and $\sin\frac{3\pi}{2} = \sin 270° = \sin(180° + 90°) = -\sin 90° = -1$]

Required shaded area = Area OAB + Area BCD

$$= \left| \int_0^{\pi} y \, dx \right| + \left| \int_{\pi}^{2\pi} y \, dx \right|$$

[Here we will have to find area OAB and Area BCD separately because $y = \sin x \geq 0$ for $0 \leq x \leq \pi$
and $\quad y = \sin x \leq 0$ for $\pi \leq x \leq 2\pi$]

Putting $y = \sin x$ from (*i*),

$$= \left| \int_0^{\pi} \sin x \, dx \right| + \left| \int_{\pi}^{2\pi} \sin x \, dx \right|$$

$$= \left| -(\cos x)_0^{\pi} \right| + \left| -(\cos x)_{\pi}^{2\pi} \right|$$

$$= | -(\cos \pi - \cos 0) | + | -(\cos 2\pi - \cos \pi) |$$

$$= | -(-1 - 1) | + | -(1 + 1) |$$

[$\because$ $\cos n\pi = (-1)^n$ for every integer n
putting $n = 1, 2$; $\cos \pi = -1$, $\cos 2\pi = 1$]

$= 2 + 2 = 4$ sq. units.

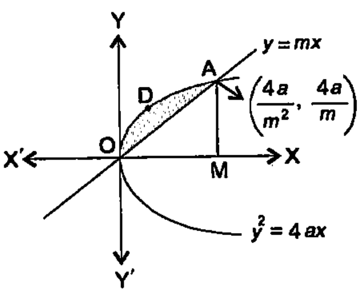
6. Find the area enclosed by the parabola $y^2 = 4ax$ and the line $y = mx$.

Sol. Step I. To draw the graphs and shade the region of integration.

Equation of the parabola is

$$y^2 = 4ax \qquad ...(i)$$

(It is rightward parabola with symmetry about x-axis)



From (*i*),

$$y = \sqrt{4ax} = 2\sqrt{a}\, x^{\frac{1}{2}} \qquad ...(ii)$$

for arc ODA of parabola in first quadrant.

Equation of the line is $y = mx \qquad ...(iii)$

We know that eqn. (*iii*) represents a straight line passing through the origin.

Step II. To find points of intersections of curves (*i*) and (*iii*), let us solve (*i*) and (*iii*) for x and y

Putting $y = mx$ from (*iii*) in (*i*),

$$m^2x^2 = 4ax \Rightarrow m^2x^2 - 4ax = 0$$

$\Rightarrow \quad x(m^2x - 4a) = 0$

$\Rightarrow$ Either $\quad x = 0$ or $m^2x - 4a = 0$ *i.e.*, $m^2x = 4a$

$\Rightarrow \quad x = 0$ or $x = \dfrac{4a}{m^2}$

When $x = 0$, from (*iii*) $y = 0$ $\therefore$ Point is O(0, 0)

When $x = \dfrac{4a}{m^2}$, $y = m \cdot \dfrac{4a}{m^2} = \dfrac{4a}{m}$

$\therefore$ Second point of intersection of parabola (*i*) and line (*iii*) is

$$A\left(\frac{4a}{m^2}, \frac{4a}{m}\right).$$

Step III. Area ODAM = Area parabola (i) and x-axis

$$= \left|\int_0^{\frac{4a}{m^2}} y\,dx\right| \left(\because \text{ At O}, x = 0 \text{ and at A}, x = \frac{4a}{m^2}\right)$$

Putting $y = 2\sqrt{a}\, x^{\frac{1}{2}}$ from (ii),

$$= \left|\int_0^{\frac{4a}{m^2}} 2\sqrt{a}\, x^{\frac{1}{2}}\, dx\right| = 2\sqrt{a}\ \frac{\left(x^{\frac{3}{2}}\right)_0^{\frac{4a}{m^2}}}{\frac{3}{2}}$$

$$= \frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{\frac{3}{2}} = 4\frac{\sqrt{a}}{3}\ \frac{4a}{m^2}\sqrt{\frac{4a}{m^2}}\quad [\because x^{\frac{3}{2}} = x\sqrt{x}\,]$$

$$= \frac{4\sqrt{a}}{3}\ \frac{4a}{m^2}\ 2\frac{\sqrt{a}}{m} = \frac{32a^2}{3m^3} \qquad ...(iv)$$

Step IV. Area of ΔOAM = Area between line (iii) and x-axis

$$= \left|\int_0^{\frac{4a}{m^2}} y\,dx\right|$$

Putting $y = mx$ from (iii), $\left|\int_0^{\frac{4a}{m^2}} mx\,dx\right|$

$$= m\left(\frac{x^2}{2}\right)_0^{\frac{4a}{m^2}} = \frac{m}{2}\left[\left(\frac{4a}{m^2}\right)^2 - 0\right]$$

$$= \frac{m}{2}\cdot\frac{16a^2}{m^4} = \frac{8a^2}{m^3} \qquad ...(v)$$

Step V. Required shaded area

= Area ODAM given by (iv)

– Area of ΔOAM given by (v)

$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{a^2}{m^3}\left(\frac{32}{3} - 8\right)$$

$$= \frac{a^2}{m^3}\left(\frac{32-24}{3}\right) = \frac{a^2}{m^3}\cdot\frac{8}{3} = \frac{8a^2}{3m^3}.$$

7. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

Sol. Equation of the parabola is $4y = 3x^2$ ...(i)

or $x^2 = \frac{4}{3}y$

It is an upward parabola with vertex at the origin and is symmetrical about y-axis.

Equation of the line is

$2y = 3x + 12$ ...(ii)

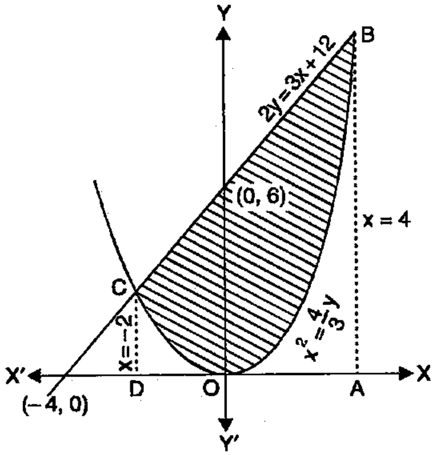
Putting $y = 0$ in (ii), $x = -4$

$\therefore$ $(-4, 0)$ is a point on line (ii)

Putting $x = 0$ in (ii), $y = 6$

$\therefore$ $(0, 6)$ is also a point on line (ii).

Joining the points $(-4, 0)$ and $(0, 6)$ we get the graph of line (ii).



To find the points of intersections, let us solve eqns. (i) and (ii), for x and y.

Putting $y = \dfrac{3x+12}{2}$ from (ii) in (i), we have

$2(3x + 12) = 3x^2$ or $3x^2 - 6x - 24 = 0$ or $x^2 - 2x - 8 = 0$

or $(x - 4)(x + 2) = 0$ $\therefore$ $x = 4, -2$.

When $x = 4$, $y = \dfrac{3x+12}{2} = 12$; When $x = -2$, $y = \dfrac{3x+12}{2} = 3$.

$\therefore$ The points of intersection are B(4, 12) and C(– 2, 3).

Area bounded by the line (ii) namely $2y = 3x + 12$ or $y = \dfrac{3}{2}x + 6$, the x-axis and the ordinates $x = -2$, $x = 4$ is ABCD

$$= \int_{-2}^{4} y\,dx = \int_{-2}^{4} \left(\frac{3}{2}x + 6\right) dx = \left[\frac{3}{4}x^2 + 6x\right]_{-2}^{4}$$

$$= (12 + 24) - (3 - 12) = 45 \qquad ...(iii)$$

Area bounded by the curve (i) namely $y = \dfrac{3}{4}x^2$, the x-axis and the ordinates $x = -2$, $x = 4$ is (area CDO + area OAB)

$$= \int_{-2}^{4} y\,dx = \int_{-2}^{4} \frac{3}{4}x^2\,dx = \left[\frac{3}{4}\cdot\frac{x^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{4}[64 - (-8)] = 18. \qquad ...(iv)$$

$\therefore$ Required area (shown shaded)

= (Area under the line – Area under the curve) between the lines $x = -2$ and $x = 4$

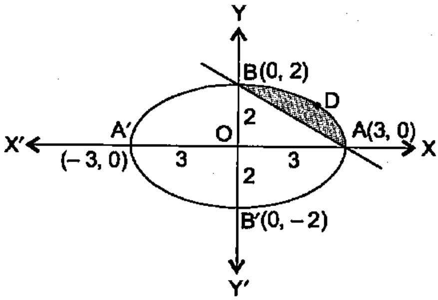
= Area given by (iii) – Area given by (iv)

= 45 – 18 = 27 sq. units.

8. Find the area of the smaller region bounded by the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and the line $\dfrac{x}{3} + \dfrac{y}{2} = 1$.

Sol. Step I. Equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad ..(i)$$



Clearly, ellipse (*i*) is symmetrical about both axes.

Intersections of ellipse (*i*) with *x*-axis.

Put $y = 0$ in (*i*),

$$\frac{x^2}{9} = 1 \Rightarrow x^2 = 9 \qquad \therefore \quad x = \pm 3$$

$\therefore$ Intersections of ellipse (*i*) with *x*-axis are A(3, 0) and A′(– 3, 0).

Similarly, intersections of ellipse (*i*) with *y*-axis (putting $x = 0$ in (*i*)) are B(0, 2) and B′(0, – 2).

Equation of the line is $\frac{x}{3} + \frac{y}{2} = 1$

$$\Rightarrow \quad \frac{y}{2} = 1 - \frac{x}{3} \Rightarrow y = 2\left(\frac{3-x}{3}\right) \quad ...(ii)$$

Table of values

x	0	3
y	2	0

$\therefore$ Graph of line (*ii*) is the line joining the points (0, 2) and (3, 0).

We have shaded the smaller region whose area is required.

Step II. From the graphs, it is clear that points of intersections of ellipse (*i*) and straight line (*ii*) are A(3, 0) and B(0, 2).

Step III. Area OADB = Area between ellipse (*i*) (arc AB of it) and *x*-axis

$$= \left|\int_0^3 y\,dx\right| \qquad \left[\text{From } (i),\ \frac{y^2}{4} = 1 - \frac{x^2}{9} = \frac{9-x^2}{9}\right.$$

$$\left.\Rightarrow y^2 = \frac{4}{9}(9 - x^2) \Rightarrow y = \frac{2}{3}\sqrt{9-x^2}\right]$$

(At point B, $x = 0$ and at point A, $x = 3$)

$$= \left|\int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx\right| = \frac{2}{3}\left|\int_0^3 \sqrt{3^2-x^2}\,dx\right|$$

$$= \frac{2}{3}\left|\left[\frac{x}{2}\sqrt{3^2-x^2} + \frac{3^2}{2}\sin^{-1}\frac{x}{3}\right]_0^3\right|$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{2}{3}\left[\frac{3}{2}\sqrt{9-9} + \frac{9}{2}\sin^{-1}1 - \left(0 + \frac{9}{2}\sin^{-1}0\right)\right]$$

$$= \frac{2}{3}\left[0+\frac{9}{2}\cdot\frac{\pi}{2}-0\right] = \frac{2}{3}\cdot\frac{9\pi}{4} = \frac{3\pi}{2} \quad ...(iii)$$

Step IV. Area of triangle OAB = Area bounded by line AB and x-axis

$$= \left|\int_0^3 y\,dx\right| = \int_0^3 \left|\frac{2}{3}(3-x)\,dx\right| \quad \text{[From (ii)]}$$

$$= \frac{2}{3}\left|\left(3x-\frac{x^2}{2}\right)_0^3\right| = \frac{2}{3}\left(\left(9-\frac{9}{2}\right)-0\right) = \frac{2}{3}\cdot\frac{9}{2} = 3 \quad ...(iv)$$

Step V. $\therefore$ Required shaded area

= Area OADB – Area OAB

$$= \frac{3\pi}{2} - 3$$

(By (iii)) (By (iv))

$$= 3\left(\frac{\pi}{2}-1\right) = \frac{3}{2}(\pi-2) \text{ sq. units.}$$

9. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and the line $\frac{x}{a}+\frac{y}{b} = 1$.

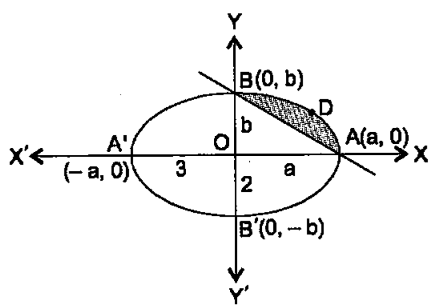
Sol. Step I. Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ ...(i)

Ellipse (i) is symmetrical about both the axes.

Intersections of ellipse (i) with x-axis ($y = 0$) are A(a, 0) and A′($-a$, 0).

Intersections of ellipse (i) with y-axis ($x = 0$) are B(0, b) and B′(0, $-b$)

Again equation of chord AB is



$$\frac{x}{a} + \frac{y}{b} = 1 \quad ...(ii)$$

Table of Values

x	0	a
y	b	0

Step II. From equation (i), $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$

$\therefore y^2 = \frac{b^2(a^2-x^2)}{a^2}$ $\therefore y = \frac{b}{a}\sqrt{a^2-x^2}$ (in first quadrant)

Area between arc AB of the ellipse and x-axis (in first quadrant)

$$= \int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \frac{b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}1 - (0+0)\right]$$

$$= \frac{b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi ab}{4} \qquad ...(iii)$$

Step III. From equation (*ii*), $\frac{y}{b} = 1 - \frac{x}{a} = \frac{a-x}{a}$

$\therefore \quad y = \frac{b}{a}(a - x)$

$\therefore$ Area between chord AB and x-axis

$$= \int_0^a y\,dx = \int_0^a \frac{b}{a}(a-x)\,dx = \frac{b}{a}\int_0^a (a-x)\,dx$$

$$= \frac{b}{a}\left[ax - \frac{x^2}{2}\right]_0^a = \frac{b}{a}\left(a^2 - \frac{a^2}{2}\right)$$

$$= \frac{b}{a}\cdot\frac{a^2}{2} = \frac{1}{2}ab \qquad ...(iv)$$

Step IV. $\therefore$ Area of smaller region bounded by ellipse (*i*) and straight line (*ii*)

= Area between arc AB and chord AB

= Area given by (*iii*) – Area given by (*iv*)

$$= \frac{\pi ab}{4} - \frac{ab}{2} = \frac{ab}{4}(\pi - 2).$$

10. Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x + 2$ and x-axis.

Sol. Step I. Equation of the parabola is $x^2 = y$ ...(*i*)

We know that equation (*i*) represents an upward parabola with symmetry about y-axis.

($\because$ Changing x to $-x$ in (*i*), keeps it unchanged)

Equation of the line is

$y = x + 2$ ...(*ii*)

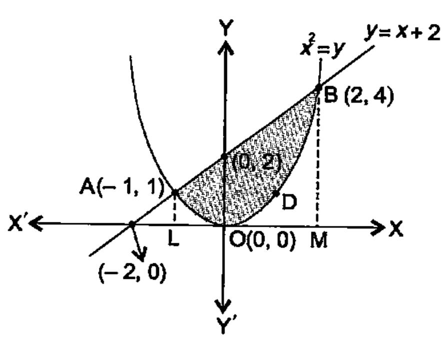


Table of values

x	0	– 2
y	2	0

$\therefore$ Graph of line (*ii*) is the line joining the points (0, 2) and (– 2, 0).

Step II. Let us solve (*i*) and (*ii*) for x and y

Putting $y = x + 2$ from (*ii*) in (*i*),

$$x^2 = x + 2$$

or $\quad x^2 - x - 2 = 0 \quad$ or $\quad x^2 - 2x + x - 2 = 0$

or $\quad x(x - 2) + 1(x - 2) = 0$

$\Rightarrow \quad (x - 2)(x + 1) = 0$

$\therefore$ Either $\quad x - 2 = 0 \quad$ or $\quad x + 1 = 0$

i.e., $\quad x = 2 \quad$ or $\quad x = -1.$

When $x = 2$, from (*i*), $\quad y = x^2 = 2^2 = 4 \quad \therefore$ Point is (2, 4)

When $x = -1$, from (*i*), $y = (-1)^2 = 1 \quad \therefore$ Point is $(-1, 1)$.

$\therefore$ The two points of intersections of parabola (*i*) and line (*ii*) are $A(-1, 1)$ and $B(2, 4)$.

Step III. Area ALODBM = Area bounded by parabola (*i*) and x-axis

$$= \left|\int_{-1}^{2} y\,dx\right| = \left|\int_{-1}^{2} x^2\,dx\right| \qquad [\because \text{ From } (i)\ y = x^2]$$

$$= \left(\frac{x^3}{3}\right)_{-1}^{2} = \frac{8}{3} - \left(\frac{-1}{3}\right) = \frac{8}{3} + \frac{1}{3} = \frac{9}{3} = 3 \qquad ...(iii)$$

Step IV. Area of trapezium ALMB = Area bounded by line (*ii*) and x-axis

$$= \int_{-1}^{2} (x+2)\,dx \qquad [\because \text{ From } (ii)\ y = x + 2]$$

$$= \left(\frac{x^2}{2} + 2x\right)_{-1}^{2} = 2 + 4 - \left(\frac{1}{2} - 2\right) = 6 - \frac{1}{2} + 2$$

$$= 8 - \frac{1}{2} = \frac{15}{2} \qquad ...(iv)$$

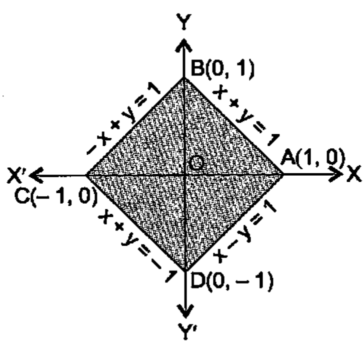
Step V. $\therefore$ Required shaded area = Area of trapezium ALMB − Area ALODBM

$$= \frac{15}{2} - 3 = \frac{9}{2} \text{ sq. units.}$$

11. Using the method of integration, find the area bounded by the curve $|x| + |y| = 1$.

Sol. Given: Equation of the curve (graph) is

$|x| + |y| = 1 \qquad ...(i)$



Curve (*i*) is symmetrical about x-axis.

[$\because$ On changing y to $-y$ in eqn. (*i*), it remains unchanged as we know that $|-y| = |y|$]

Similarly, curve (*i*) is symmetrical about y-axis.

We know that, for first quadrant; $x \geq 0$ and $y \geq 0$

$\Rightarrow \quad |x| = x$ and $|y| = y$

$\therefore$ Eqn. (*i*) becomes $x + y = 1 \qquad ...(ii)$

which is the equation of a straight line.

Table of values

x	0	1
y	1	0

$\therefore$ Graph of $x + y = 1$ is the straight line joining the points (0, 1) and (1, 0).

We know that for second quadrant, $x \le 0$ and $y \ge 0$

$\Rightarrow$ $|x| = -x$ and $|y| = y$

$\therefore$ Equation (*i*) becomes $-x + y = 1$

which represents a straight line.

Table of values

x	0	-1
y	1	0

$\therefore$ Graph of $-x + y = 1$ is the straight line joining the points (0, 1) and (– 1, 0).

We know that for third quadrant, $x \le 0$ and $y \le 0$.

$\Rightarrow$ $|x| = -x$ and $|y| = -y$

$\therefore$ Eqn. (*i*) becomes $-x - y = 1$ or $x + y = -1$

which represents a straight line.

Table of values

x	0	-1
y	-1	0

$\therefore$ Graph of $x + y = -1$ is the straight line joining the points (0, – 1) and (– 1, 0).

We know that for fourth quadrant $x \ge 0$ and $y \le 0$.

$\Rightarrow$ $|x| = x$ and $|y| = -y$

$\Rightarrow$ Equation (*i*) becomes $x - y = 1$ which again represents a straight line.

Table of values

x	0	1
y	-1	0

$\therefore$ Graph of $x - y = 1$ is the straight line joining the points (0, – 1) and (1, 0).

$\therefore$ Graph of Eqn. (*i*) is the square ABCD.

$\therefore$ Area bounded by curve (*i*)

= Area of square ABCD

$= 4 \times \Delta OAB$

$= 4 \times$ Area bounded by line (*ii*) namely $x + y = 1$ and the coordinate axes

$$= 4 \left| \int_0^1 y\, dx \right| = 4 \left| \int_0^1 (1 - x)\, dx \right|$$

$[\because\ x + y = 1 \ \Rightarrow\ y = 1 - x]$

$$= 4 \left(x - \frac{x^2}{2} \right)_0^1 = 4\left[\left(1 - \frac{1}{2}\right) - 0\right] = 4 \times \frac{1}{2} = 2 \text{ sq. units.}$$

12. Find the area bounded by the curves

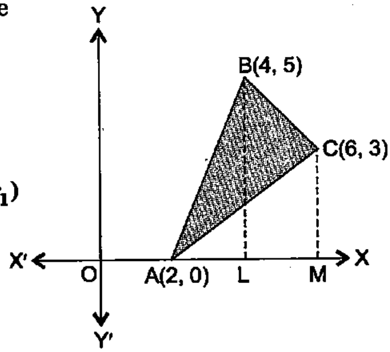
$$\{(x, y) : y \ge x^2 \text{ and } y = |x|\}$$

Sol. It is same as Q. No. 9, Exercise 8.1, page 558.

13. Using the method of integration find the area of the triangle whose vertices are A(2, 0), B(4, 5) and C(6, 3).

Sol. Vertices of the given triangle are

A(2, 0), B(4, 5) and C(6, 3).



Now, equation of side AB is

$$y - 0 = \frac{5-0}{4-2}(x - 2)$$

$$\left| y - y_1 = \frac{(y_2 - y_1)}{x_2 - x_1}(x - x_1) \right.$$

$\Rightarrow$ $$y = \frac{5}{2}(x - 2)$$

$\therefore$ Area of ΔALB bounded by line AB and x-axis

$$= \left| \int_2^4 y\,dx \right| = \left| \int_2^4 \frac{5}{2}(x-2)\,dx \right| = \frac{5}{2}\left(\frac{x^2}{2} - 2x\right)_2^4$$

$$= \frac{5}{2}[(8 - 8) - (2 - 4)] = \frac{5}{2}(0 + 2)$$

$$= \frac{5}{2} \times 2 = 5 \text{ sq. units.} \quad ...(i)$$

Again equation of side BC is

$$y - 5 = \frac{3-5}{6-4}(x - 4) \qquad \left| y - y_1 = \frac{(y_2 - y_1)}{x_2 - x_1}(x - x_1) \right.$$

$\Rightarrow$ $y - 5 = -(x - 4)$

$\Rightarrow$ $y = 5 - x + 4 = 9 - x$

$\therefore$ Area of trapezium BLMC bounded by line BC and x-axis

$$= \left| \int_4^6 y\,dx \right| = \left| \int_4^6 (9 - x)\,dx \right| = \left| \left(9x - \frac{x^2}{2}\right)_4^6 \right|$$

$$= |\, 54 - 18 - (36 - 8)\,| = |\, 36 - 36 + 8\,|$$

$$= 8 \quad ...(ii)$$

Again equation of line AC is

$$y - 0 = \frac{3-0}{6-2}(x - 2) \Rightarrow y = \frac{3}{4}(x - 2)$$

$\therefore$ Area of ΔAMC bounded by line AC and x-axis

$$= \left| \int_2^6 y\,dx \right| = \left| \int_2^6 \frac{3}{4}(x-2)\,dx \right|$$

$$= \frac{3}{4}\left(\frac{x^2}{2} - 2x\right)_2^6 = \frac{3}{4}[18 - 12 - (2 - 4)]$$

$$= \frac{3}{4}(6 + 2) = \frac{3}{4} \times 8 = 6 \text{ sq. units.} \quad ...(iii)$$

We observe from the above figure that

Area of ΔABC = Area of ABL + Area of trapezium BLMC – Area of AMC

$= \underset{(\text{by } (i))}{5} + \underset{(\text{by } (ii))}{8} - \underset{(\text{by } (iii))}{6}$

= 7 sq. units.

14. Using the method of integration find the area of the region bounded by lines:

$$2x + y = 4,\ 3x - 2y = 6 \quad \text{and} \quad x - 3y + 5 = 0$$

Sol. Equation of one line is $2x + y = 4$ ...(*i*)

Equation of second line is $3x - 2y = 6$ ...(*ii*)

Equation of third line is $x - 3y + 5 = 0$ ...(*iii*)

Let ABC be triangle (region) bounded by the given lines (*i*), (*ii*), (*iii*). Let us find point of intersection A of lines (*i*) and (*ii*) i.e. solve (*i*) and (*ii*) for x and y. Eqn (*i*) × 2 + Eqn (*ii*) gives $4x + 2y + 3x - 2y = 8 + 6$

or $7x = 14$ or $x = 2$

Putting $x = 2$ in (*i*) $4 + y = 4$ $\therefore y = 0$

$\therefore$ point A is (2,0)

Let us find point of intersection B of lines (*ii*) and (*iii*) *i.e.,* solve (*ii*) and (*iii*) for x and y.

Eqn. (*ii*) – 3 × eqn. (*iii*) gives

$$3x - 2y - 6 - 3(x - 3y + 5) = 0$$

i.e., $3x - 2y - 6 - 3x + 9y - 15 = 0$

or $7y - 21 = 0 \Rightarrow 7y = 21$

$\Rightarrow$ $y = 3$

Putting $y = 3$ in (*ii*), $3x - 6 = 6 \Rightarrow 3x = 12 \Rightarrow x = 4$

$\therefore$ Point B is (4, 3).

Let us find point of intersection C of lines (*i*) and (*iii*) *i.e.,* solve (*i*) and (*iii*) for x and y.

Eqn. (*i*) – 2 × eqn. (*iii*) gives

$$2x + y - 4 - 2(x - 3y + 5) = 0$$

$\Rightarrow 2x + y - 4 - 2x + 6y - 10 = 0$

$\Rightarrow 7y - 14 = 0 \Rightarrow 7y = 14 \Rightarrow y = 2$

Putting $y = 2$ in (*i*), $2x + 2 = 4$ or $2x = 2$ or $x = 1$.

$\therefore$ Point C is (1, 2)

$\therefore$ Vertices A, B, C of triangle (region) ABC are A(2, 0), B(4, 3) and C(1, 2).

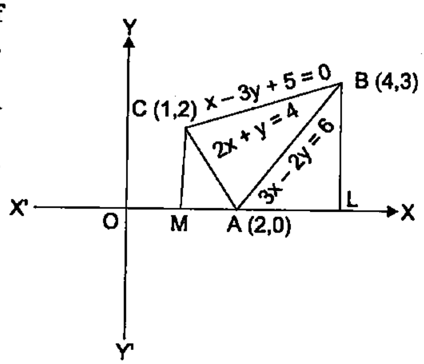
Join of A and C is the graph of line (*i*) $2x + y = 4$.

($\because$ (*i*) intersects (*ii*) at A and (*iii*) at C)

Similarly A B and BC.

Now area of ΔACM bounded by line (*i*) *i.e.,* AC and x-axis.

$$= \left| \int_1^2 y\,dx \right|$$



(At point C, $x = 1$ and at point A, $x = 2$)
Putting $y = 4 - 2x$ from (*i*),

$$= \left| \int_1^2 (4-2x)\,dx \right| = \left| \left(4x - \frac{2x^2}{2}\right)_1^2 \right|$$

$$= (8 - 4) - (4 - 1) = 4 - 3 = 1 \qquad ...(iv)$$

Now area of ΔABL, bounded by line (*ii*) *i.e.*, AB and x-axis

$$= \left| \int_2^4 y\,dx \right| = \left| \int_2^4 \frac{3}{2}(x-2)\,dx \right|$$

[$\because$ From (*ii*), $-2y = -3x + 6$

$\Rightarrow y = \frac{-1}{2}(-3x + 6) = \frac{3}{2}(x - 2)$]

$$= \frac{3}{2}\left| \left(\frac{x^2}{2} - 2x\right)_2^4 \right| = \frac{3}{2}\,|\,(8 - 8) - (2 - 4)\,|$$

$$= \frac{3}{2}(2) = 3 \qquad ...(v)$$

Now area of trapezium CMLB bounded by line (*iii*) *i.e.*, BC and x-axis

$$= \left| \int_1^4 y\,dx \right| = \left| \int_1^4 \frac{1}{3}(x+5)\,dx \right|$$

[$\because$ From (*iii*), $x + 5 = 3y \Rightarrow y = \frac{1}{3}(x + 5)$]

$$= \frac{1}{3}\left| \left(\frac{x^2}{2} + 5x\right)_1^4 \right| = \frac{1}{3}\left[8 + 20 - \left(\frac{1}{2} + 5\right)\right]$$

$$= \frac{1}{3}\left(28 - \frac{11}{2}\right) = \frac{1}{3}\left(\frac{56-11}{2}\right) = \frac{1}{3}\left(\frac{45}{2}\right)$$

$$= \frac{15}{2} \qquad ...(vi)$$

$\therefore$ Required area of region (triangle) bounded by the three given lines

= Area of trapezium CLMB – Area of ΔACM – Area of ΔABL

$$= \underset{\text{(by (vi))}}{\frac{15}{2}} - \underset{\text{(by (iv))}}{1} - \underset{\text{(by (v))}}{3}$$

$$= \frac{15}{2} - 4 = \frac{7}{2} \text{ sq. units.}$$

15. Find the area of the region

$$\{(x, y) : y^2 \le 4x \text{ and } 4x^2 + 4y^2 \le 9\}$$

Sol. The required area is the area common to the interiors of the parabola $y^2 = 4x$ ...(i)

[Parabola (i) is a rightward parabola with vertex at origin and is symmetrical about x-axis.]

and the circle $4x^2 + 4y^2 = 9$ ...(ii)

Dividing every term of eqn. (ii) by 4,

$$x^2 + y^2 = \frac{9}{4} = \left(\frac{3}{2}\right)^2$$

which is a circle whose centre is origin and radius is $\frac{3}{2}$.

This circle is symmetrical about both the axes.

To find the points of intersection, let us solve (i) and (ii) for x and y.

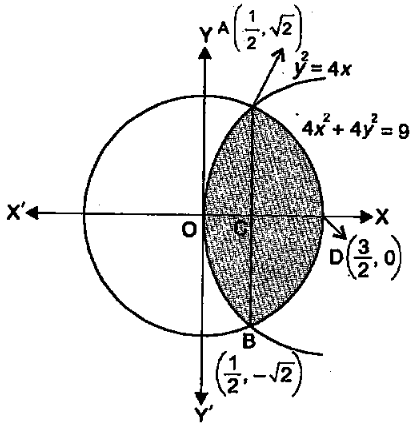
Putting $y^2 = 4x$ from (i) in (ii), we have

$$4x^2 + 16x - 9 = 0$$

$$\therefore \quad x = \frac{-16 \pm \sqrt{256 + 144}}{8}$$

$$\left[\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$= \frac{-16 \pm 20}{8} = \frac{1}{2}, -\frac{9}{2}$$



When $x = -\frac{9}{2}$, from (i), $y^2 = -18$ is negative and hence y is imaginary and hence impossible.

Therefore $x = -\frac{9}{2}$ is rejected.

When $x = \frac{1}{2}$, from (i), $y^2 = 4x = 4 \times \frac{1}{2} = 2$

$\therefore \quad y = \pm\sqrt{2}$

$\therefore$ The two points of intersection of parabola (i) and circle (ii) are

$$A\left(\frac{1}{2}, \sqrt{2}\right) \text{ and } B = \left(\frac{1}{2}, -\sqrt{2}\right).$$

For the parabola (i), $y = 2\sqrt{x}$ in the first quadrant.

For the circle (ii), $4y^2 = 9 - 4x^2$ or $y^2 = \frac{9}{4} - x^2$

or $\quad y = \sqrt{\frac{9}{4} - x^2}$ in first quadrant.

Required area OADBO (Area of the circle which is interior to the parabola) (shaded)

= 2 × Area OADO = 2 [Area OAC + Area CAD]

= 2 [Area between parabola (*i*) and *x*-axis in first quadrant + Area between circle (*ii*) and *x*-axis in first quadrant]

$$= 2\left[{}^*\int_0^{1/2} 2\sqrt{x}\,dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\,dx\right] \qquad [\text{Area} = \int y\,dx]$$

$$= 2\left[\left\{2 . \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\}_0^{\frac{1}{2}} + \left\{\frac{x\sqrt{\frac{9}{4} - x^2}}{2} + \frac{\frac{9}{4}}{2}\sin^{-1}\frac{x}{\frac{3}{2}}\right\}_{\frac{1}{2}}^{\frac{3}{2}}\right]$$

$$\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left[\frac{4}{3} \times \frac{1}{2\sqrt{2}} + \frac{9}{8}\sin^{-1} 1 - \frac{\frac{1}{2}\sqrt{2}}{2} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right] \quad \left[\because \left(\frac{1}{2}\right)^{3/2} = \frac{1}{2}\sqrt{\frac{1}{2}} = \frac{1}{2\sqrt{2}}\right]$$

$$= 2\left[\frac{\sqrt{2}}{3} + \frac{9}{8} . \frac{\pi}{2} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right] = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} + \frac{\sqrt{2}}{6}.$$

$$\left|\because \frac{2}{3} - \frac{2}{4} = \frac{1}{6}\right.$$

16. Choose the correct answer:
Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

(A) -9 **(B)** $\frac{-15}{4}$ **(C)** $\frac{15}{4}$ **(D)** $\frac{17}{4}$.

Sol. Equation of the curve is $y = x^3$ ...(*i*)

Let us draw the graph of curve (*i*) for values of x from $x = -2$ to $x = 1$.

Table of Values for $y = x^3$

x	-2	-1	0	1
y	-8	-1	0	1

We are to find the area of the total shaded region.

We will have to find the two shaded areas OBN and OAM separately because from the table,

*Limits of integration for parabola are $x = 0$ to x of point of intersection and for circle are x of point of intersection to x = radius of circle.

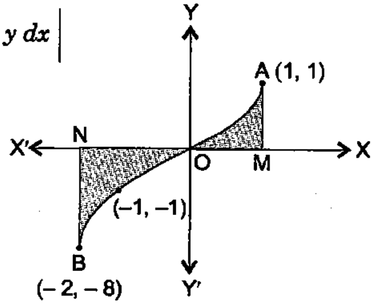
$y = x^3 \le 0$ for $-2 \le x \le 0$ for the region OBN
and $y = x^3 \ge 0$ for $0 \le x \le 1$ for the region OAM

Now area of region OBN $= \left|\int_{-2}^{0} y\,dx\right|$

$= \left|\int_{-2}^{0} x^3\,dx\right|$ (By (*i*))

$= \left|\left(\frac{x^4}{4}\right)_{-2}^{0}\right|$

$= \left|0 - \frac{16}{4}\right| = |-4| = 4$ ...(*ii*)



Again area of region OAM $= \left|\int_{0}^{1} y\,dx\right|$

$= \left|\int_{0}^{1} x^3\,dx\right|$ (By (*i*))

$= \left|\left(\frac{x^4}{4}\right)_{0}^{1}\right| = \left|\frac{1}{4} - 0\right| = \frac{1}{4}$ ...(*iii*)

Adding areas (*ii*) and (*iii*), the total required shaded area

$= 4 + \frac{1}{4} = \frac{16+1}{4} = \frac{17}{4}$ sq. units

∴ Option (D) is the correct answer.

17. Choose the correct answer:
The area bounded by the curve $y = x\,|\,x\,|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by

(A) 0 **(B) $\frac{1}{3}$** **(C) $\frac{2}{3}$** **(D) $\frac{4}{3}$.**

Sol. Equation of the curve is

$y = x\,|\,x\,| = x(x) = x^2$ if $x \ge 0$ ...(*i*)

and $y = x\,|\,x\,| = x(-x)$

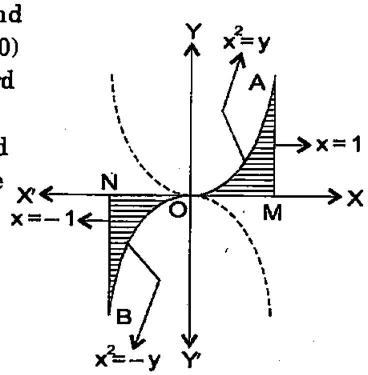
$= -x^2$ if $x \le 0$ ...(*ii*)

Eqn. (*i*) namely $x^2 = y$ $(x \ge 0)$ represents the arc of the upward parabola in first quadrant and equation (*ii*) namely $x^2 = -y$ $(x \le 0)$ represents the arc of the downward parabola in the third quadrant.

We are to find the area bounded by the given curve, x-axis and the ordinates

$x = -1$ and $x = 1$.

Required area = Area ONBO + Area OAMO



$= \left|\int_{-1}^{0} y\, dx\right|$ (for y given by (*ii*))

$+ \left|\int_{0}^{1} y\, dx\right|$ (for y given by (*i*))

$= \left|\int_{-1}^{0} -x^2\, dx\right| + \left|\int_{0}^{1} x^2\, dx\right|$

$= \left|\left(\frac{-x^3}{3}\right)_{-1}^{0}\right| + \left|\left(\frac{x^3}{3}\right)_{0}^{1}\right| = 0 - \left(\frac{-1}{3}\right) + \frac{1}{3} - 0 = \frac{2}{3}$

$\therefore$ Option (C) is the correct answer.

18. Choose the correct answer:

The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$

(A) $\frac{4}{3}(4\pi - \sqrt{3})$ **(B)** $\frac{4}{3}(4\pi + \sqrt{3})$

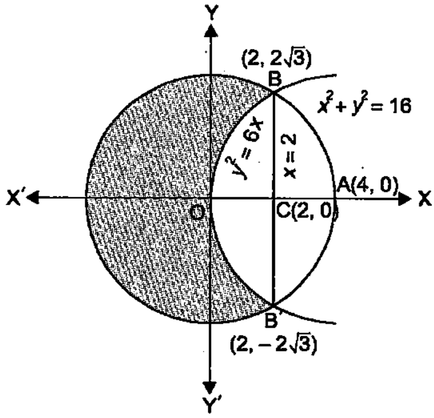
(C) $\frac{4}{3}(8\pi - \sqrt{3})$ **(D)** $\frac{4}{3}(8\pi + \sqrt{3})$.

Sol. Equation of circle is

$x^2 + y^2 = 16$ ...(*i*)

and that of parabola is

$y^2 = 6x$ ...(*ii*)



Now (*i*) is the circle with centre at O(0, 0) and radius 4.

$\therefore$ A $\leftrightarrow$ (4, 0)

Also, this circle is symme-trical about x-axis ($\because$ on changing y to $-y$, its equation remains unaltered.)

Also Circle (*i*) is symmetrical about y-axis.

Equation (*ii*) represents a rightward parabola with vertex at origin O. It is also symmetrical about x-axis.

To find the points of intersection of the two curves, let us solve them for x and y.

Putting $y^2 = 6x$ from (*ii*) in (*i*),

$$x^2 + 6x - 16 = 0 \text{ or } (x + 8)(x - 2) = 0$$

$\Rightarrow$ $x = -8$ or 2

When $x = -8$, from (*ii*), $y^2 = -48 < 0$ so $x = -8$ is not possible.

When $x = 2$, from (*ii*), $y^2 = 12 \Rightarrow y = \pm 2\sqrt{3}$

$\therefore$ The two points of intersection are B$(2, 2\sqrt{3})$ and B′ $(2, -2\sqrt{3})$.

Required area (shaded) = Area of circle – area of circle interior to the parabola

$= \pi \times 4^2$ – area OBAB′O

(∵ area of circle $= \pi r^2$, here $r = 4$)

$= 16\pi - 2 \times$ area OBACO ...(*iii*)

(∵ the two curves are symmetrical about x-axis.)

Now area OBACO = area OBCO + area BACB

= (area under arc OB of parabola and x-axis)

+ (area under arc BA of circle and x-axis)

$$= \int_0^2 \sqrt{6x}\, dx + \int_2^4 \sqrt{16 - x^2}\, dx$$

from (*ii*)   from (*i*)

$$= \sqrt{6} \cdot \left[\frac{x^{3/2}}{3/2}\right]_0^2 + \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{2}{3}\sqrt{6}\ (2\sqrt{2}) + 8 \sin^{-1} 1 - \sqrt{12} - 8 \sin^{-1} \frac{1}{2}$$

or area OBACO $= \dfrac{8}{\sqrt{3}} + 8 \cdot \dfrac{\pi}{2} - 2\sqrt{3} - 8 \cdot \dfrac{\pi}{6}$

$$\left[\because \sin\frac{\pi}{2} = 1 \text{ and } \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$$= \frac{8}{\sqrt{3}} - 2\sqrt{3} + 8\pi\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$= \frac{8-6}{\sqrt{3}} + 8\pi\left(\frac{3-1}{6}\right) = \frac{2}{\sqrt{3}} + \frac{8\pi}{3}$$

Putting this value of area OBACO in (*i*),

$$\text{Required area} = 16\pi - 2\left(\frac{2}{\sqrt{3}} + \frac{8\pi}{3}\right)$$

$$= 16\pi - \frac{4}{\sqrt{3}} - \frac{16\pi}{3}$$

$$= 16\pi\left(1 - \frac{1}{3}\right) - \frac{4}{\sqrt{3}} = \frac{32\pi}{3} - \frac{4}{\sqrt{3}}$$

$$= \frac{32\pi}{3} - \frac{4\sqrt{3}}{3} = \frac{4}{3}(8\pi - \sqrt{3}) \text{ sq. units.}$$

∴ Option (C) is the correct answer.

19. Choose the correct answer:
The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \dfrac{\pi}{2}$ is

(A) $2(\sqrt{2} - 1)$ **(B)** $\sqrt{2} - 1$ **(C)** $\sqrt{2} + 1$ **(D)** $\sqrt{2}$.

Sol. We are to find the area bounded by y-axis, $y = \cos x$, $y = \sin x$ when $0 \le x \le \dfrac{\pi}{2}$.

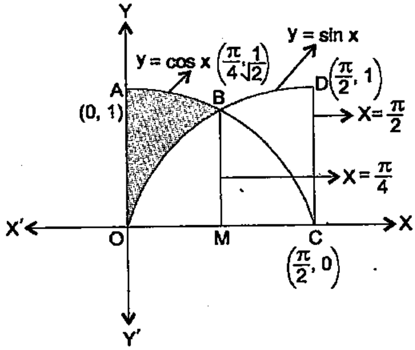


Table of values for $y = \cos x$ $\left(0 \le x \le \frac{\pi}{2}\right)$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

Table of values for $y = \sin x$ $\left(0 \le x \le \frac{\pi}{2}\right)$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

From the two tables of values, we observe that graphs of $y = \sin x$ and $y = \cos x$ $\left(0 \le x \le \frac{\pi}{2}\right)$ have a common point *i.e.*, intersect at the point B $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

Now required shaded area OAB

= Area OABM – Area OBM

= (Area bounded by the curve $y = \cos x$, x-axis and the vertical lines $x = 0$ to $x = \frac{\pi}{4}$)

– (Area bounded by the curve $y = \sin x$, x-axis and the vertical lines $x = 0$ to $x = \frac{\pi}{4}$)

$$\int_0^{\pi/4} \cos x \, dx - \int_0^{\pi/4} \sin x \, dx = (\sin x)_0^{\pi/4} - (-\cos x)_0^{\pi/4}$$

$$= \sin \frac{\pi}{4} - \sin 0 + (\cos \frac{\pi}{4} - \cos 0)$$

$$= \frac{1}{\sqrt{2}} - 0 + \frac{1}{\sqrt{2}} - 1 = \frac{2}{\sqrt{2}} - 1 = \left(\sqrt{2}-1\right) = \text{ sq units.}$$

∴ Option (B) is the correct answer.

Remark. We were required to find area bounded by y-axis. The second possible solution was:

$$\text{Required area} = \left|\int_0^{1/\sqrt{2}} x\,dy\right| \text{ where } x = \sin^{-1} y \text{ from } y = \sin x$$

$$+ \left|\int_{1/\sqrt{2}}^{1} x\,dy\right| \text{ where } x = \cos^{-1} y \text{ from } y = \cos x$$

Since it is laborious to evaluate $\int \sin^{-1} y\,dy$

$= \int \sin^{-1} y\,.1\,dy$ and $\int \cos^{-1} y\,dy = \int \cos^{-1} y\,.1\,dy$,

so, we have chosen to the solution by first method.